Symmetric Topological Phases

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Plan:

• Today:
The introduction of symmetry fractionalization:
  (1) AKLT chain
  (2) Generalized symmetry fractionalizations for:
    topological defects (dislocations in topological insulators)
    topological excitations in topologically ordered phases.

• Tomorrow:
  (1) Quantum spin liquid phases in frustrated magnets, and related experiments in materials
  (2) Parton constructions of quantum spin liquids, and symmetry fractionalization
The modern view of gapped quantum phases

- Landau phases
  - Ising ferromagnet
  - Ising paramagnet

- + Topological Phases

- "Standard model"
The modern view of gapped quantum phases

- Landau phases
  - Ising ferromagnet
  - Ising paramagnet
  - ...

  "Standard model"

- Generalization of IQH phases
- Generalization of FQH phases

+ Topological Phases
The modern view of gapped quantum phases

Landau phases

- Ising ferromagnet
- Ising paramagnet

+ Topological Phases

"Standard model"

symmetry protected topological phases

- Top. Insulator
- Top. superconductor

- Ordinary bulk excitation
- Symmetry-protected gapless edge modes

Generalization of FQH phases
The modern view of gapped quantum phases

Landau phases
- Ising ferromagnet
- Ising paramagnet

+ Topological Phases

“Standard model”

Symmetry protected topological phases
- Top. Insulator
- Top. superconductor

- Ordinary bulk excitation
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Topologically ordered phases
- Gapped spin liquid
- Fractional Chern insulator

- Anyon bulk excitation
- Topological ground state degeneracy
- Robust even without any symmetry
The modern view of gapped quantum phases

- Landau phases
  - Ising ferromagnet
  - Ising paramagnet

+ Topological Phases
  - Ordinary bulk excitation
  - Symmetry-protected gapless edge modes

- Topologically ordered phases
  - Gapped spin liquid
  - Fractional Chern insulator

- Standard model

- Symmetry protected topological phases
  - Top. Insulator
  - Top. superconductor

- Topological ground state degeneracy
- Robust even without any symmetry
Symmetry protected topological phases

- These characteristic gapless edge states are “anomalous”.
  Namely, they can never be realized in a (d-1)-dimension system, assuming certain global symmetry is respected.
Example: Anomalous edge states

- Chiral edge state of integer quantum hall liquid cannot be realized in 1-spatial dimension:

  Easy to show: \[ j = \frac{e}{h} \cdot \mu \]

  Namely, a position dependent chemical potential will break current conservation in 1D.
Example: Anomalous edge states

- Chiral edge state of integer quantum hall liquid cannot be realized in 1-spatial dimension:

  \[ j = \frac{e}{h} \cdot \mu \]

  Easy to show:

  Namely, a position dependent chemical potential will break current conservation in 1D.

  This is just the hall effect at 2D boundary.
SPT phases --- Key feature:

• Gapped bulk. Conventional bulk excitations.

• Anomalous edge states that cannot be realized in local (d-1)-dimensional quantum systems (assuming certain global symmetries).

• In some sense, this is precisely why these d-dimensional topological phases are robust: One CANNOT think about the system as a trivial bulk glued with a (d-1)-dimensional gapless system.
Symmetry fractionalization in 1D SPT phases

• Next, I will present the first example of symmetry protected topological phases beyond quantum hall liquids
  --- the AKLT spin-1 chain model. Affleck, Kennedy, Lieb, Tasaki (PRL 1987)

• AKLT model has a gapped bulk, but gapless spin-1/2 edge states.
  --- Edge states are also anomalous. No way to realize in 0-d. symmetry become “fractionalized”.

• Confirmed in experiments: (e.g., in NENP spin-1 chain)
Spin-1 chain: the AKLT model

- Consider an antiferromagnetic spin-1 chain:

  ![Spin-1 Chain Diagram]

- Let’s modify the usual Heisenberg model a little bit:

  \[ H = K \sum_i [S_i \cdot S_{i+1} + \beta (S_i \cdot S_{i+1})^2] \quad (K > 0) \]

  I will show:
  when \( \bar{\gamma} = 1/3 \), the model is (quasi-)exactly solvable, with interesting ground state. Affleck, Kennedy, Lieb, Tasaki (PRL 1987)

  Here “(quasi-)” means that one can solve the ground state(s) exactly, but not the excited states.
Spin-1 chain: the AKLT model

• Consider an antiferromagnetic spin-1 chain:

\[ S_i \quad S_{i+1} \]

• Let’s modify the usual Heisenberg model a little bit:

\[
H = K \sum_i [S_i \cdot S_{i+1} + \beta (S_i \cdot S_{i+1})^2] \quad (K > 0)
\]

Note that unlike the spin-1/2 case, for spin-1, \((S_i \cdot S_j)^2\) is an independent operator.

Let \( \vec{J} \equiv \vec{S}_i + \vec{S}_j \), \( J^2 = j(j+1) = \begin{cases} 0 \text{ or } 2 & \text{if } S = \frac{1}{2} \\ 0, 2, 6 & \text{if } S = 1 \end{cases} \)

\[
\vec{S}_i \cdot \vec{S}_j = \frac{1}{2} (\vec{J}^2 - \vec{S}_i^2 - \vec{S}_j^2)
\]

\[
= \begin{cases} \frac{1}{2} (\vec{J}^2 - \frac{3}{2}) & \text{if } S = \frac{1}{2} \\ \frac{1}{2} (\vec{J}^2 - 4) & \text{if } S = 1 \end{cases}
\]
Spin-1 chain: the AKLT model

- Consider an antiferromagnetic spin-1 chain:

  ![Image of a spin-1 chain]

  $S_i \quad S_{i+1}$

- Let's modify the usual Heisenberg model a little bit:

  $$H = K \sum_i [S_i \cdot S_{i+1} + \beta (S_i \cdot S_{i+1})^2]$$

  for $S=1$, $\vec{J} = \vec{S}_i + \vec{S}_j$

<table>
<thead>
<tr>
<th>$j=0$</th>
<th>$j=1$</th>
<th>$j=2$</th>
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<tbody>
<tr>
<td>$\vec{J}^2$</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>$\vec{S}_i \cdot \vec{S}_j$</td>
<td>-2</td>
<td>-1</td>
</tr>
<tr>
<td>$(\vec{S}_i \cdot \vec{S}_j)^2$</td>
<td>4</td>
<td>1</td>
</tr>
</tbody>
</table>

  $\frac{1}{2} [\vec{S}_i \cdot \vec{S}_j + \frac{1}{3} (\vec{S}_i \cdot \vec{S}_j)^2 + \frac{\beta}{3}]$ | 0 | 0 | 1

  This is why $\beta = \frac{1}{3}$ is special! Projector into $j=2$.
Spin-1 chain: the AKLT model

- Consider an antiferromagnetic spin-1 chain:

  ![Spin-1 chain diagram]

  \( \mathbf{S}_i \quad \mathbf{S}_{i+1} \)

- Let’s modify the usual Heisenberg model a little bit:

  \[
  H = \frac{K}{2} \sum_i \delta[(s_i + s_{i+1}) = 2] + \text{const}
  \]

  for \( S = 1 \):

  \[
  \mathbf{J} = \mathbf{S}_i + \mathbf{S}_j
  \]

<table>
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<th>( j )</th>
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</tr>
</tbody>
</table>

  \( \frac{1}{2} \left[ \mathbf{S}_i \cdot \mathbf{S}_j + \frac{1}{3} (\mathbf{S}_i \cdot \mathbf{S}_j)^2 + \frac{2}{3} \right] \)

  \( \frac{1}{2} \left[ \mathbf{S}_i \cdot \mathbf{S}_j + \frac{1}{3} (\mathbf{S}_i \cdot \mathbf{S}_j)^2 + \frac{2}{3} \right] \)

  \( 0 \quad 0 \quad 1 \)

  this is why \( \beta = \frac{1}{3} \)

  is special!

  projector into \( j=2 \)
Spin-1 chain: the AKLT model

- Consider an antiferromagnetic spin-1 chain:

\[ S_i \quad S_{i+1} \]

- Let’s modify the usual Heisenberg model a little bit:

\[
H = \frac{K}{2} \sum_i \delta[(S_i + S_{i+1}) = 2] + \text{const}
\]

If we can find a quantum state \(|a>\), such that the combined spin of nearest neighbors can only be 0 or 1, then \(|a>\) will certainly be one ground state.

Surprisingly, it is quite easy to write down such a state \(|a>\).
Spin-1 chain: the AKLT model

• Consider an antiferromagnetic spin-1 chain:

\[ S_i \quad S_{i+1} \]

• Let’s modify the usual Heisenberg model a little bit:

\[
H = \frac{K}{2} \sum_i \delta[(S_i + S_{i+1}) = 2] + \text{const}
\]

If we can find a quantum state |^a>, such that the combined spin of nearest neighbors can only be 0 or 1, then |^a> will certainly be one ground state.

The idea to write down |^a> is to split each spin-1 into two auxiliary spin-1/2’s:

Argument: for every two n.n. sites, 2 of the 4 spin-1/2’s form singlet, the other two can only form J=0 or 1.
AKLT model: the exact ground state(s)

\[ H = \frac{K}{2} \sum_i \delta[(s_i + s_{i+1}) = 2] + \text{const} \]

Let's construct the ground state \( |\alpha\rangle \) explicitly:

represent 3 states @ site-\( i \) as, using 2 spin-\( \frac{1}{2} \)

\[ |\phi_{\uparrow\uparrow}\rangle = |\uparrow\rangle \otimes |\uparrow\rangle, \quad |\phi_{\uparrow\downarrow}\rangle = |\uparrow\rangle \otimes |\downarrow\rangle + |\downarrow\rangle \otimes |\uparrow\rangle \]

namely:

\[ |\phi_{\uparrow\uparrow}\rangle = |m = +1\rangle \]

\[ |\phi_{\uparrow\downarrow}\rangle = |m = -1\rangle \]

\[ |\phi_{\uparrow\downarrow}\rangle = \sqrt{2}/2 \cdot |m = 0\rangle. \]
AKLT model: the exact ground state(s)

\[ H = \frac{K}{2} \sum_i \delta[(s_i + s_{i+1}) = 2] + \text{const} \]

Let's construct the ground state \( |\alpha\rangle \) explicitly:

Represent 3 states @ site-\( i \) as, using 2 spin-\( \frac{1}{2} \)

\[ |\Phi_{\uparrow\uparrow}\rangle = |\uparrow\rangle \otimes |\uparrow\rangle, \quad |\Phi_{\uparrow\downarrow}\rangle = |\uparrow\rangle \otimes |\downarrow\rangle \]
\[ |\Phi_{\downarrow\uparrow}\rangle = |\downarrow\rangle \otimes |\uparrow\rangle, \quad |\Phi_{\downarrow\downarrow}\rangle = \frac{1}{2} (|\uparrow\rangle \otimes |\downarrow\rangle + |\downarrow\rangle \otimes |\uparrow\rangle) \]

Namely:

\[ |\Phi_{+1}\rangle = |m=+1\rangle \]
\[ |\Phi_{-1}\rangle = |m=-1\rangle \]
\[ |\Phi_{0}\rangle = \sqrt{\frac{1}{2}}, \quad |m=0\rangle \]

Under spin-1 rotation:

\[ \hat{S} |\Phi_{\alpha\beta}\rangle = (\frac{\sqrt{3}}{2})_{\alpha',\beta'} |\Phi_{\alpha',\beta'}\rangle \]

\[ + (\frac{\sqrt{3}}{2})_{\beta',\beta} |\Phi_{\alpha'\beta}\rangle \]

Exactly like 2 spin-\( \frac{1}{2} \).
AKLT model: the exact ground state(s)

\[ H = \frac{K}{2} \sum_i \delta([s_i + s_{i+1}) = 2] + \text{const} \]

Let's construct the ground state \( |\psi> \) explicitly:

Represent 3 states at site-\( i \) as, using 2 spin-\( \frac{1}{2} \):

\[ |\psi_{\uparrow \uparrow}> = |\uparrow> \otimes |\uparrow> \quad |\psi_{\uparrow \downarrow}> = |\uparrow> \otimes |\downarrow> \quad |\psi_{\downarrow \uparrow}> = \frac{1}{2} (|\uparrow> \otimes |\downarrow> + |\downarrow> \otimes |\uparrow>) \]

E.g. \( \hat{S}^z |\psi_{\alpha \beta}> = (S^z_{\alpha} + S^z_{\beta}) |\psi_{\alpha \beta}> \)

1. \( S^+ |\psi_{\uparrow \downarrow}> = S^+ |m=-1> \)
   \[ = \sqrt{2} |m=0> = 2 |\psi_{\uparrow \downarrow}> \]

2. \( \hat{S}^+ |\phi_{\alpha \beta}> = \left( \frac{\hat{\sigma}^+}{2} \right)_{\alpha \beta} |\phi_{\alpha \beta}> \)

   \[ = \frac{1}{2} \left( |\phi_{\alpha \beta}> + |\phi_{\beta \alpha}> \right) \]

Under spin-1 rotation:

\[ \hat{S} |\phi_{\alpha \beta}> = \left( \frac{\hat{\sigma}^+}{2} \right)_{\alpha \beta} |\phi_{\alpha \beta}> + \left( \frac{\hat{\sigma}^-}{2} \right)_{\beta \alpha} |\phi_{\alpha \beta}> \]

Exactly like 2 spin-\( \frac{1}{2} \).
AKLT model: the exact ground state(s)

\[ H = \frac{K}{2} \sum_i \delta[(s_i + s_{i+1}) = 2] + \text{const} \]

Let's construct the ground state \( |^{a} \rangle \) explicitly:

Consider two-site chain:

\[ \text{G.S.: } |^{4}_{\alpha\beta} \rangle = \sum_{\beta, \alpha_2} |^{1}_{\alpha_1\beta_1} \otimes |^{2}_{\alpha_2\beta_2} \rangle \cdot \delta_{\beta_1, \alpha_2} \]

only form \( J = 0 \) or \( 1 \)

\[ \sum_{\nu, \nu'} = -3 \nu \uparrow \nu = 1. \]
AKLT model: the exact ground state(s)

\[ H = \frac{K}{2} \sum_i \delta[(s_i + s_{i+1}) = 2] + \text{const} \]

Let's construct the ground state \(|\alpha\rangle\) explicitly:

Consider two-site chain:

\[ \text{G.S.: } |4_{\alpha\beta}\rangle = \sum_{\alpha_1, \alpha_2} |\phi_{\alpha_1\beta_{\alpha_1}}\rangle \otimes |\phi_{\alpha_2\beta_{\alpha_2}}\rangle \cdot \Sigma_{\beta_{\alpha_1}, \beta_{\alpha_2}} \quad (\alpha, \beta = \uparrow, \downarrow) \]

Proof:

\[ (S_i + S_{i+1}) |4_{\alpha\beta}\rangle = \left(\frac{\hat{S}_z}{2}\right)_{\alpha_1\beta_1} |4_{\alpha\beta}\rangle + \left(\frac{\hat{S}_z}{2}\right)_{\beta_1\beta_2} |4_{\alpha\beta}\rangle \]

\[ + \left(\frac{\hat{S}_z}{2}\right)_{\beta_2\beta_1} |\phi_{\alpha_1\beta_{\alpha_1}}\rangle \otimes |\phi_{\alpha_2\beta_{\alpha_2}}\rangle \cdot \Sigma_{\beta_1, \beta_2} \]

\[ + \left(\frac{\hat{S}_z}{2}\right)_{\alpha_1\beta_2} |\phi_{\alpha_1\beta_{\alpha_1}}\rangle \otimes |\phi_{\alpha_2\beta_{\alpha_2}}\rangle \cdot \Sigma_{\beta_1, \beta_2} \]

\[ = -\left(\frac{\hat{S}_z}{2}\right)_{\alpha_1\alpha_2} |4_{\alpha\beta}\rangle \]

\[ = \left(\frac{\hat{S}_z}{2}\right)_{\alpha_1\alpha_2} |4_{\alpha\beta}\rangle \]

\[ = \left(\frac{\hat{S}_z}{2}\right)_{\alpha_1\alpha_2} \cdot \Sigma_{\alpha_1, \alpha_2} = 1 \]

\[ \sum_{\uparrow \downarrow} = -\Sigma_{\uparrow \downarrow} = 1 \]
AKLT model: the exact ground state(s)

\[ H = \frac{K}{2} \sum_i \delta[(s_i + s_{i+1}) = 2] + \text{const} \]

Let's construct the ground state \(|\alpha\rangle\) explicitly:

Consider two-site chain:

\[ \text{G.S.: } |\Psi_{\alpha\beta}\rangle = \sum_{\beta, \alpha_2} |\Phi_{\alpha_1 \beta_1}\rangle \otimes |\Phi_{\alpha_2 \beta}\rangle > \cdot \Sigma \beta, \alpha_2 \]

Proof:

\[ (\vec{S}_1 + \vec{S}_2) |\Psi_{\alpha\beta}\rangle = (\frac{\vec{S}}{2})_{\alpha \alpha} |\Psi_{\alpha\beta}\rangle + (\frac{\vec{S}}{2})_{\beta \beta} |\Psi_{\alpha\beta}\rangle \]

Be cautious: \(|\Psi_{\alpha\beta}\rangle\) are NOT orthonormal.
But enough to prove \(\delta(\vec{S}_1 + \vec{S}_2 = 2) = 0\).
AKLT model: the exact ground state(s)

\[ H = \frac{K}{2} \sum_i \delta[(s_i + s_{i+1}) = 2] + \text{const} \]

Let's construct the ground state \( |\alpha\rangle \) explicitly:

**general G.S. (totally 4 of them)**

\[ |\Psi_{\alpha\beta}\rangle \equiv \sum_{\beta_1, \alpha_1} \sum_{\beta_2, \alpha_2} \sum_{\beta_3, \alpha_3} \cdots \sum_{\beta_N, \alpha_N} \phi_{\alpha_1 \beta_1} \phi_{\alpha_2 \beta_2} \phi_{\alpha_3 \beta_3} \cdots \phi_{\alpha_N \beta_N} \]

**total spin rotation:**

\[ \sum_i \frac{\hat{s}_i}{2} |\Psi_{\alpha\beta}\rangle = \left( \frac{\hat{\sigma}_z}{2} \right)_{\alpha'\beta'} |\Psi_{\alpha'\beta'}\rangle + \left( \frac{\hat{\sigma}_z}{2} \right)_{\beta'\beta} |\Psi_{\beta'\beta}\rangle \]

exactly like two spin-\( \frac{1}{2} \) at ends.
Is the spin-1/2 fake or real?

\[
\left(\sum_i \vec{\sigma}_i\right) |\psi_{\alpha\beta}\rangle = (\frac{\sigma}{2})_{d'd'} |\psi_{d'\beta}\rangle + (\frac{\sigma}{2})_{\beta'\beta} |\psi_{\alpha\beta}\rangle
\]

Although formally looks like two spin-1/2 at the edges:

- Two things to worry about:
  1. $|\psi_{\alpha\beta}\rangle$ are not orthonormal.
  2. I have not constructed a local operator acting only on one edge that implements the spin-1/2 rotation.
Local spin rotations

• Two things to worry about:

\( | \psi_{\alpha\beta} \rangle \) are not orthonormal.

--- orthonormal up to exponentially small error as \( L \) increases

\[
\begin{align*}
\langle \psi_{\uparrow\uparrow} | \psi_{\uparrow\uparrow} \rangle &= \langle \psi_{\downarrow\downarrow} | \psi_{\downarrow\downarrow} \rangle \\
\approx \langle \psi_{\uparrow\downarrow} | \psi_{\downarrow\uparrow} \rangle &= \langle \psi_{\downarrow\uparrow} | \psi_{\downarrow\uparrow} \rangle \sim (\frac{3}{2})^L \\
\text{and } \langle \psi_{\downarrow\uparrow} | \psi_{\downarrow\uparrow} \rangle \sim (\frac{1}{2})^L. & \quad \text{other overlaps are zero.}
\end{align*}
\]
Local spin rotations

- Two things to worry about:

1. \( |\psi_{\alpha}\beta\rangle\) are not orthonormal.
   --- orthonormal up to exponentially small error as \(L\) increases

2. I have not constructed a local operator acting only on one edge that implements the spin-1/2 rotation.
   --- result in (1) allows us to construct such an operator:

Consider a long chain:
Define local unitary (almost) operators for the two edge segments:

\[
\begin{align*}
\sqrt{S}_L \mid \psi_{\alpha}, \rho \rangle &= \left( \frac{\sigma}{2} \right)_{\alpha \alpha} \mid \psi_{\alpha}, \rho \rangle \\
\sqrt{S}_R \mid \psi_{\alpha-1}, \rho \rangle &= \left( \frac{\sigma}{2} \right)_{\beta \beta} \mid \psi_{\alpha-1}, \rho \rangle
\end{align*}
\]
Key feature of symmetry fractionalization

We find:

\[ H = \frac{K}{2} \sum_i \delta[(s_i + s_{i+1}) = 2] + \text{const} \]

\[ \sum_i \]

\[ \alpha \beta_1 \quad \alpha_2 \beta_2 \quad \alpha_3 \beta_3 \quad \cdots \quad \beta_{n-1} \alpha_n \beta \\
\]

\[ \text{two } S=1/2 \quad \text{Spin singlet bond} \]

Total spin rotation:

\[ \left( \sum_i \hat{s}_i \right) | \Psi_{ab} \rangle = \left( \frac{\hbar}{2} \right) \alpha' \alpha \left| \Psi_{\alpha' \beta} \right\rangle + \left( \frac{\hbar}{2} \right) \beta' \beta \left| \Psi_{\alpha \beta'} \right\rangle \]

\[ \left( \sum_L \right) | \Psi_{ab} \rangle + \left( \sum_R \right) | \Psi_{ab} \rangle. \]

Local operator.

\[ \hat{U}(\hat{\eta}, \phi) \left| \Psi_{ab} \right\rangle = \hat{U}_L(\hat{\eta}, \phi) \circ \hat{U}_R(\hat{\eta}, \phi) \left| \Psi_{ab} \right\rangle \]

Spin rotation.
Key feature of symmetry fractionalization

We find:

$$H = \frac{K}{2} \sum_i \delta[(s_i + s_{i+1}) = 2] + \text{const}$$

This is striking: we started from SO(3) spin-1 model, but we got spin-1/2 on edges.
AKLT model: the exact ground state(s)

\[ H = \frac{K}{2} \sum_i \delta[(s_i + s_{i+1}) = 2] + \text{const} \]

We find 4 ground states for open chain. The ground state degeneracy comes from the unpaired spin-1/2 on each ends of the chain.

One can further show that these are the only four ground states, and the many-body excitation spectrum of the chain has a FINITE energy gap:
AKLT model: the exact ground state(s)

$$H = \frac{K}{2} \sum_i \delta[(s_i + s_{i+1}) = 2] + \text{const}$$

We find 4 ground states for open chain. The ground state degeneracy comes from the unpaired spin-1/2 on each ends of the chain!

However if the chain is a closed loop (periodic boundary condition), there is only a UNIQUE ground state:
The crucial feature of symmetry fractionalization

\[ H = \frac{K}{2} \sum_i \delta[(s_i + s_{i+1}) = 2] + \text{const} \]

Symmetry fractionalization at chain edges --- Crucial feature

In a degenerate sector of energy eigenstates, global symmetry is implemented by product of two local operators spatially far away from each other:

\[ \hat{U}(g) | \psi_a > = \hat{U}_L(g) \cdot \hat{U}_R(g) | \psi_a > \]

\( g \in SG \) (symmetry group)
The crucial feature of symmetry fractionalization

\[ H = \frac{K}{2} \sum_i \delta[(s_i + s_{i+1}) = 2] + \text{const} \]

This is why it is possible to have spin-1/2 edge states in a spin-1 model --- although a single spin-1/2 is NOT a representation of SO(3) symmetry, the product of two spin-1/2’s is a representation: \( \frac{1}{2} \times \frac{1}{2} = 0 + 1 \)

\[ \hat{U}(g) | \psi_a > = \hat{U}_L(g) \cdot \hat{U}_R(g) | \psi_a > \]

\( g \in SG \) (symmetry group)
The crucial feature of symmetry fractionalization

\[ H = \frac{K}{2} \sum_i \delta[(s_i + s_{i+1}) = 2] + \text{const} \]

This is also why the 4-fold degeneracy is robust: consider a local SO(3) symmetric perturbation \( V \):

\[ [\hat{V}, \hat{U}(g)] = 0 \implies [\hat{V}, \hat{U}_L(g)] = 0 \text{ AND } [\hat{V}, \hat{U}_R(g)] = 0 \]

The two SU(2) symmetries are individually respected!

\[ \hat{U}(g) | \Phi_a > = \hat{U}_L(g) \cdot \hat{U}_R(g) | \Phi_a > \]

\( g \in SG \text{ (symmetry group)} \)
The crucial feature of symmetry fractionalization

\[ H = \frac{K}{2} \sum_i \delta[(s_i + s_{i+1}) = 2] + \text{const} \]

This is also why the 4-fold degeneracy is robust: consider a local SO(3) symmetric perturbation \( V \):

\[ [\hat{V}, \hat{U}(g)] = 0 \quad \Rightarrow \quad [\hat{V}, \hat{U}_L(g)] = 0 \quad \text{AND} \quad [\hat{V}, \hat{U}_R(g)] = 0 \]

The two SU(2) symmetries are individually respected!

However if perturbation is not SO(3) symmetric, the 4-fold degeneracy is gone.

Such spin-1/2 edge states are characteristic feature in the whole AKLT phase. You can kill the spin-1/2 edge states only by (1) a phase transition OR (2) removing the symmetry. ---- called Symmetry Protected Topological Phase
How to generally understand 1D SPT phases?

• From AKLT chain, we know, under internal symmetry $g$:

$$\hat{U}(g) |\Psi\rangle = \hat{U}_L(g) \circ \hat{U}_R(g) |\Psi\rangle$$

There is a potential phase ambiguity in the definition of $\hat{U}_A(g)$. They do not have to form reps of SG. Instead they can be “projective representation” of SG. (e.g. spin-1/2 is projective rep of SO(3))

$$\hat{U}_L(g_1) \circ \hat{U}_L(g_2) = e^{i \theta(g_1, g_2)} \hat{U}_L(g_1 \circ g_2)$$
How to generally understand 1D SPT phases?  

Turner, Pollmann, Berg, Oshikawa, Chen, Gu, Wen....

- From AKLT chain, we know, under internal symmetry g:

\[ \hat{U}(g) |4> = \hat{U}_L(g) \circ \hat{U}_R(g) |4> \]

There is a potential phase ambiguity in the definition of \( \hat{U}_L(g) \). They do not have to form reps of SG. Instead they can be “projective representation” of SG. (e.g. spin-1/2 is projective rep of SO(3))

\[ \hat{U}_L(g_1) \circ \hat{U}_L(g_2) = e^{i \theta(g_1, g_2)} \hat{U}_L(g_1 g_2) \]

- Mathematically, the phase factor here is called a factor system. It is a function of group elements and have to satisfy consistency condition:

\[ \left[ \hat{U}_L(g_1) \circ \hat{U}_L(g_2) \right] \circ \hat{U}_L(g_3) = \hat{U}_L(g_1) \circ \left[ \hat{U}_L(g_2) \circ \hat{U}_L(g_3) \right] \]

\[ \Rightarrow e^{i \theta(g_1, g_2)} \cdot e^{i \theta(g_1 g_2, g_3)} = e^{i \theta(g_1, g_2 g_3)} e^{i \theta(g_2, g_3)} \]

Mathematically this is 2-cocycle condition. Inequivalent proj. reps are classified by 2\textsuperscript{nd} cohomology group:

\[ H^2(SG, U(1)) \]
A simple example:

\[ \hat{U}_L(g_1) \circ \hat{U}_L(g_2) = e^{i \theta(g_1, g_2)} \hat{U}_L(g_1 \circ g_2) \]

\[ \left[ \hat{U}_L(g_1) \circ \hat{U}_L(g_2) \right] \circ \hat{U}_L(g_3) = \hat{U}_L(g_1) \circ \left[ \hat{U}_L(g_2) \circ \hat{U}_L(g_3) \right] \]

\[ \Rightarrow e^{i \theta(g_1, g_2)} \cdot e^{i \theta(g_1 g_2, g_3)} = e^{i \theta(g_1, g_2 g_3)} \cdot e^{i \theta(g_2, g_3)} \]

Consider \( SG = \mathbb{Z}_2 \times \mathbb{Z}_2 \)

\[ H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)) = \mathbb{Z}_2. \]

\[ \{1, \sigma\} \times \{1, \tau\}. \]

\[ \mathbb{Z}_2 = 0: \text{ trivial}, \]

\[ \mathbb{Z}_2 = 1: \Rightarrow \quad U_L(\sigma) \cdot U_L(\tau) = -U_L(\tau) \cdot U_L(\sigma) \]

"Spinh -\( \frac{1}{2} \)"
How to generally understand 1D SPT phases?

Turner, Pollmann, Berg, Oshikawa, Chen, Gu, Wen….

- $2^{\text{nd}}$ cohomology group can be used to classify 1D (bosonic) SPT phases.

$$H^2[SG, U(1)]$$

<table>
<thead>
<tr>
<th>Symmetry of Hamiltonian</th>
<th>Number of Different Phases</th>
</tr>
</thead>
<tbody>
<tr>
<td>None</td>
<td>1</td>
</tr>
<tr>
<td>$SO(3)$</td>
<td>2</td>
</tr>
<tr>
<td>$D_2$</td>
<td>2</td>
</tr>
<tr>
<td>$T$</td>
<td>2</td>
</tr>
<tr>
<td>$SO(3) + T$</td>
<td>4</td>
</tr>
<tr>
<td>$D_2 + T$</td>
<td>16</td>
</tr>
</tbody>
</table>

Chen, Gu, Wen (2010)
Summary of discussion so far, and outlook

• SPT phase protected by local symmetries: trivial bulk + gapless anomalous edge states.
  
  higher dimensions:

  → Fermion: Integer quantum hall states, topological insulators
      (can be realized even with weak interaction)
  
  Boson: Bosonic integer quantum hall states, bosonic topological insulators
      (require strong interaction to realize)

• In 1D, we find SPT phases host symmetry fractionalized edge states.
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- In 1D, we find SPT phases host symmetry fractionalized edge states.

Next, let’s attempt to generalize these concepts.
Some imaginations

• The essence of SPT phases are: anomalous lower dimensional gapless states are realized at the edge of the system.

What is so special about edges? Can one realize these anomalous lower dimensional states in other situations?
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What is so special about edges? Can one realize these anomalous lower dimensional states in other situations?

Edges are special because:

(1) In 1d, edges must be created in pairs. (This is why fractional quantum number can be realized.)

(2) In 2d, edge must form a closed loop. (This is why non-stoppable helical/chiral modes can be realized.)

(3) In 3d, edge must form a closed surface. (This is why single Dirac-cone can be realized in TI with time-reversal symmetry.)
Some imaginations

- Can we replace the edges by some other objects?
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replacing by point-like topological objects:
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• These objects in principle also could host anomalous lower dimensional states! Indeed, there are already lots of examples.

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Topological defects:

• Vortex hosted majorana modes in p+ip 2d superconductor.
  (Read, Green, Ivanov, Fu, Kane....)

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Topological dynamical excitations: --- “symmetry enriched” phenomena
- Symmetry fractionalizations for gauge charges
  Laughlin state e* = e/3
  Spin-charge separation in gapped quantum spin liquids.
- A large class of exactly solvable models: (Mesaros&YR, 2012)
  Showing: gauge charge hosted symmetry fractionalization in 2d
gauge flux loop hosted anomalous line states in 3d...
Plan:

- Tomorrow I will talk about symmetry fractionalization for gauge charge excitations in quantum spin liquids.

- Today, if time is allowed, let’s have a simple derivation of the so-called “worm-hole” effect in 3D TI. (G. Rosenberg, H.-M. Guo, M. Franz, 2010)
Plan:

• Tomorrow I will talk about symmetry fractionalization for gauge charge excitations in quantum spin liquids.

• Today, if time is allowed, let’s have a simple derivation of the so-called “worm-hole” effect in 3D TI. (G. Rosenberg, H.-M. Guo, M. Franz, 2010)

---If a pi-flux loop (TR sym.) is threaded through the bulk 3D strong TI, the loop is topological bound with helical modes: (same as the anomalous edge state of 2D TI).

---If one interpret the pi-flux as a dynamical $\mathbb{Z}_2$ gauge flux excitation, namely if the TI are not formed by electrons, but by fermions carrying $\mathbb{Z}_2$ gauge charge, this effect is an example of symmetry-enriched phenomena.
The wormhole effect

- A simple model of the pi-flux loop:

Step (1): Cut the 3D TI

Two surfaces-- two sets of Dirac nodes:

\[ H = \mathbf{P} \cdot \mathbf{\sigma} \mu_z \]

\[ (\mu_z = \pm 1 : \text{L/R surfaces}) \]
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• A simple model of the pi-flux loop:

Step (1): Cut the 3D TI

Two surfaces—two sets of Dirac nodes:

\[ H = \vec{p} \cdot \vec{\sigma} m_Z \]

\( m_Z = \pm 1: \ L/R \) surfaces

Step (2): Gluing back, but trapping a pi flux (black) line in the middle

Hopping from left to right: \( m \cdot \vec{M}_x \)

m<0 in red region
m>0 in uncolored region
The wormhole effect

- A simple model of the pi-flux loop:

\[
H = (k_x \sigma_x + k_y \sigma_y) \mu_z + m(x) \mu_x
\]

\[
m(x) = \begin{cases} 
  +m & \text{if } x > 0 \\
  -m & \text{if } x < 0 
\end{cases}
\]
The wormhole effect

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@ \( k_y = 0 \), midgap modes:
\[ \psi_{\pm}(x) = e^{-\int_{0}^{x} m(x') dx'} \cdot \psi_{0,\pm} \]
where \( \sigma \times m_y \cdot \psi_{0,\pm} = \psi_{0,\pm} \).

Adding \( k_y \neq 0 \) \( \Rightarrow \) helical modes.
Dislocations in 3D TI

- Although magnetic pi-flux is difficult to realize in TI, even here in magnetic lab, the crystalline topological defects — dislocations can have similar effect. (YR, Zhang, Vishwanath 2009)

Condition for existence of helical modes:

\[ \vec{B} \cdot \vec{M}_\nu = \pi \text{(mod} 2\pi) \]

- \( \vec{B} \): Burger’s vector in real space.
- \( \vec{M}_\nu \): Weak index vector in momentum space

Realized in TI with nonzero weak index: e.g., SmB6....
Plan:

I was mentioning $\mathbb{Z}_2$ gauge excitations, like flux loops, and their symmetry enriched phenomena.

But can these be realized in materials?

• Tomorrow:

(1) Quantum spin liquid phases in frustrated magnets, and related experiments in materials

(2) Parton constructions of quantum spin liquids, and symmetry fractionalization.