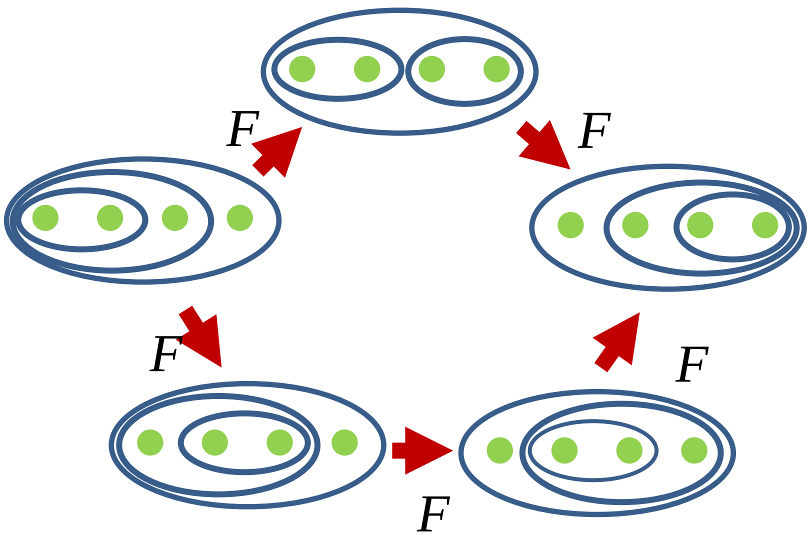
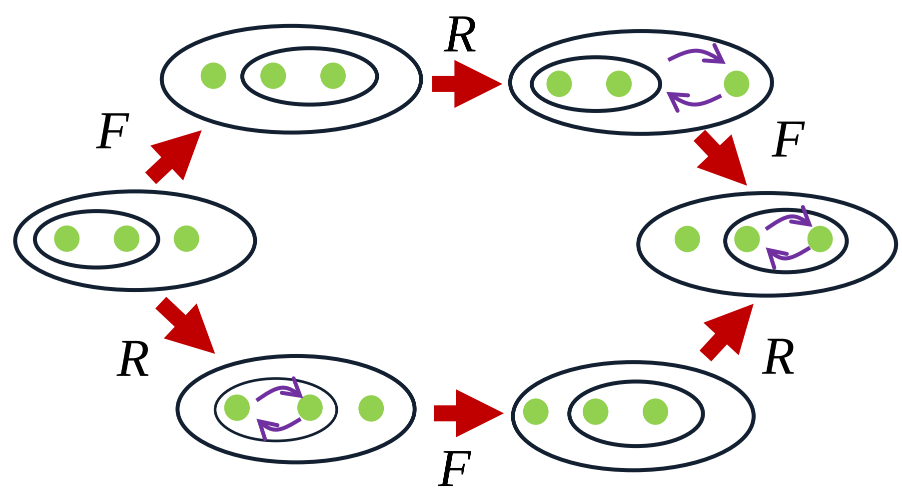
Topological Quantum Computing

Nick Bonesteel

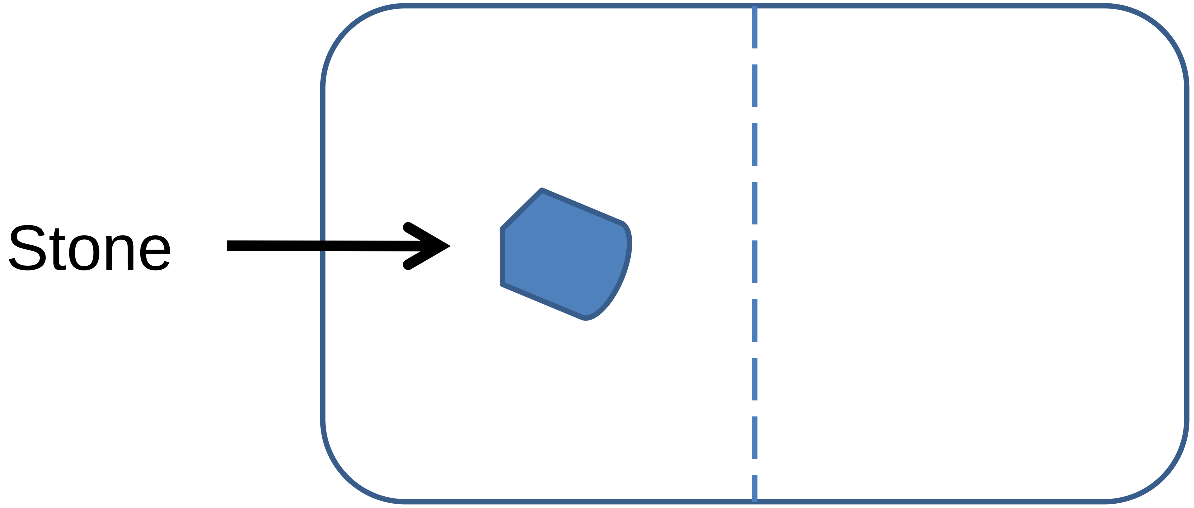
Florida State University





NHMFL 2014 Winter School, Jan. 8, 2014

Early Digital Memory



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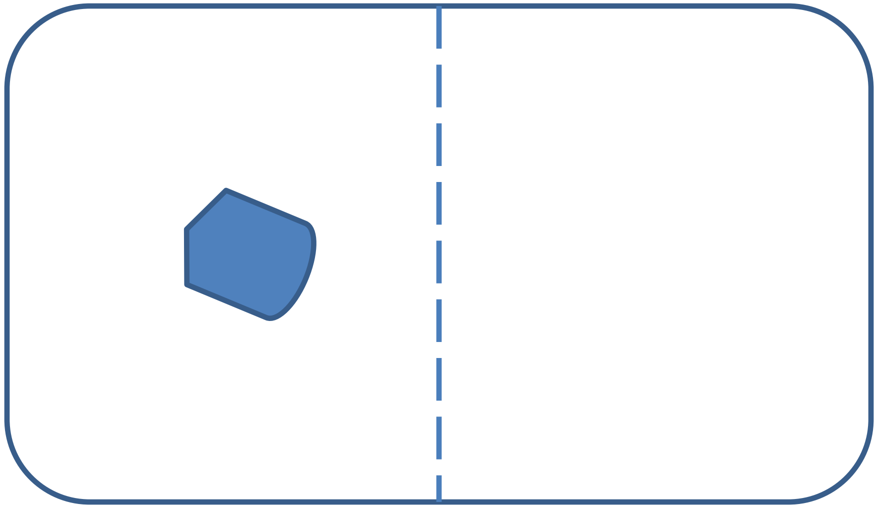
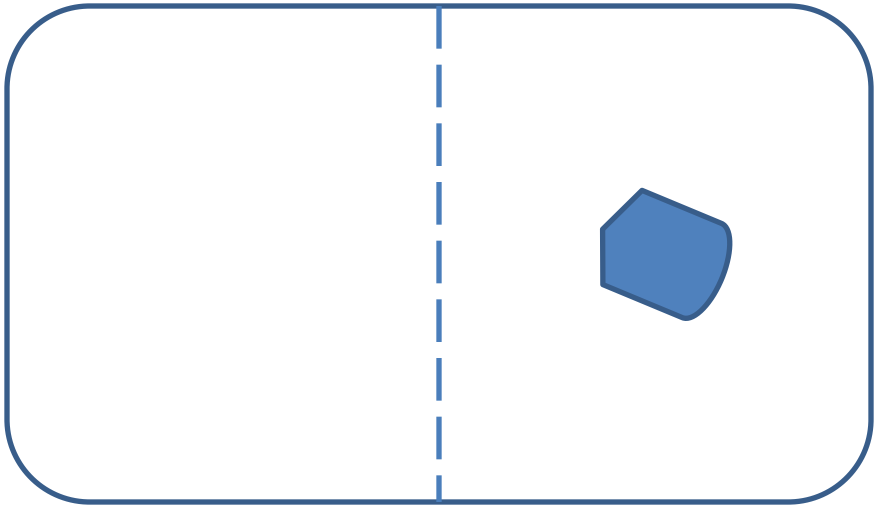


= 0

Early Digital Memory



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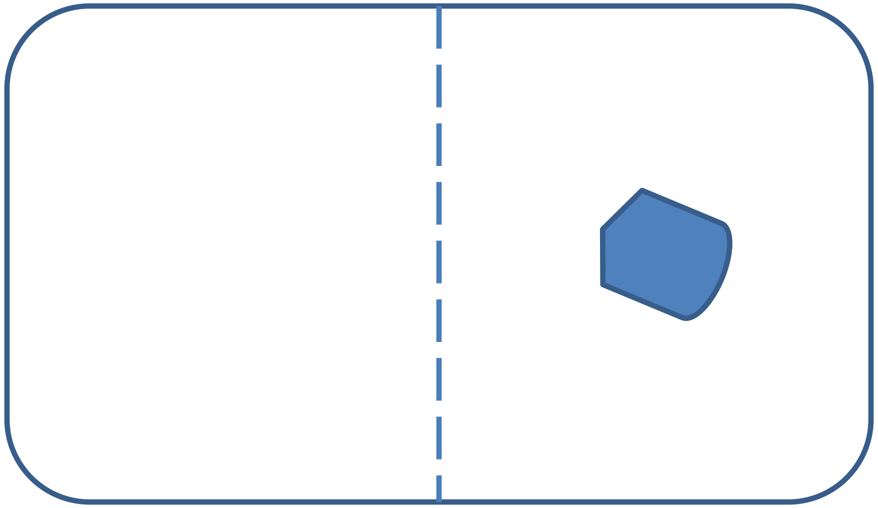
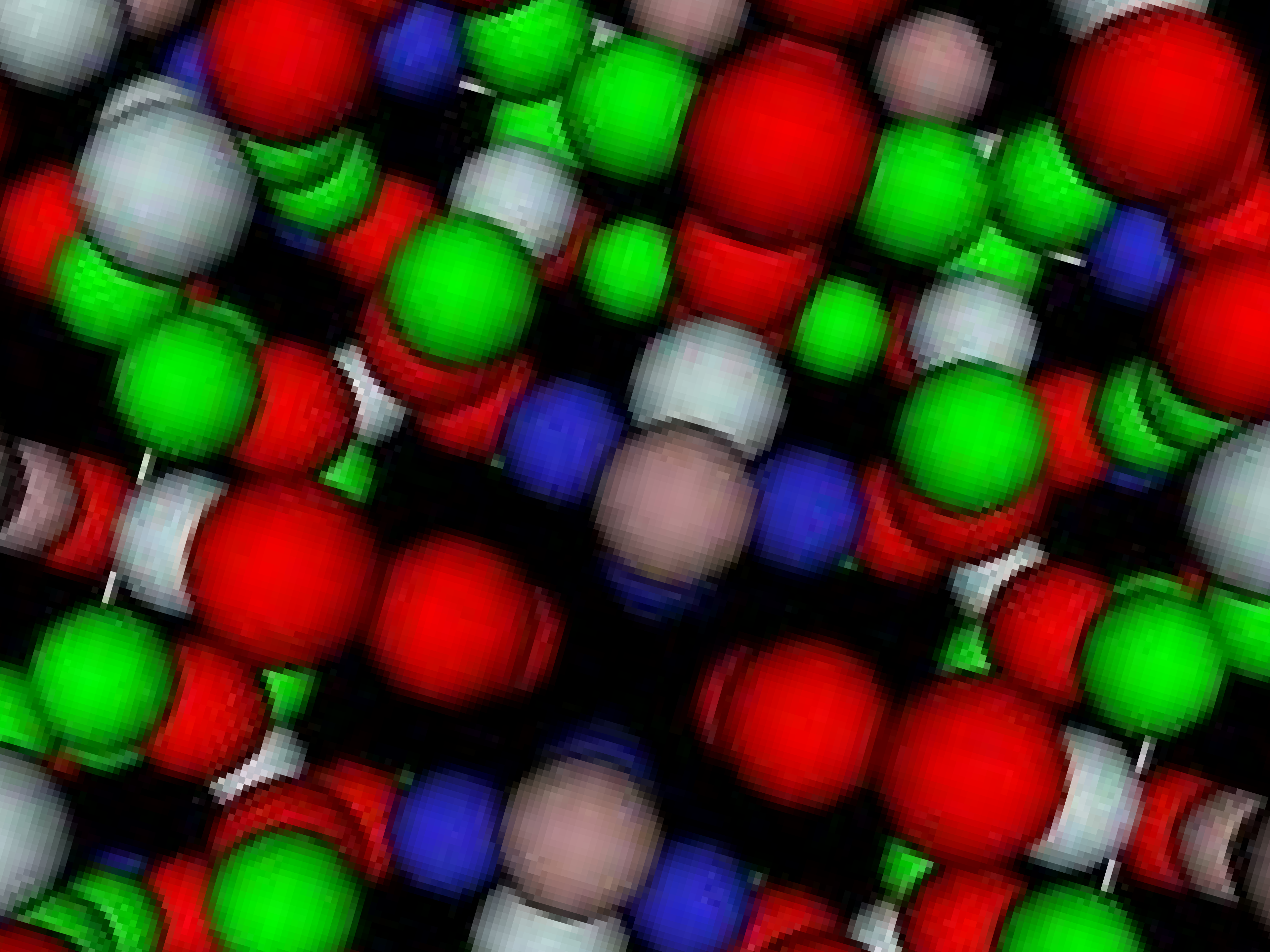
Early Digital Memory

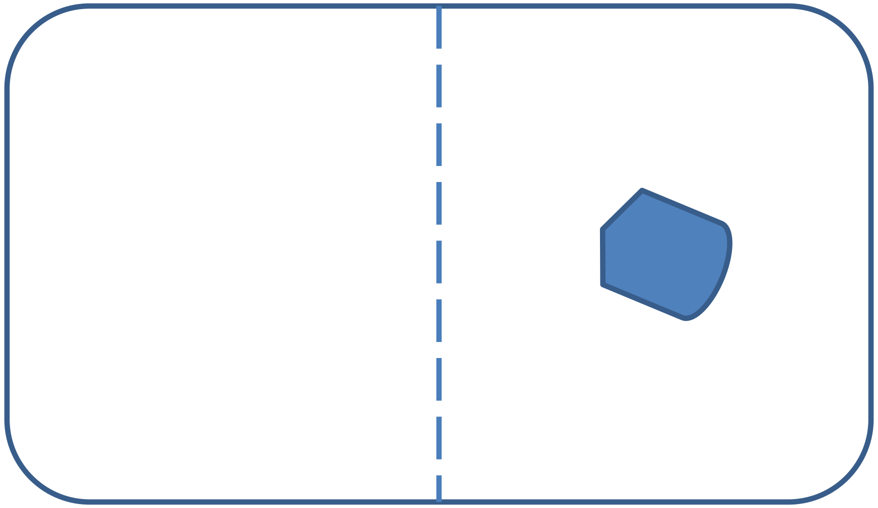


= 1

The iStone

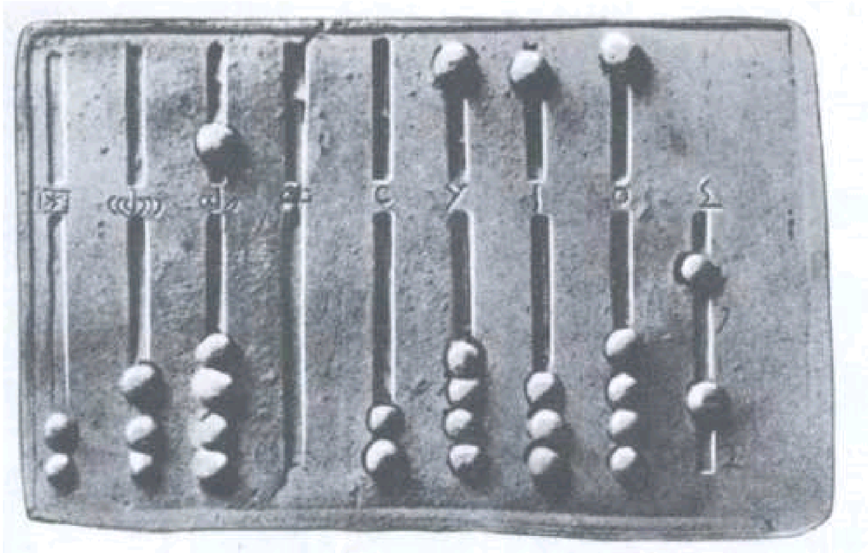


Early Digital Memory



The iStone: 1 bit

Early Digital Memory



The iStone 5: ~ 20 bits

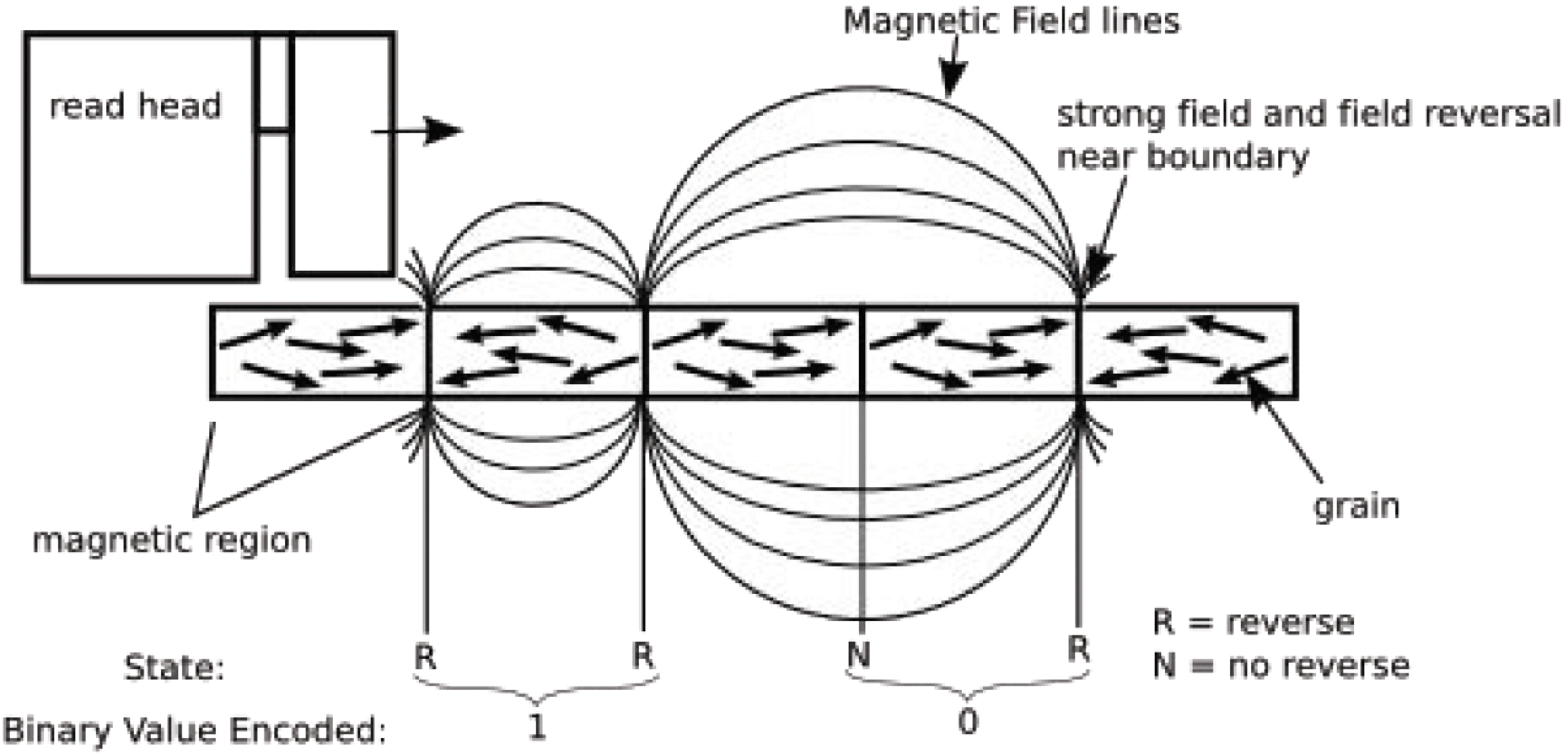
Modern Digital Memory

The iPhone 5: ~ 5.5×10^{11} bits

Modern Digital Memory

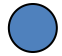
The iPod: ~ 1.4 x 10^{12} bits

Modern Digital Memory

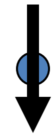


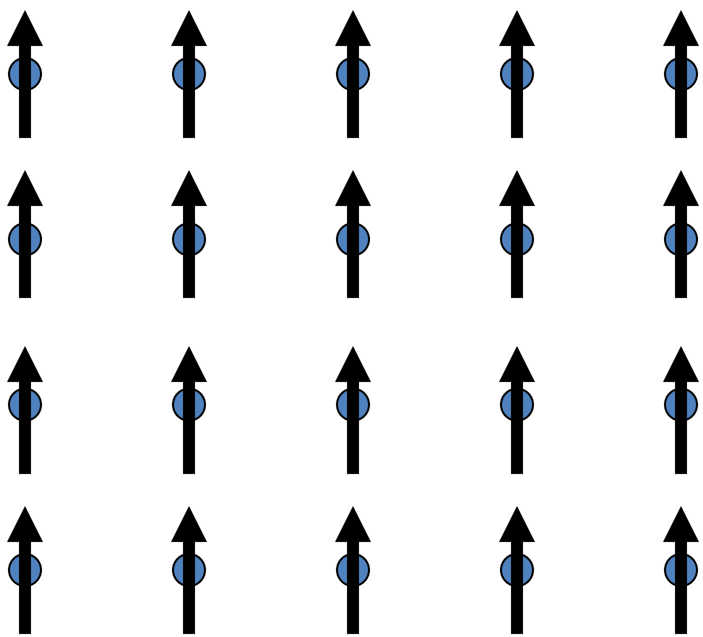
http://en.wikipedia.org/wiki/Hard_disk_drive

Magnetic Order

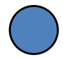
A spin-1/2 particle: 

 “spin up”

 “spin down”



Magnetic Order

A spin-1/2 particle: 

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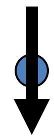
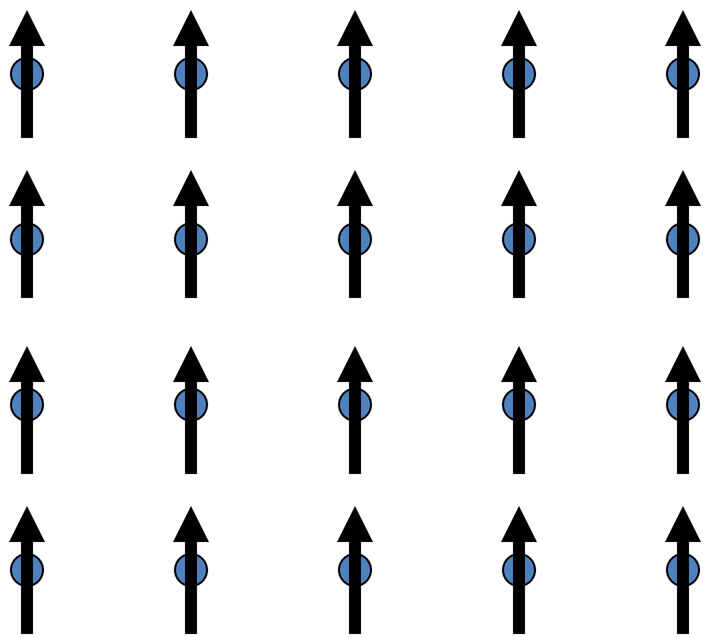
 “spin down”



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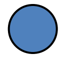
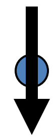
Magnetic Order

A spin-1/2 particle: 

“spin up”



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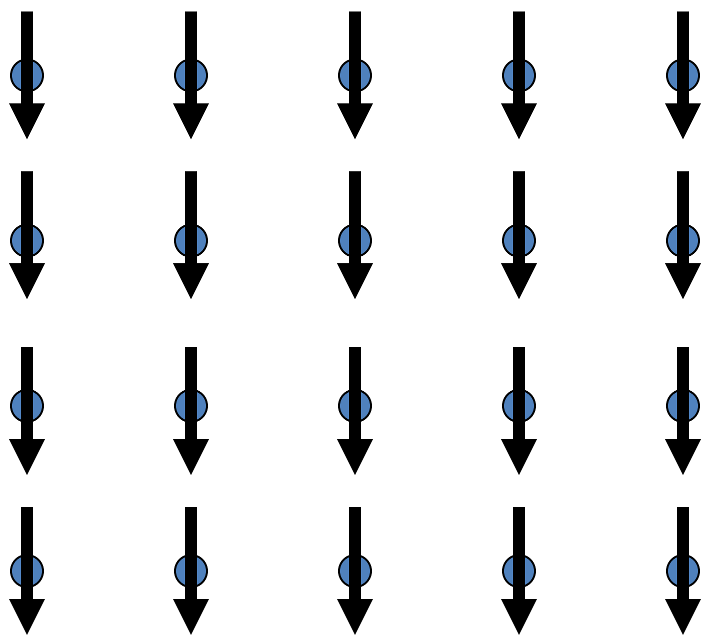
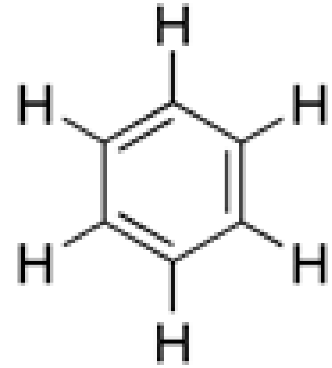


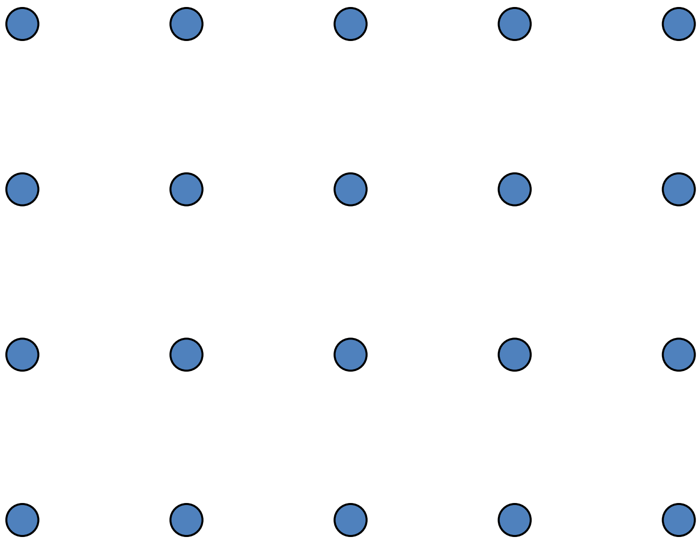
= 1

Another Kind of Order

A valence bond:

$$= \frac{1}{\sqrt{2}}\left(\uparrow\downarrow - \downarrow\uparrow\right)$$





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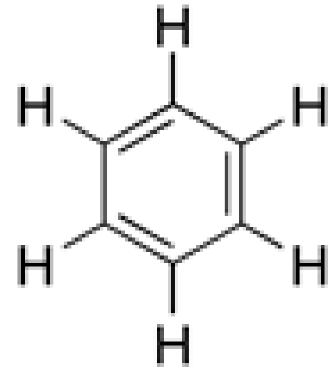

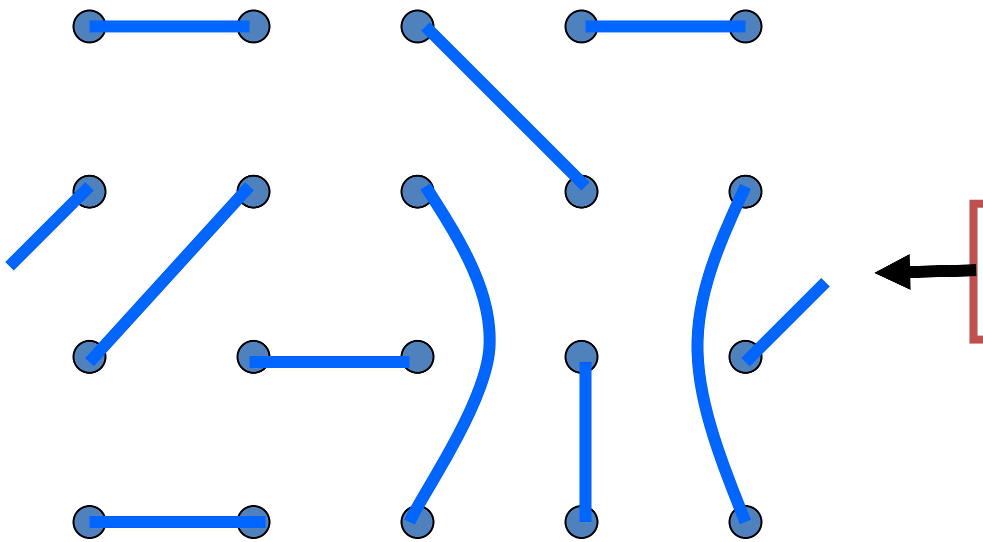

Use periodic boundary conditions

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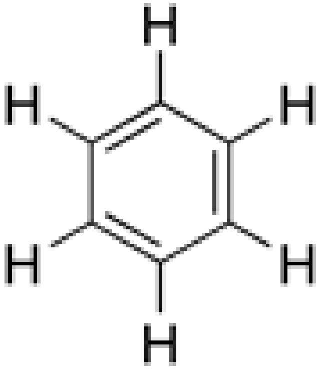

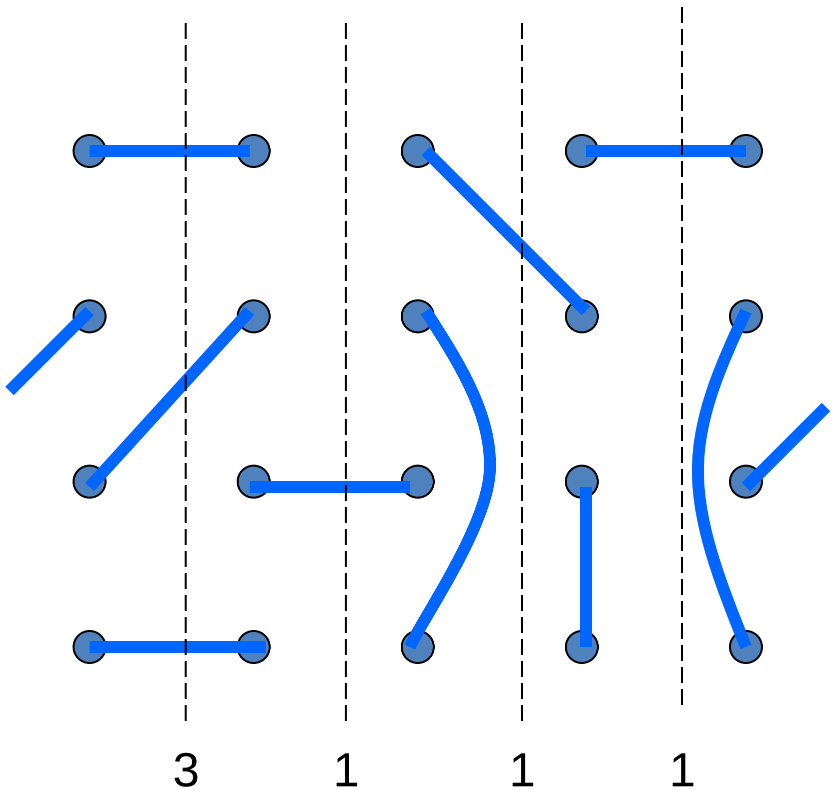

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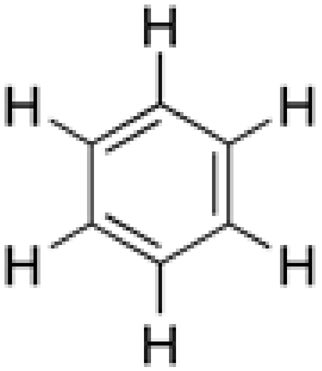

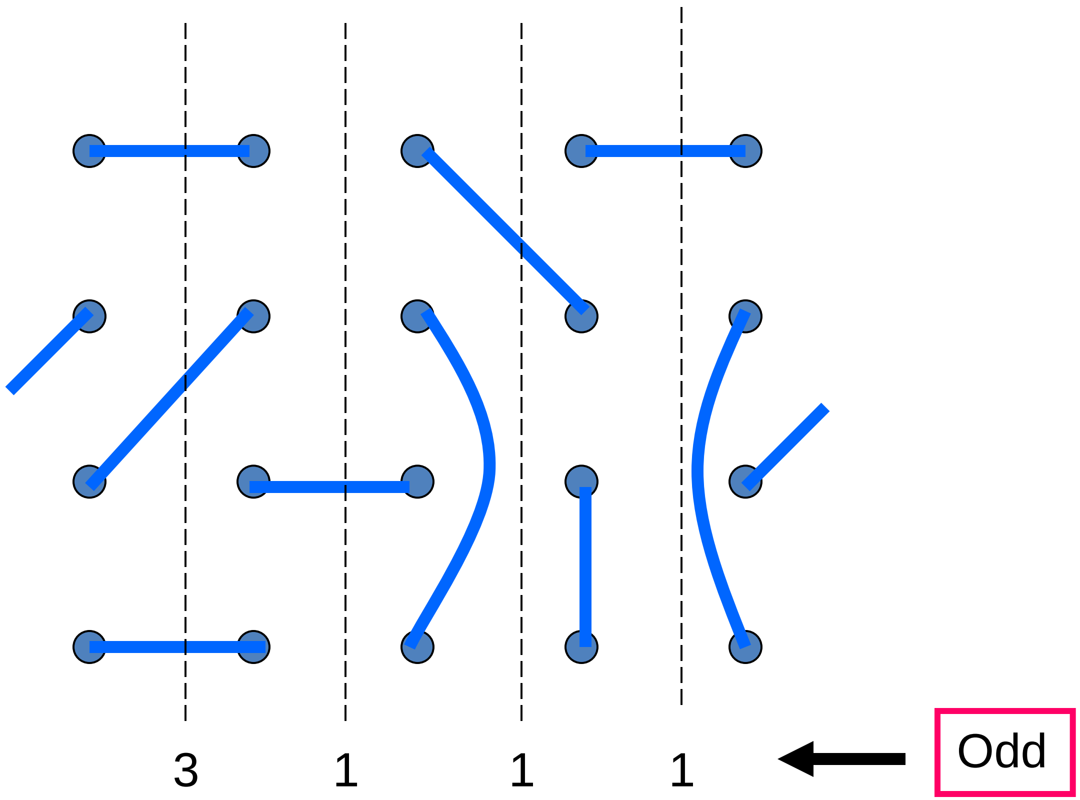

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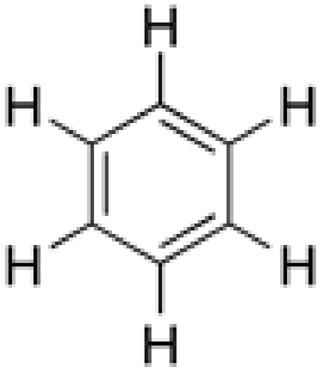

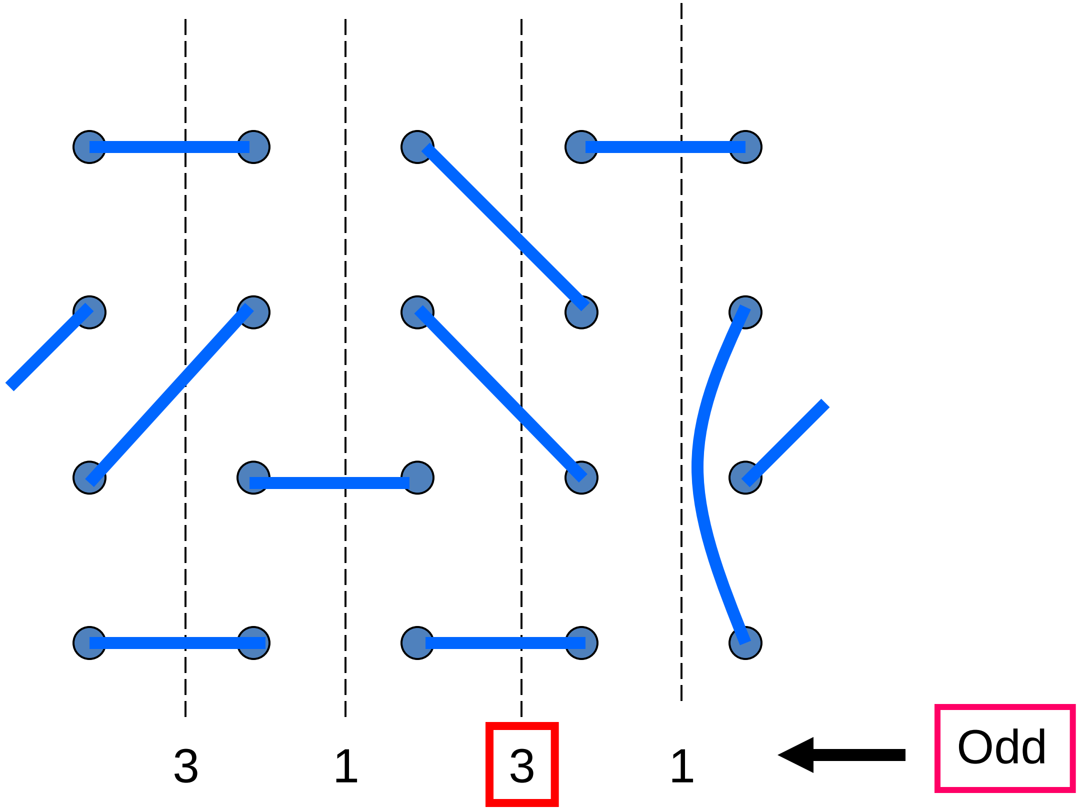

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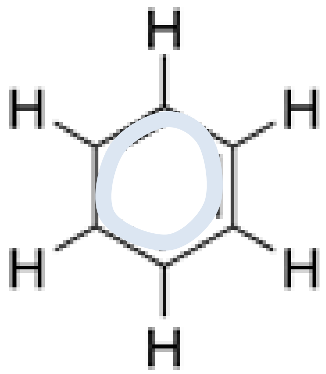

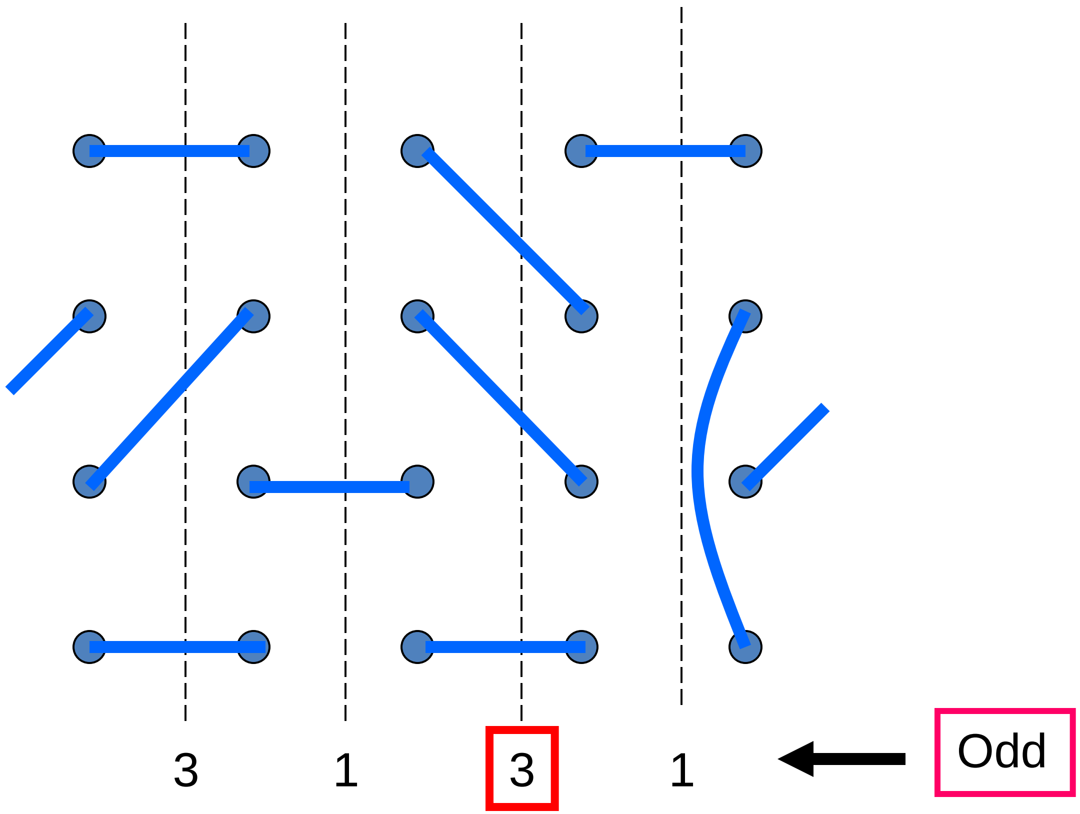

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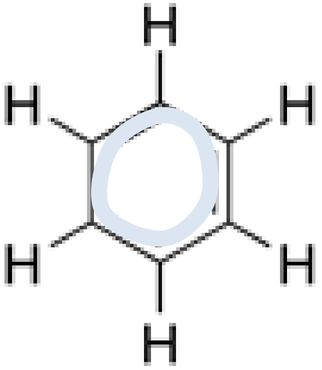

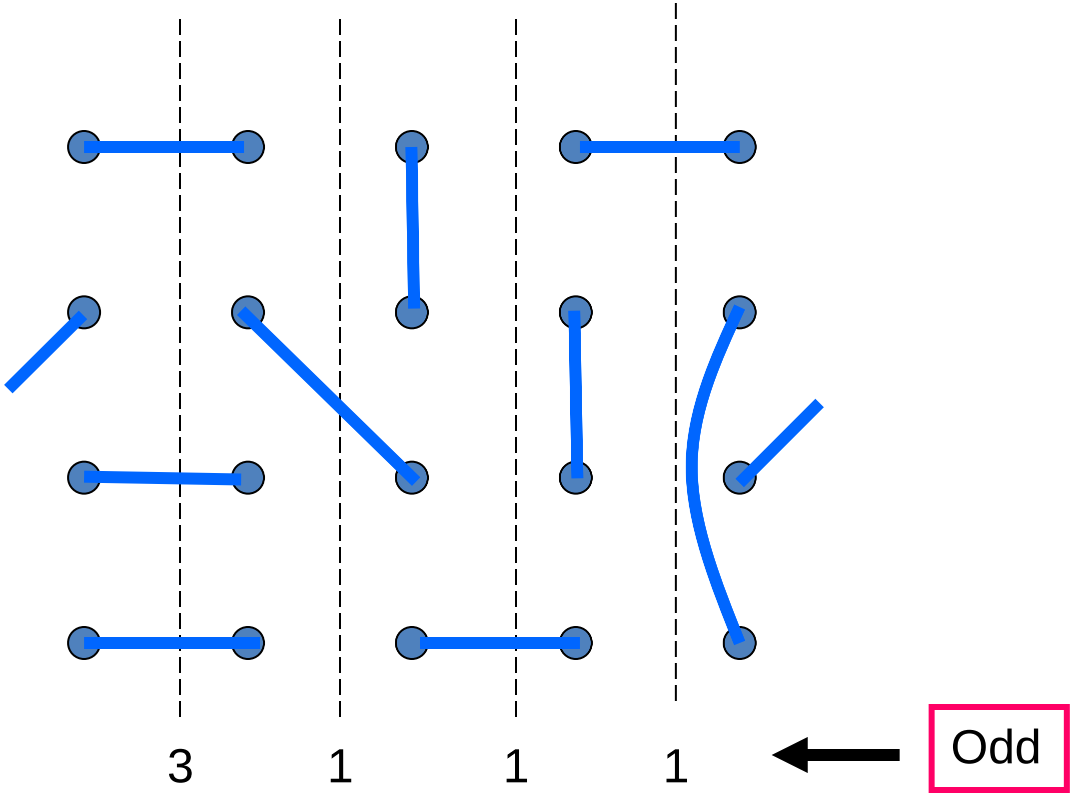

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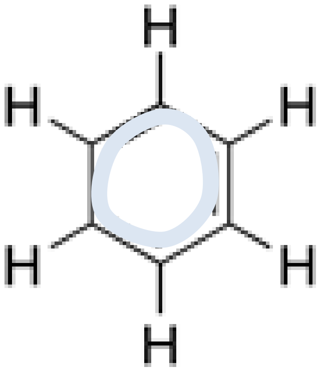

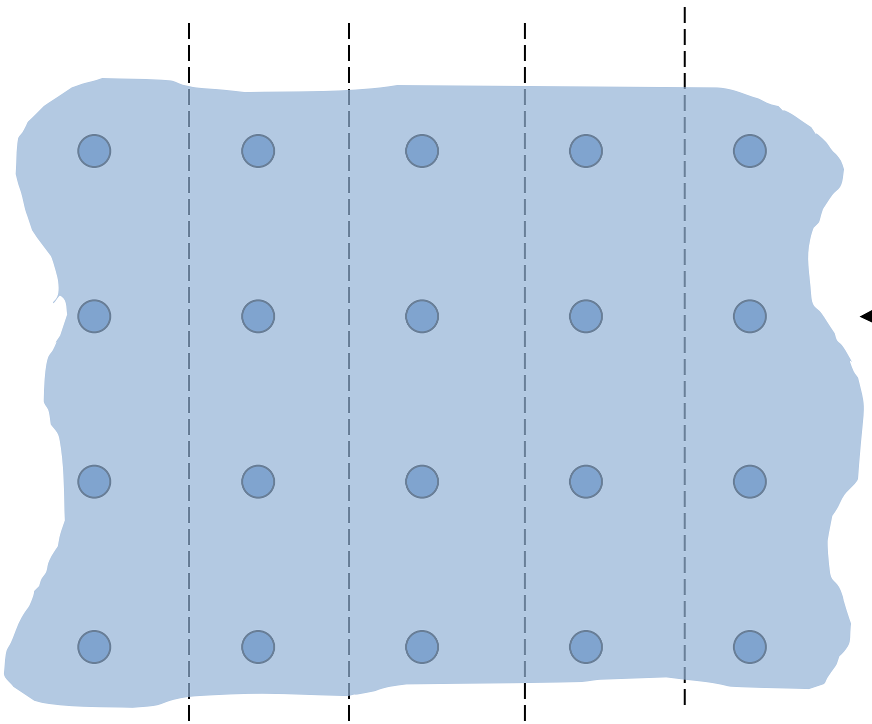

Quantum superposition of valence-bond states. A "**spin liquid**."

Odd

Another Kind of Order

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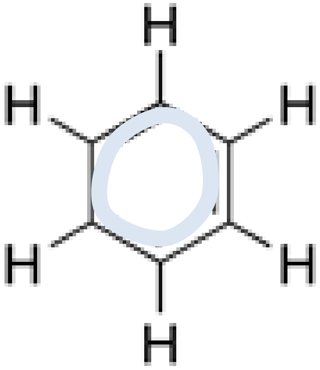
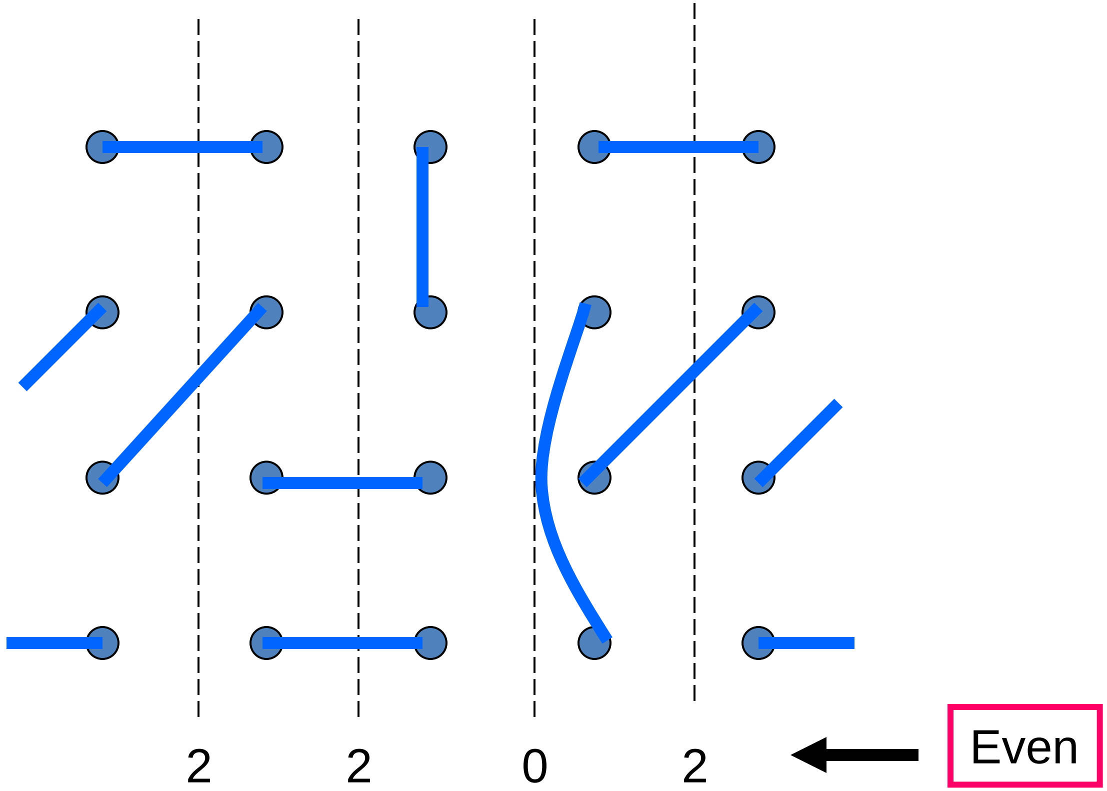
 $= \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)$





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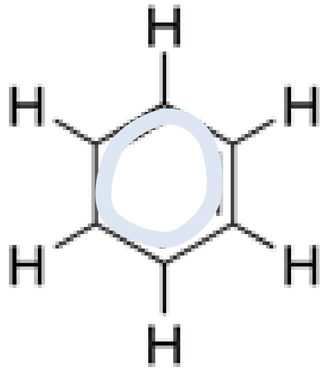

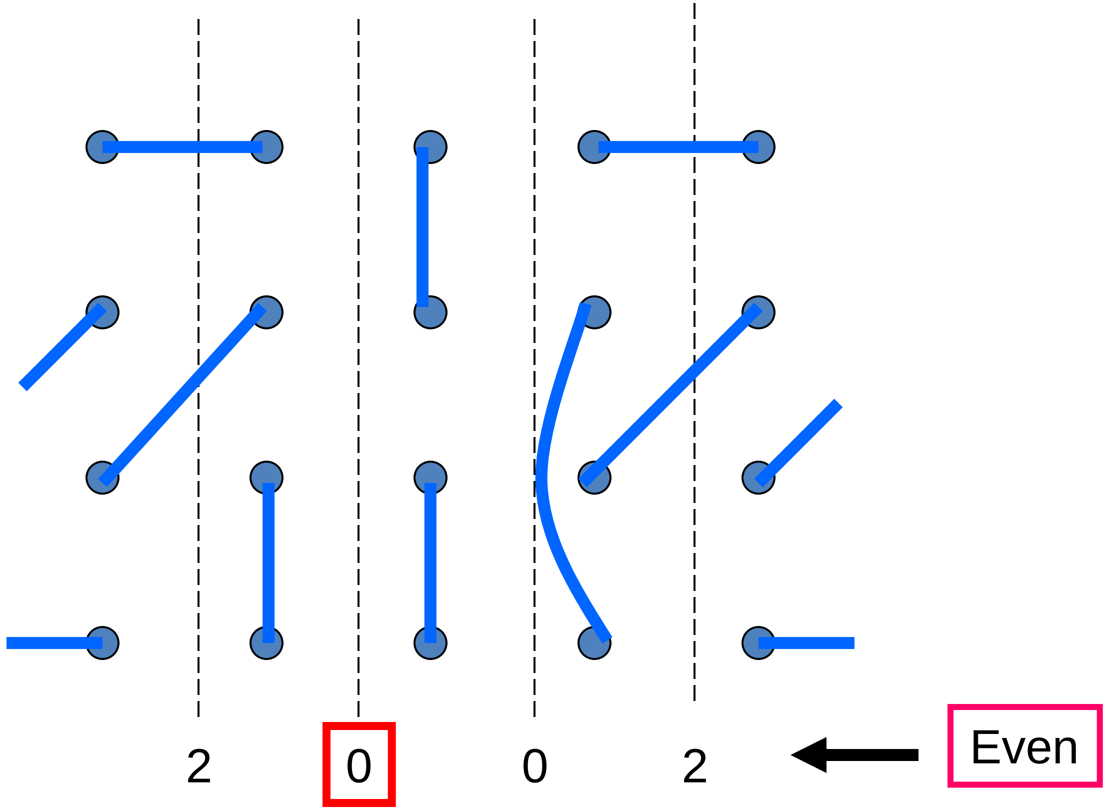

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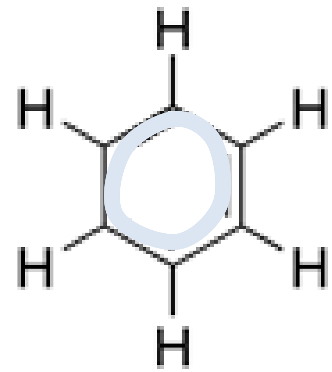

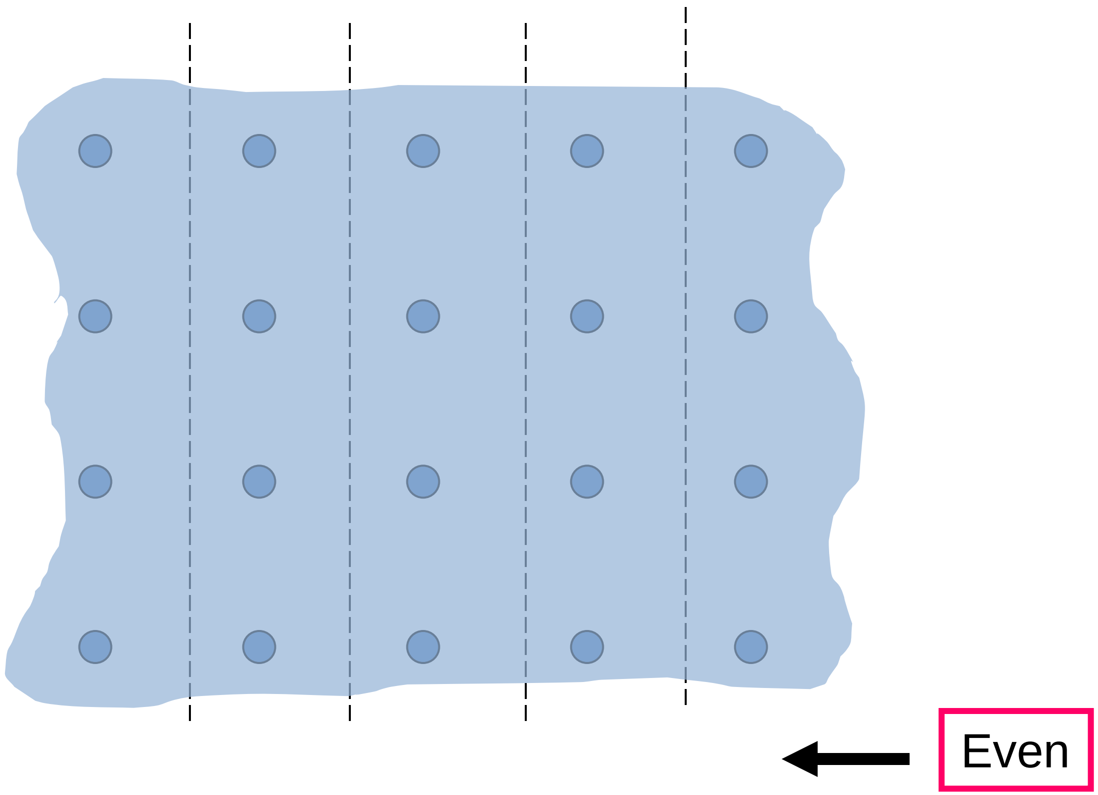

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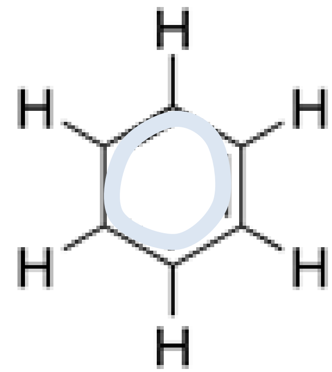
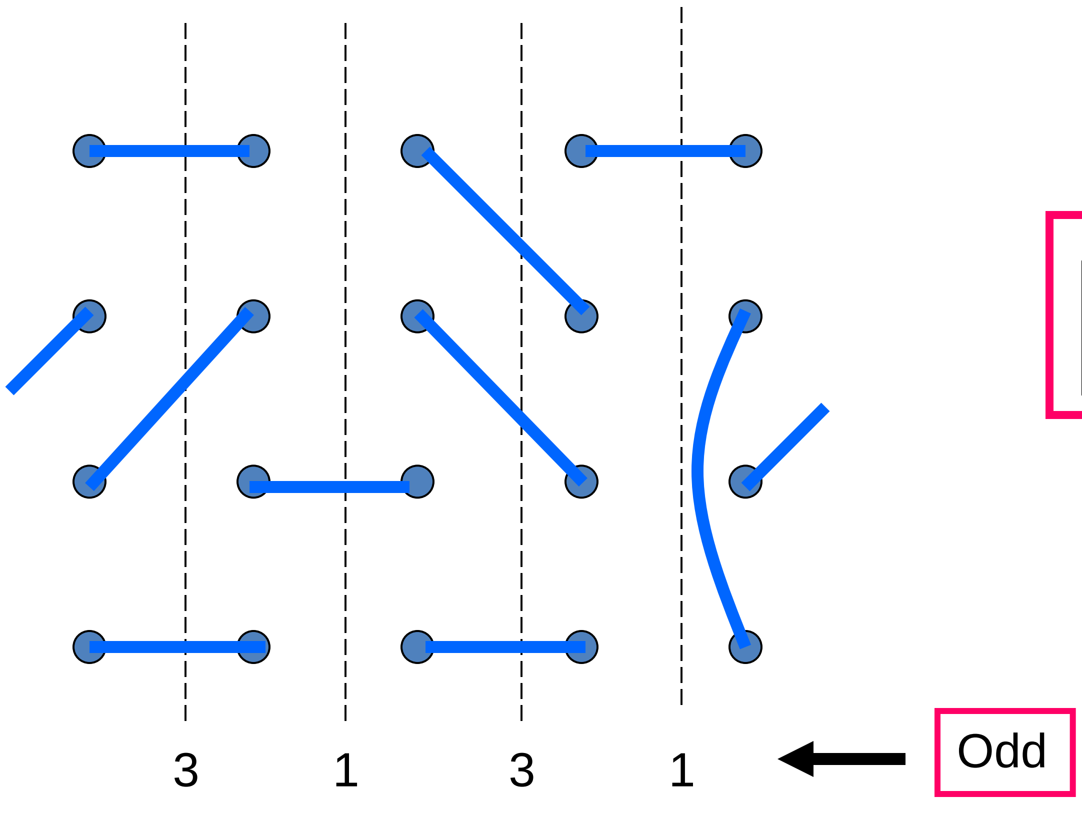


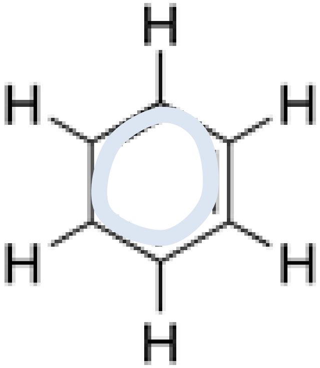


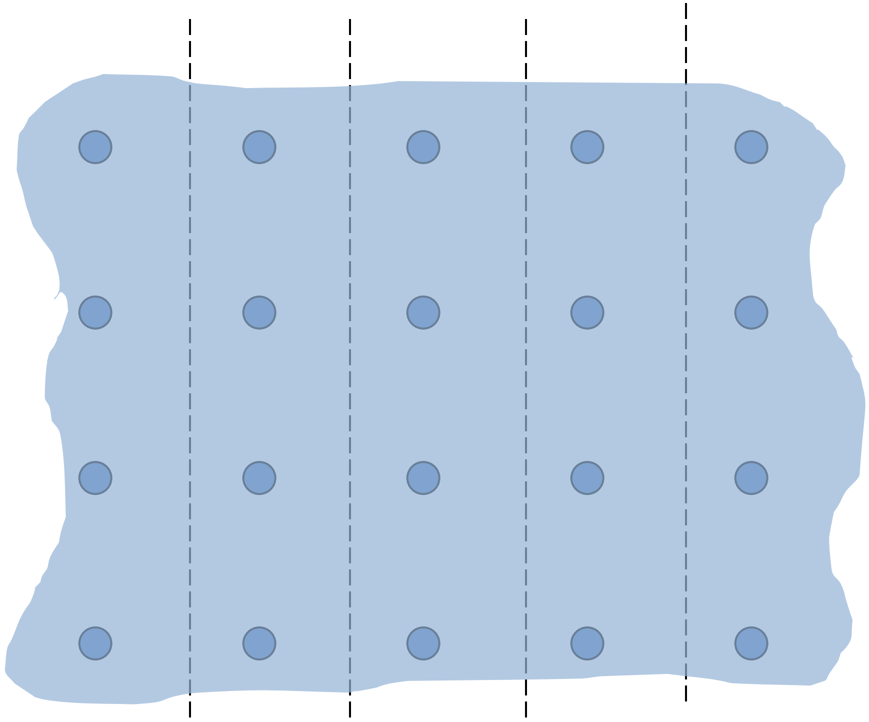
$|0\rangle$

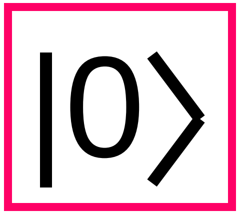
Another Kind of Order

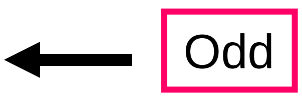
A valence bond:

$$= \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)$$









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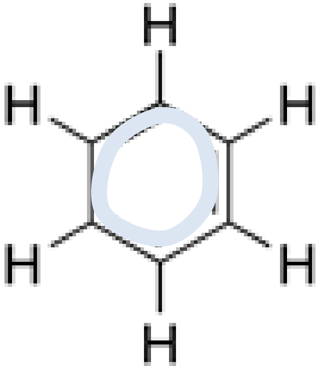
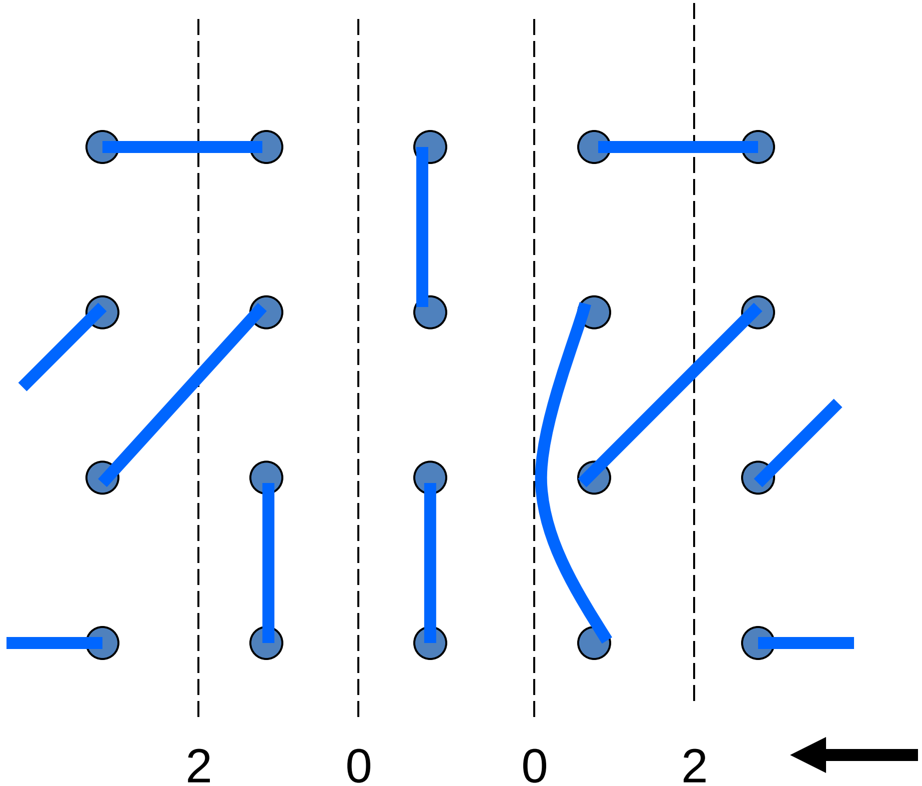




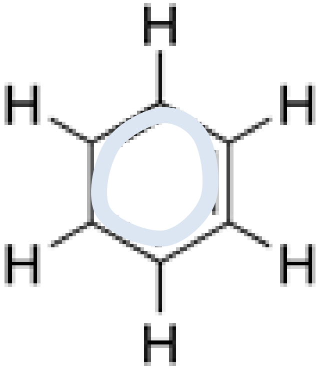
$|1\rangle$

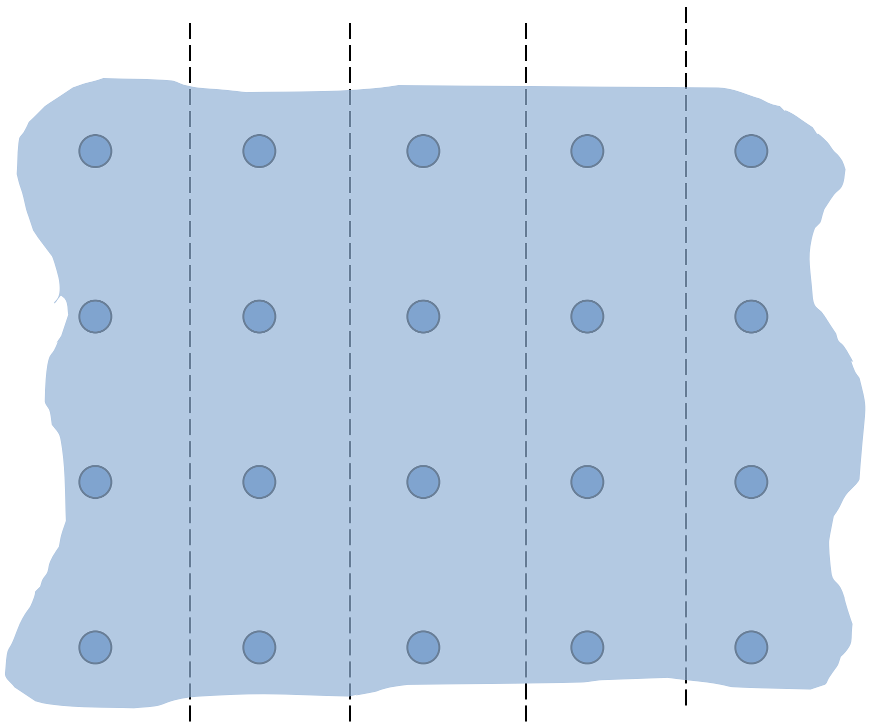
Even

Another Kind of Order

A valence bond:

$$= \frac{1}{\sqrt{2}}\left(\uparrow\downarrow - \downarrow\uparrow\right)$$



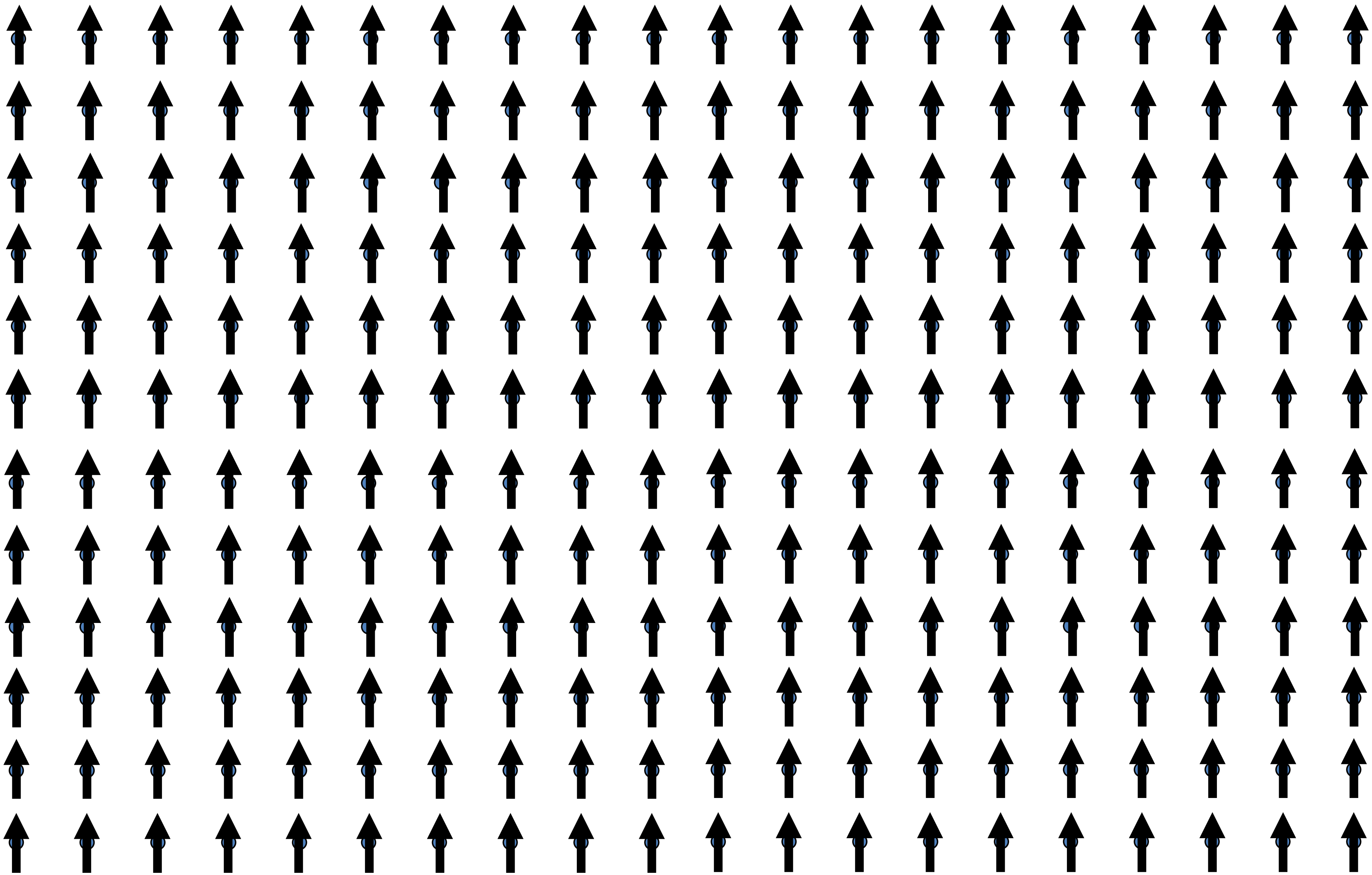


$|1\rangle$

← Even

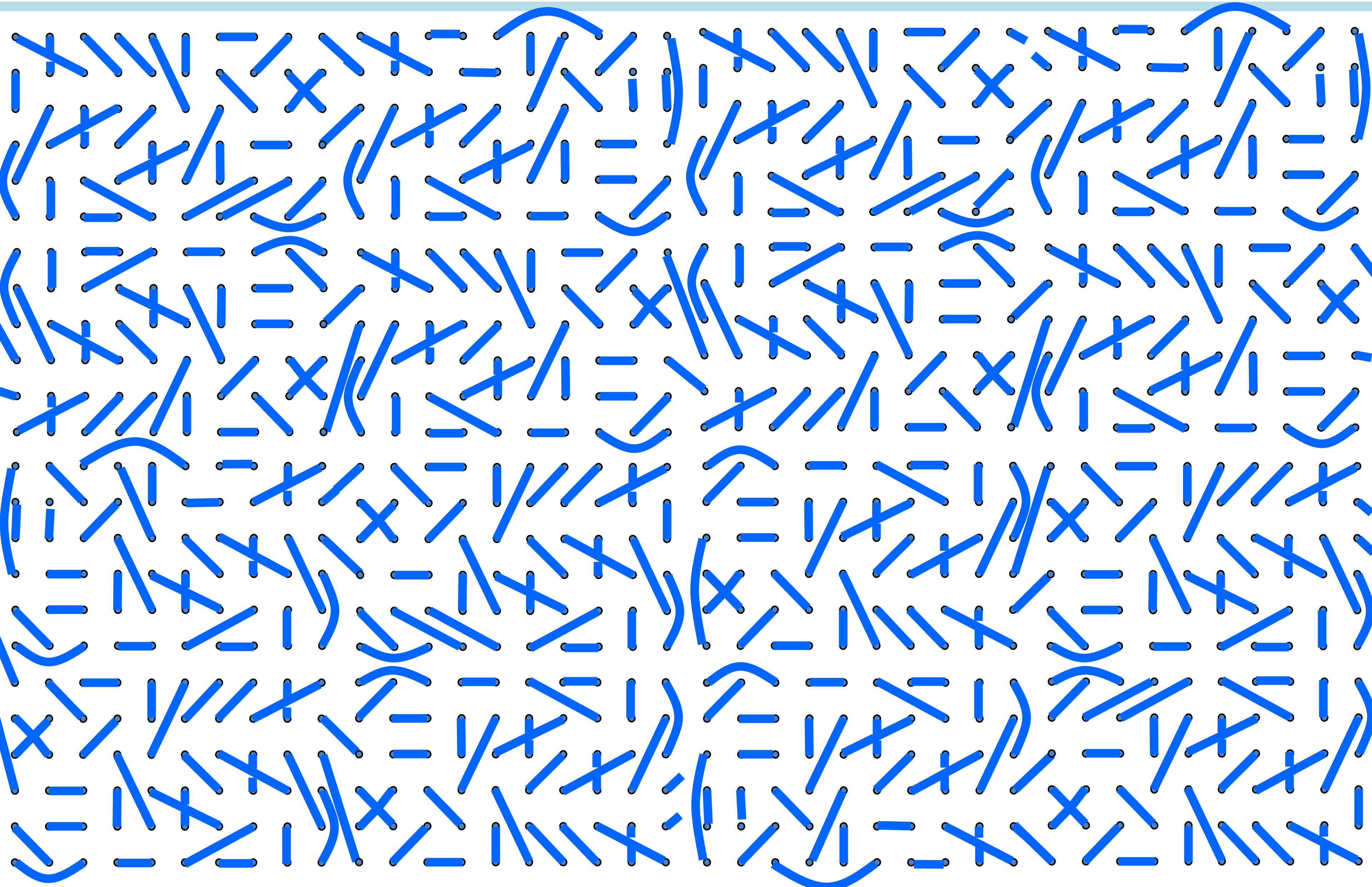
Is it a 0 or a 1?

Is it a 0 or a 1?



Is it a $|0\rangle$ or a $|1\rangle$?

Is it a |0❭ or a |1❭?



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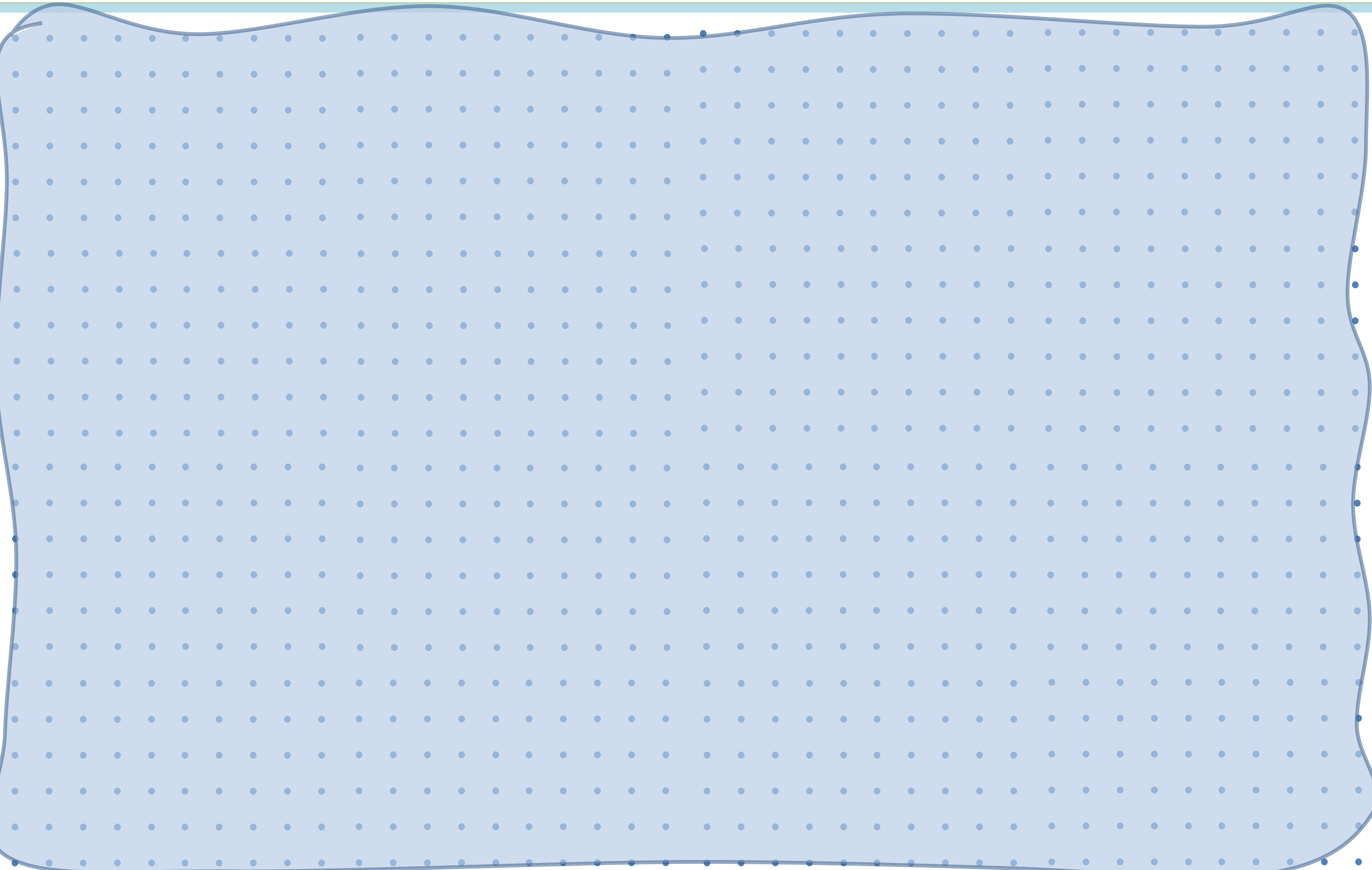
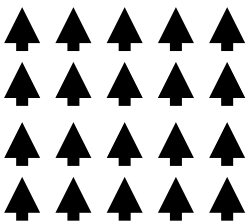


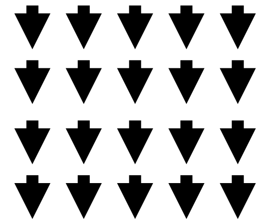
Topological Order (Wen & Niu, PRB 41, 9377 (1990))

Conventionally Ordered States: Multiple “broken symmetry” ground states characterized by a locally observable order parameter.



magnetization

$$m = \langle S_z \rangle = +\frac{1}{2}$$

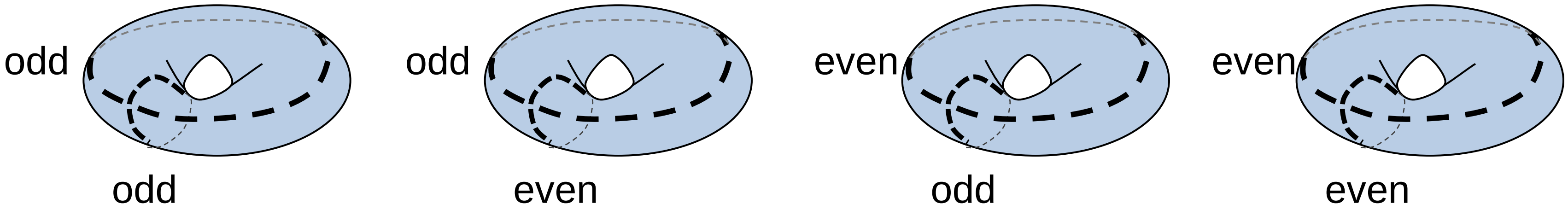


magnetization

$$m = \langle S_z \rangle = -\frac{1}{2}$$

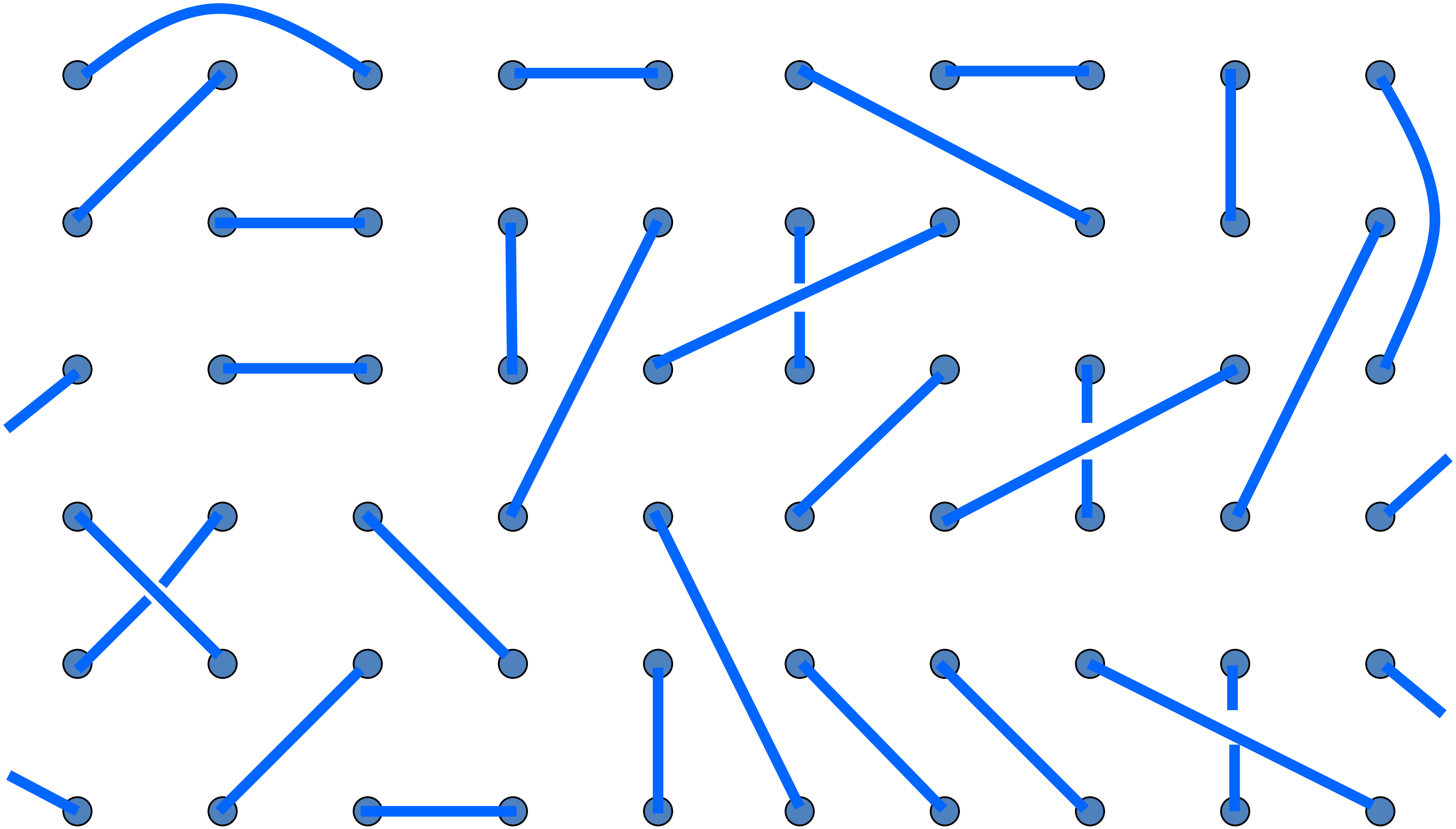
Nature’s classical error correction

Topologically Ordered States: Multiple ground states on topologically nontrivial surfaces with no locally observable order parameter.

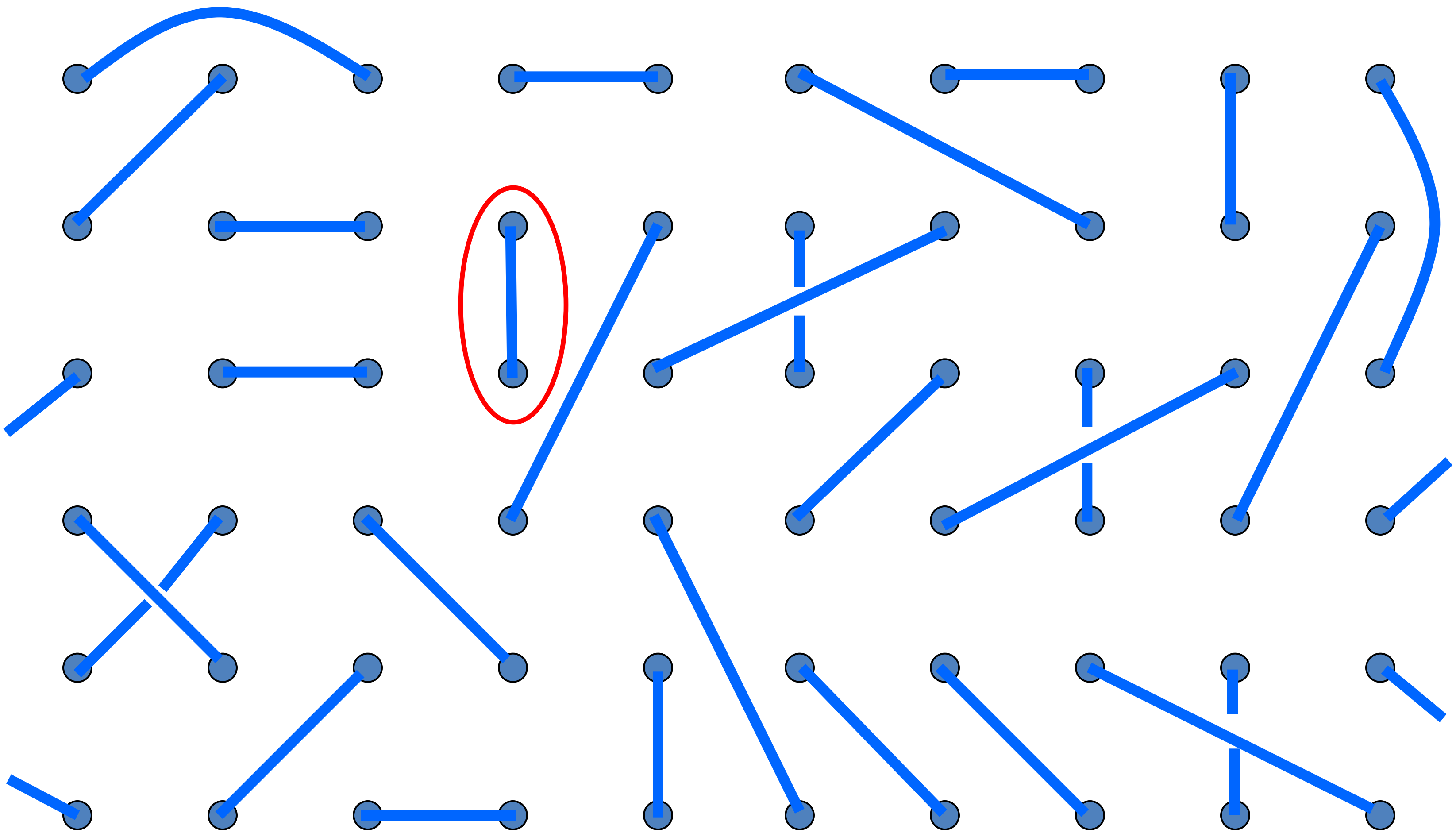


Nature’s quantum error correction

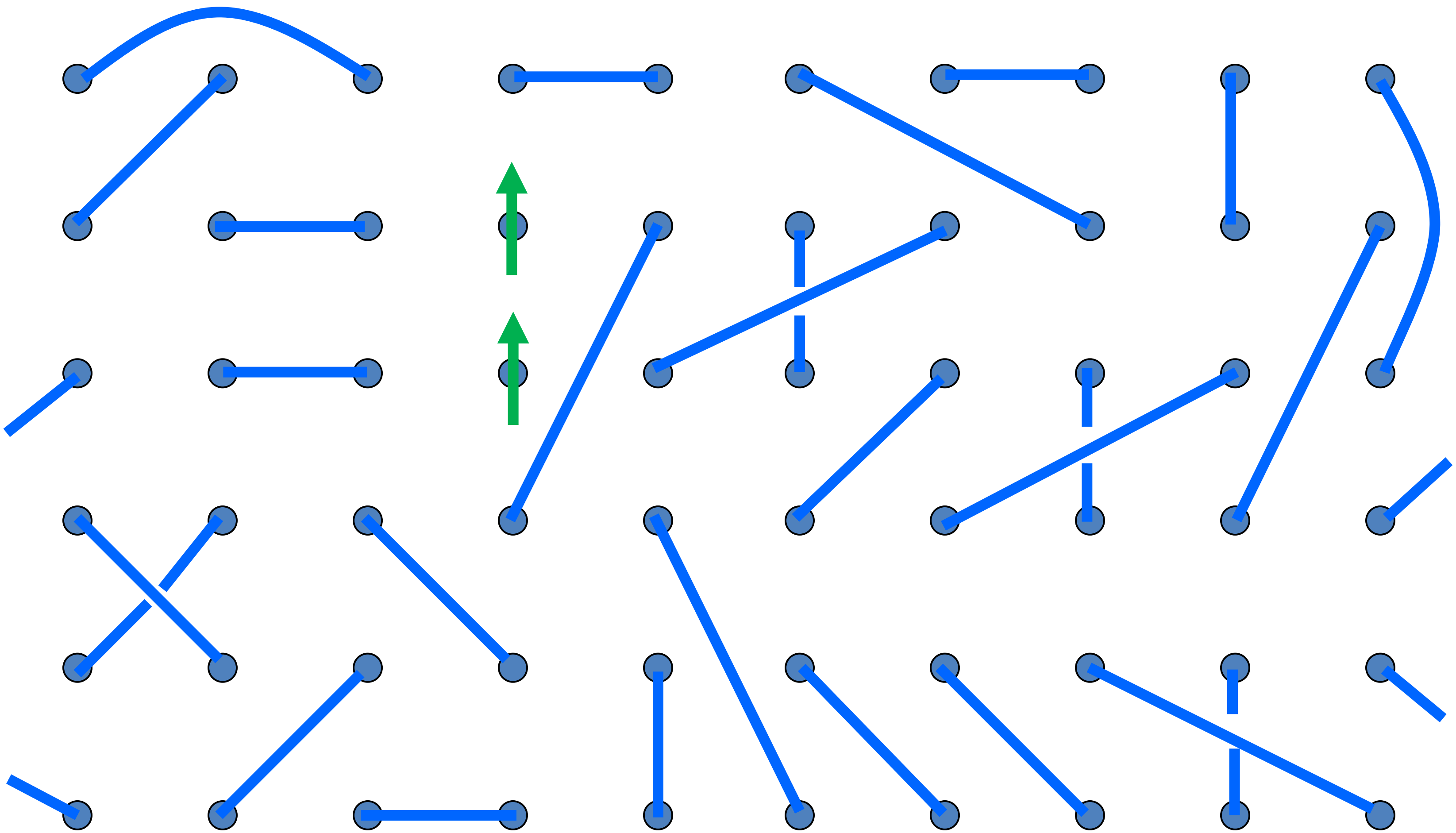
Topological Order: Excitations



Topological Order: Excitations

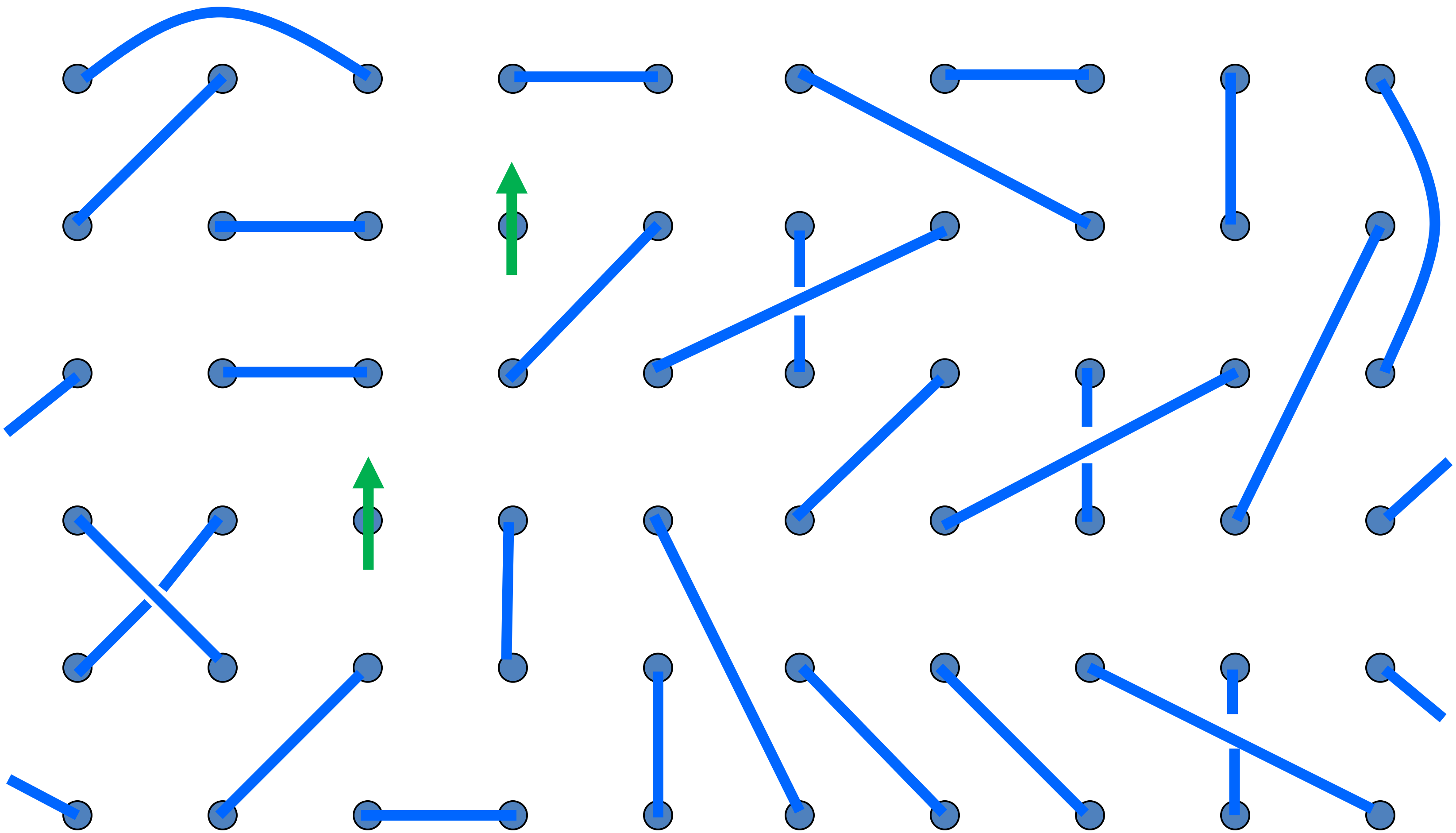


Topological Order: Excitations



Breaking a bond creates an excitation with $S_z = 1$

Topological Order: Excitations



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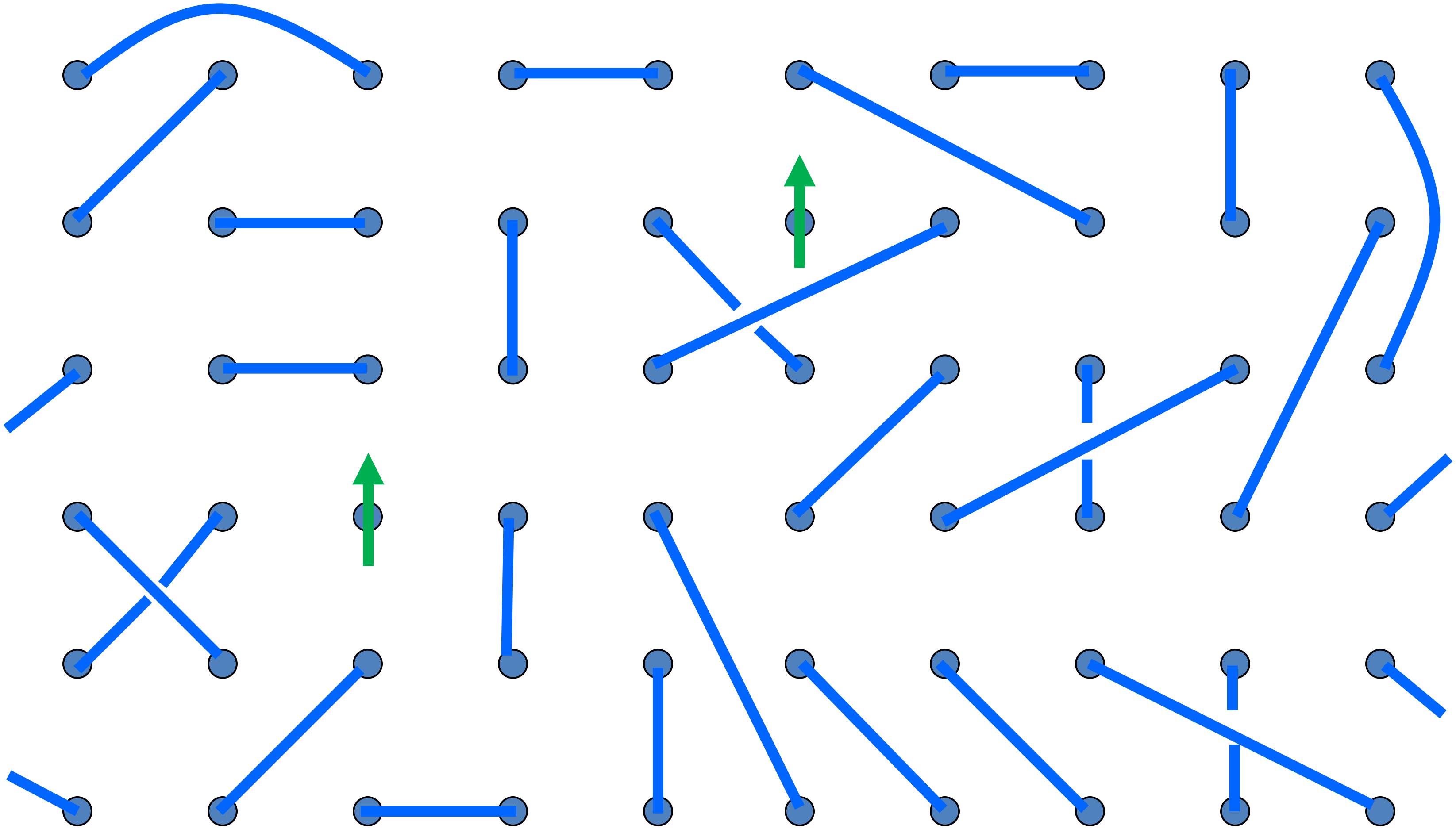
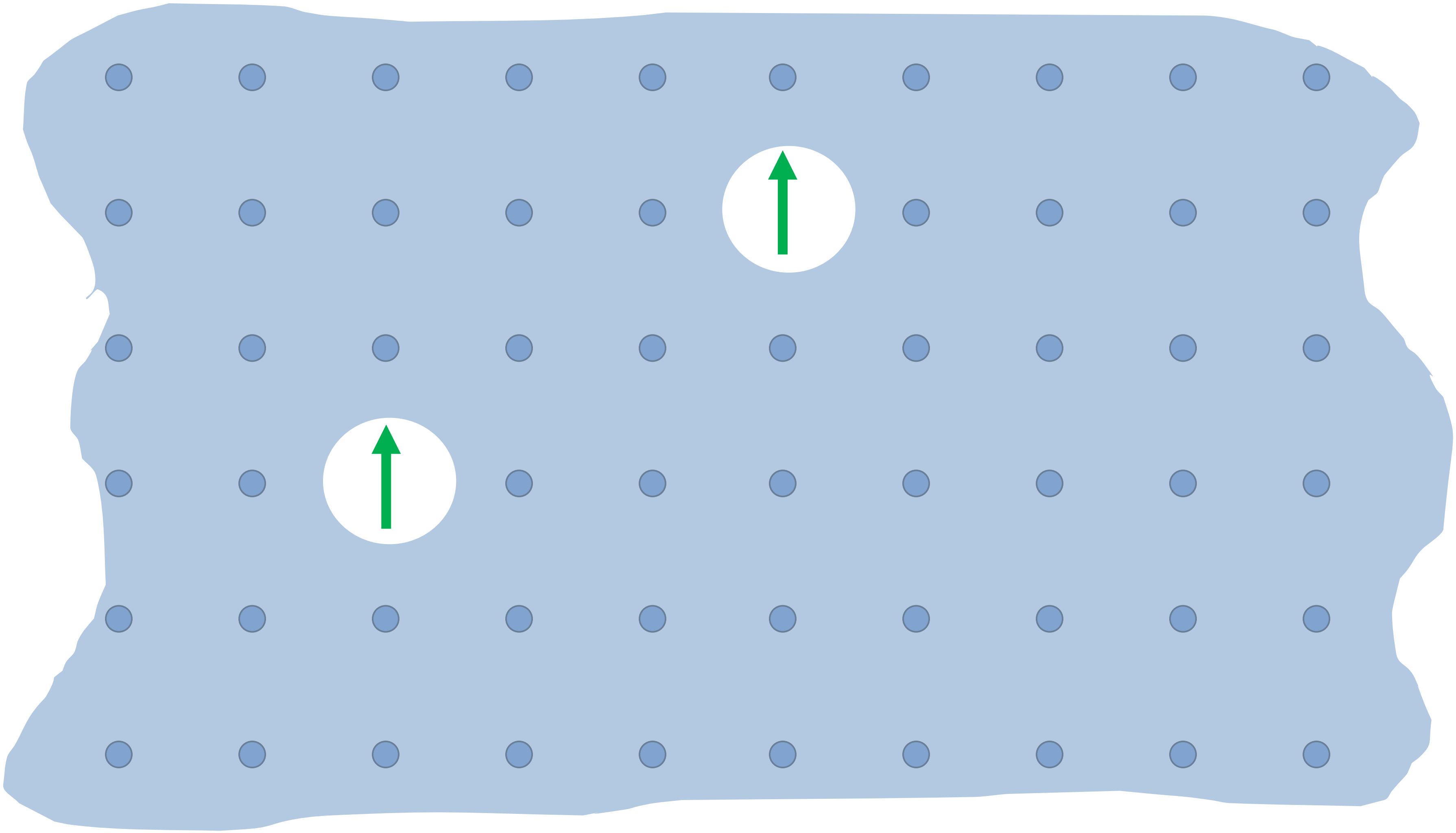


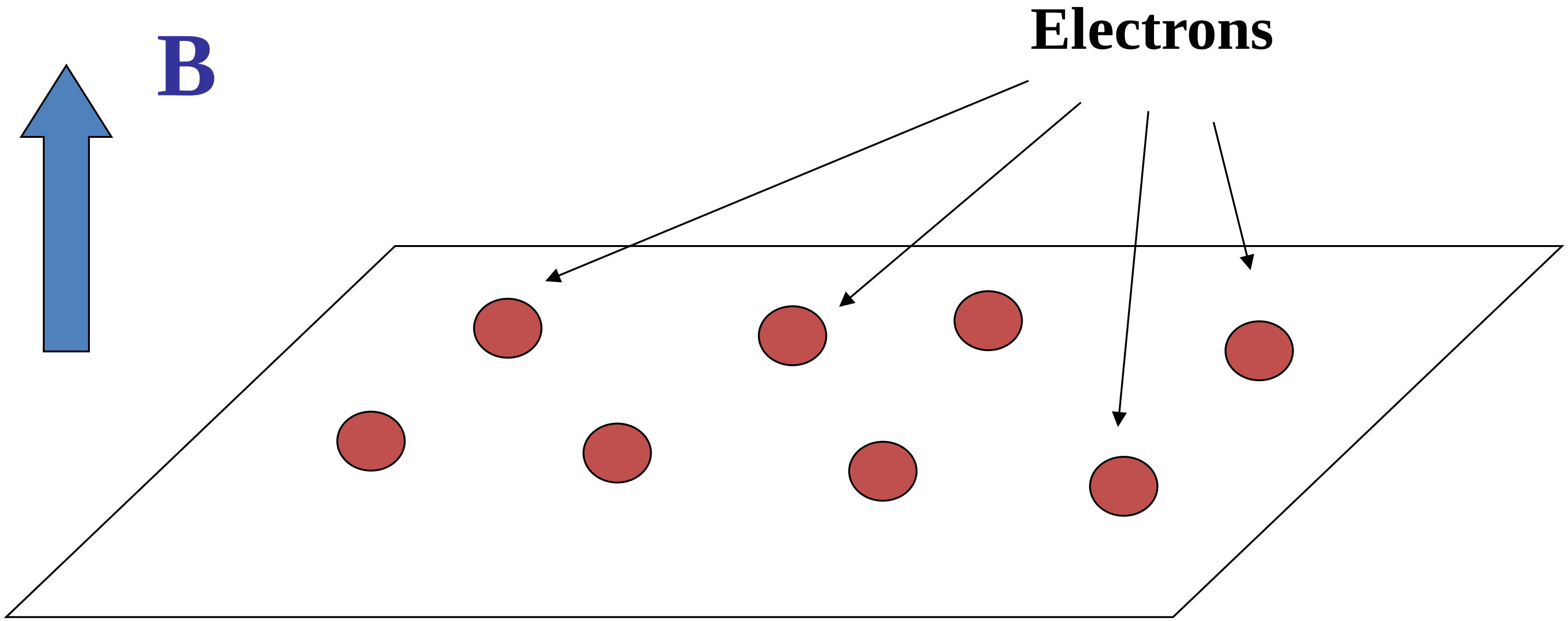
Breaking a bond creates an excitation with $S_z = 1$

Fractionalization



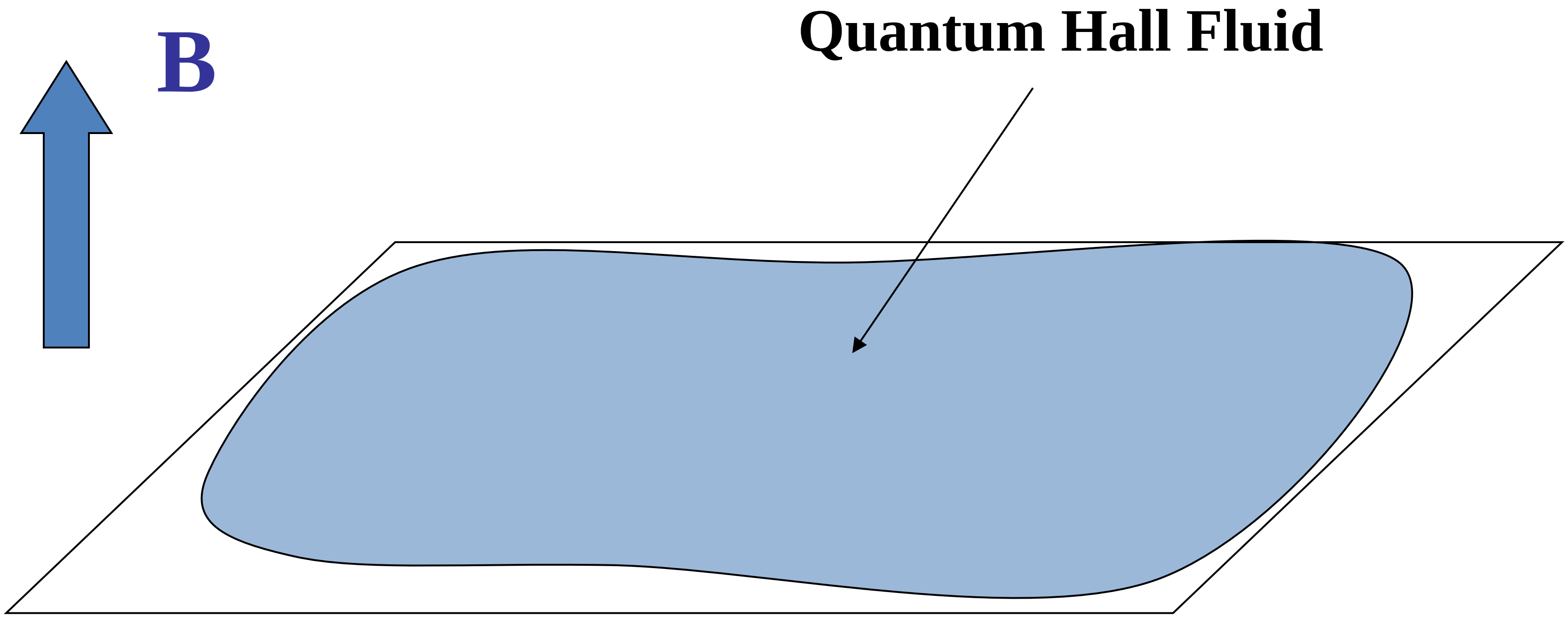
S_z = 1 excitation ***fractionalizes*** into two S_z = ½ excitations

Fractional Quantum Hall States



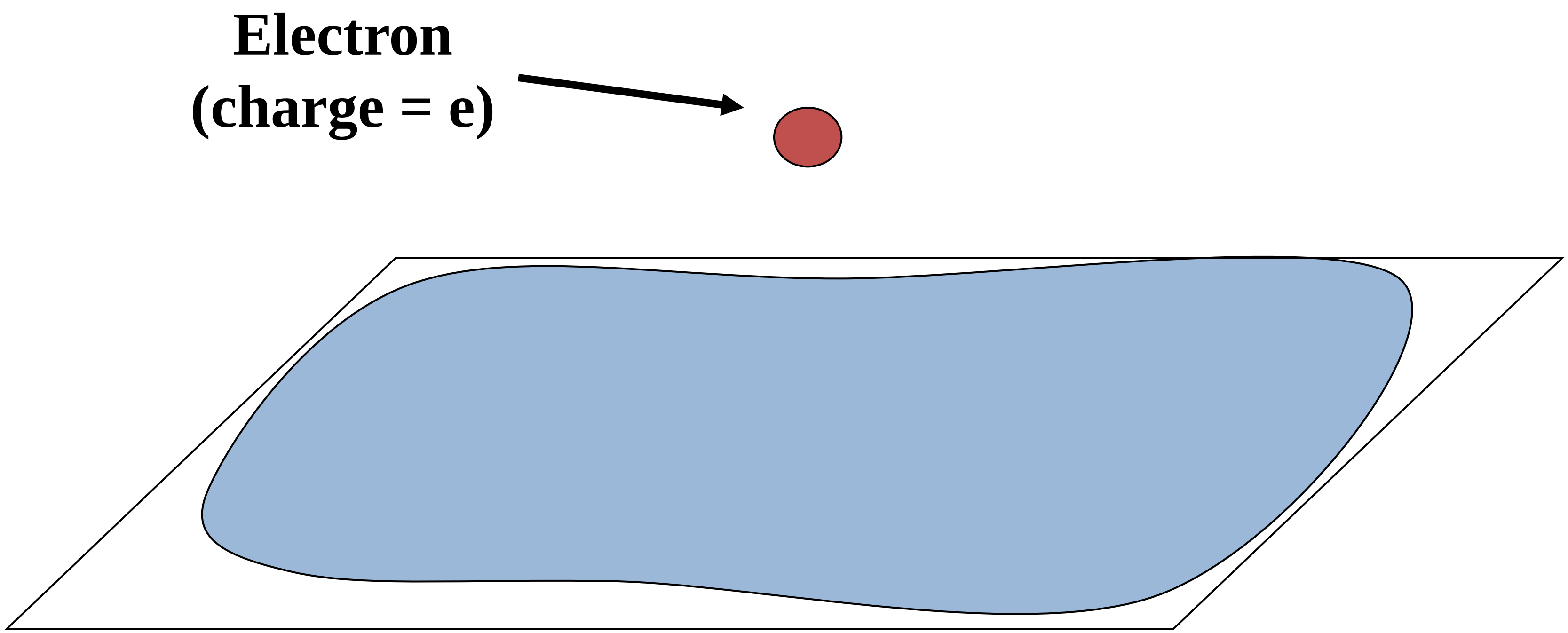
A two dimensional gas of electrons in a strong magnetic field **B**.

Fractional Quantum Hall States



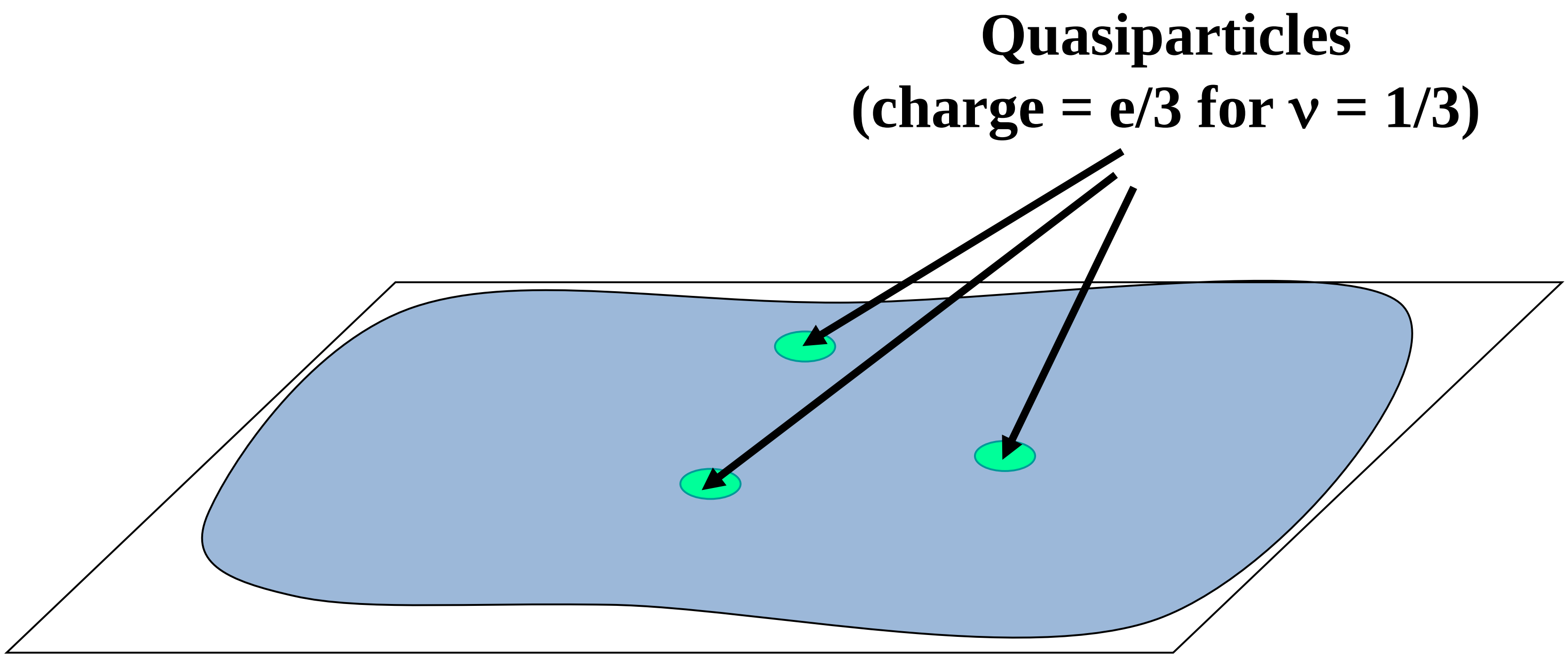
An **incompressible quantum liquid** can form when the Landau level filling fraction $\nu = n_{elec}(hc/eB)$ is a rational fraction.

Charge Fractionalization



When an electron is added to a FQH state it can be **fractionalized** --- i.e., it can break apart into **fractionally charged quasiparticles.**

Charge Fractionalization

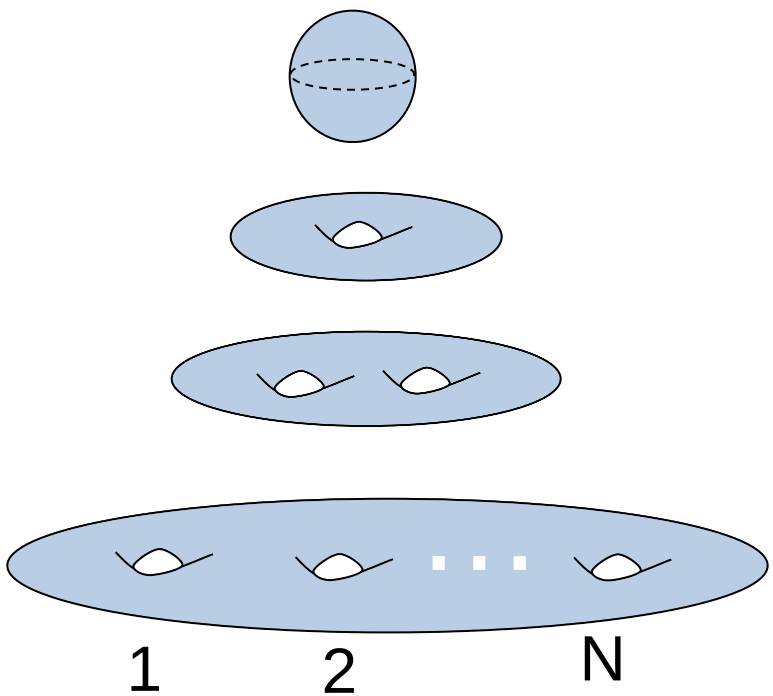


When an electron is added to a FQH state it can be **fractionalized** --- i.e., it can break apart into **fractionally charged quasiparticles.**

Topological Degeneracy

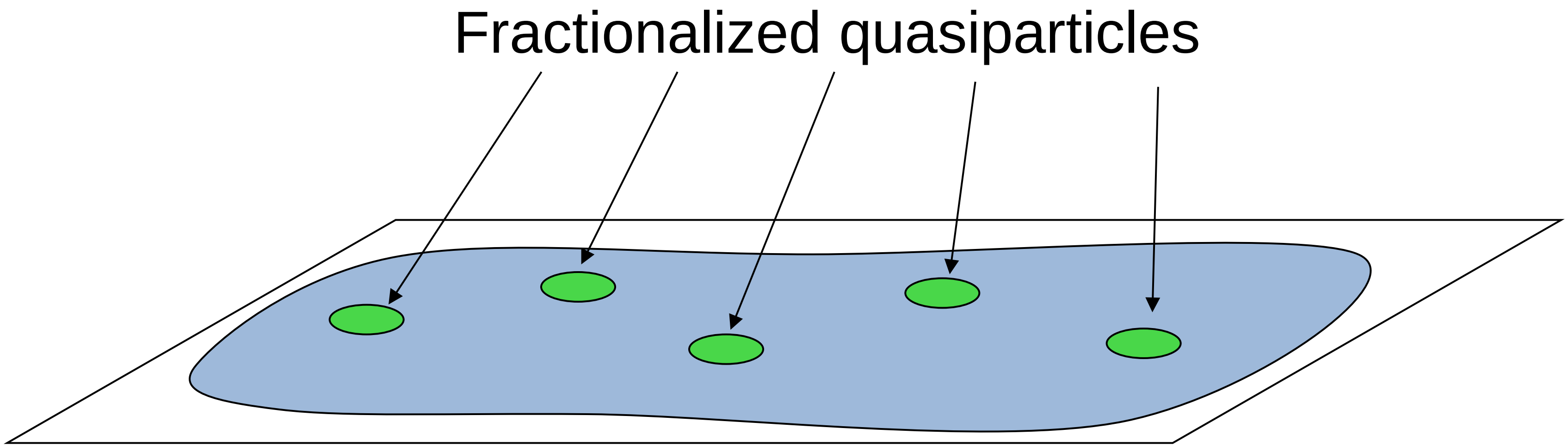
As in our spin-liquid example, FQH states on **topologically nontrivial surfaces** have degenerate ground states which **can only be distinguished by global measurements**.

For the ν **= 1/3** state:



Degeneracy
1
3
9
3^N

“Non-Abelian” FQH States (Moore & Read ‘91)



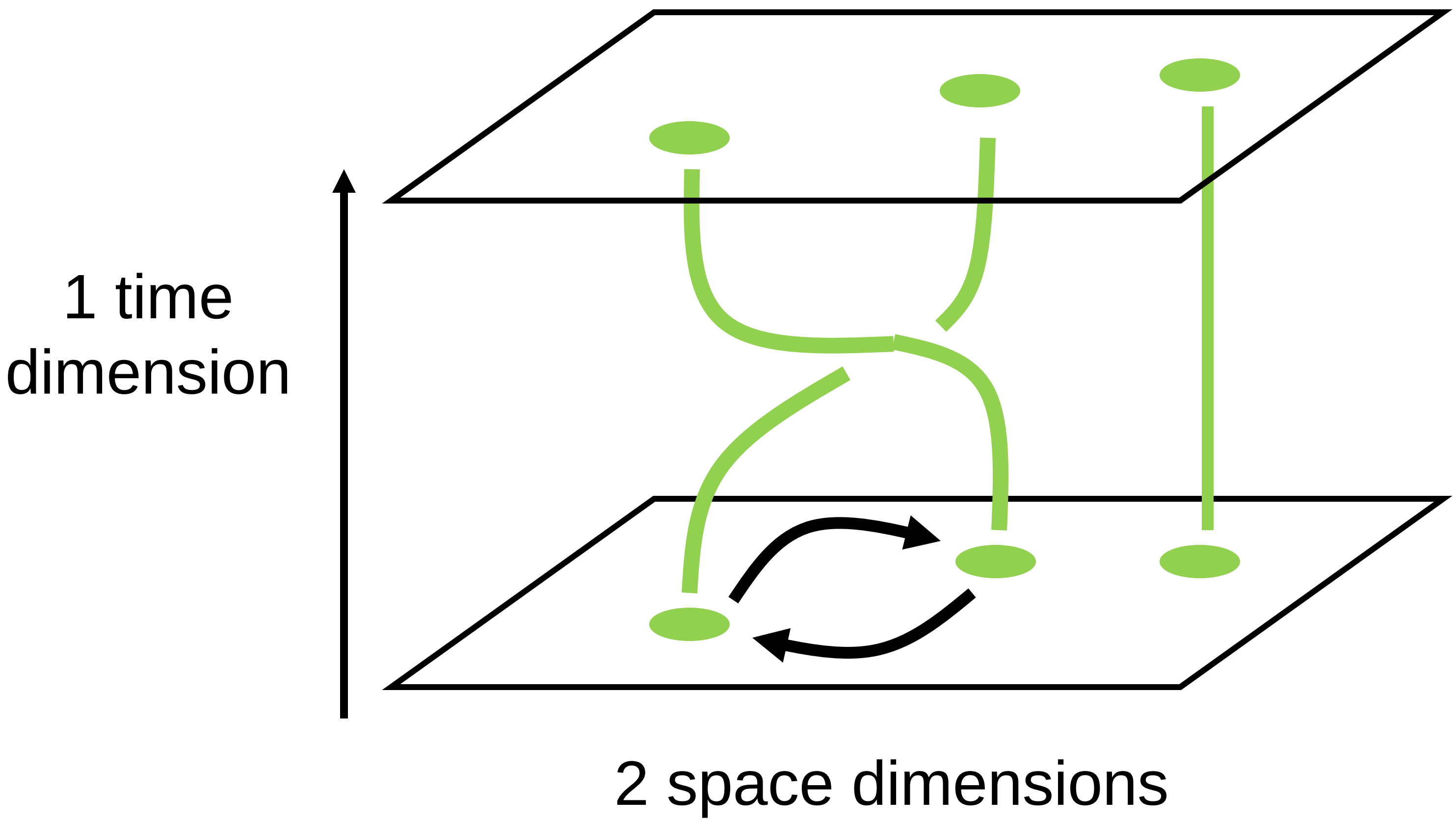
Essential features:

A degenerate Hilbert space whose dimensionality is **exponentially large in the number of quasiparticles**.

States in this space **can only be distinguished by global measurements** provided quasiparticles are far apart.

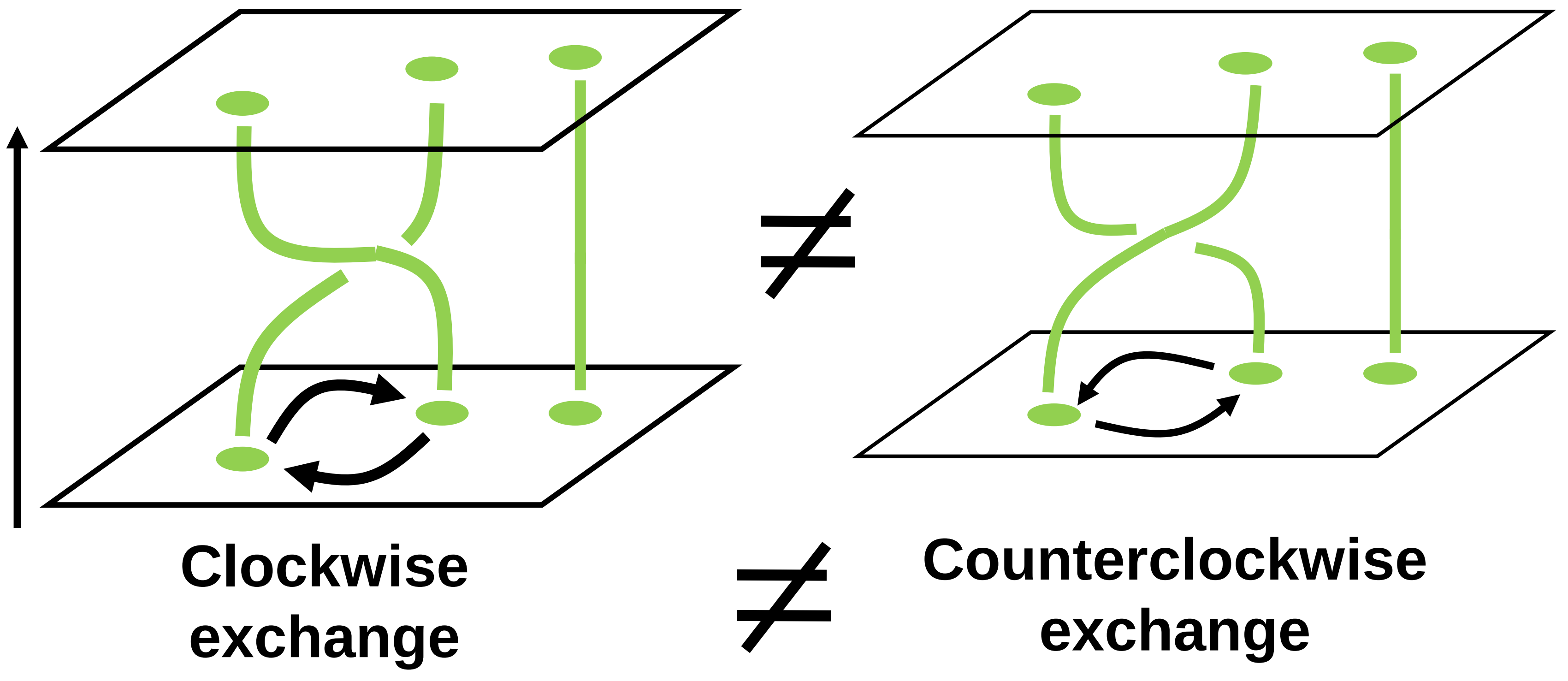
→ ***A perfect place to hide quantum information!***

Exchanging Particles in 2+1 Dimensions



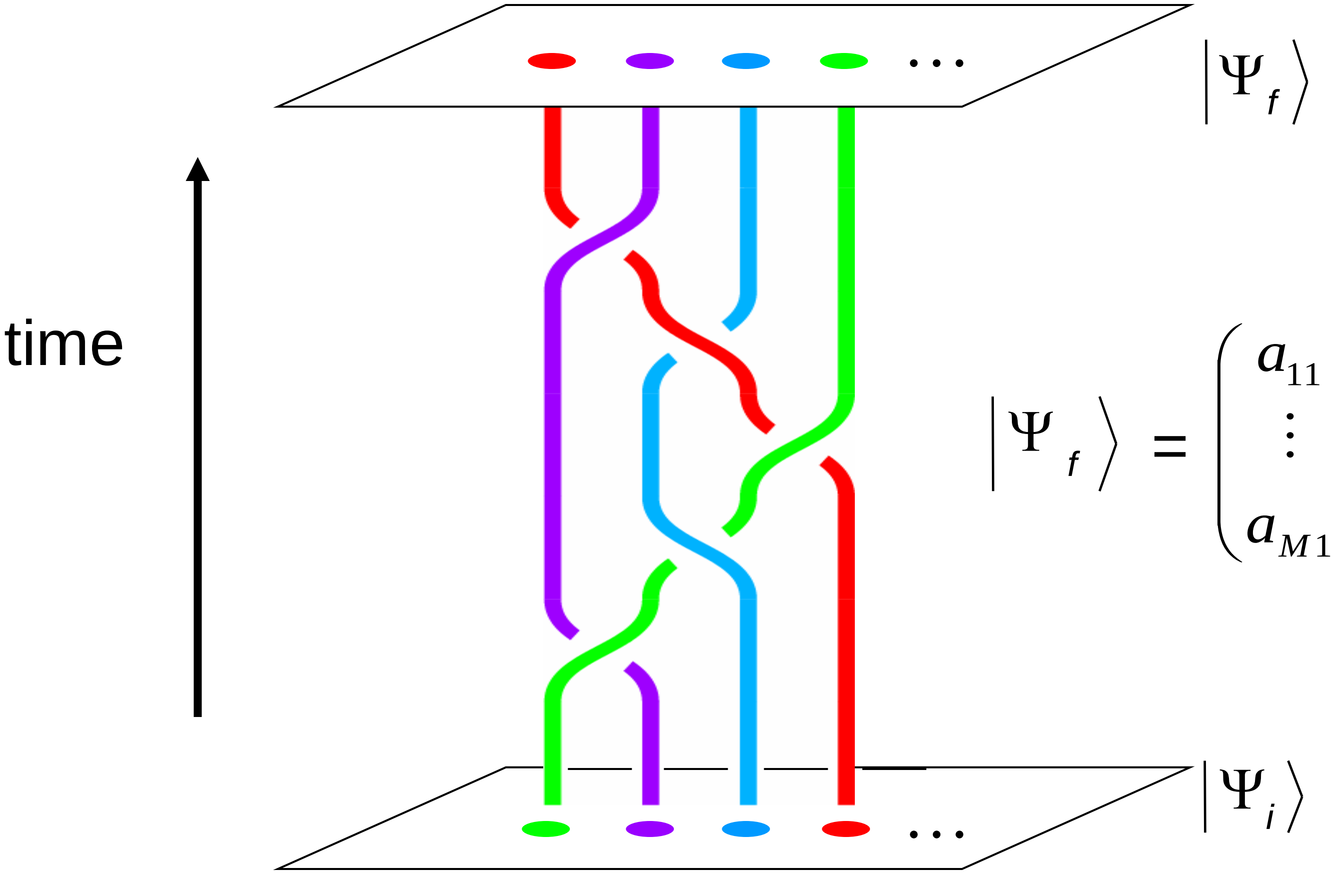
Particle “world-lines” form **braids** in 2+1 (=3) dimensions

Exchanging Particles in 2+1 Dimensions



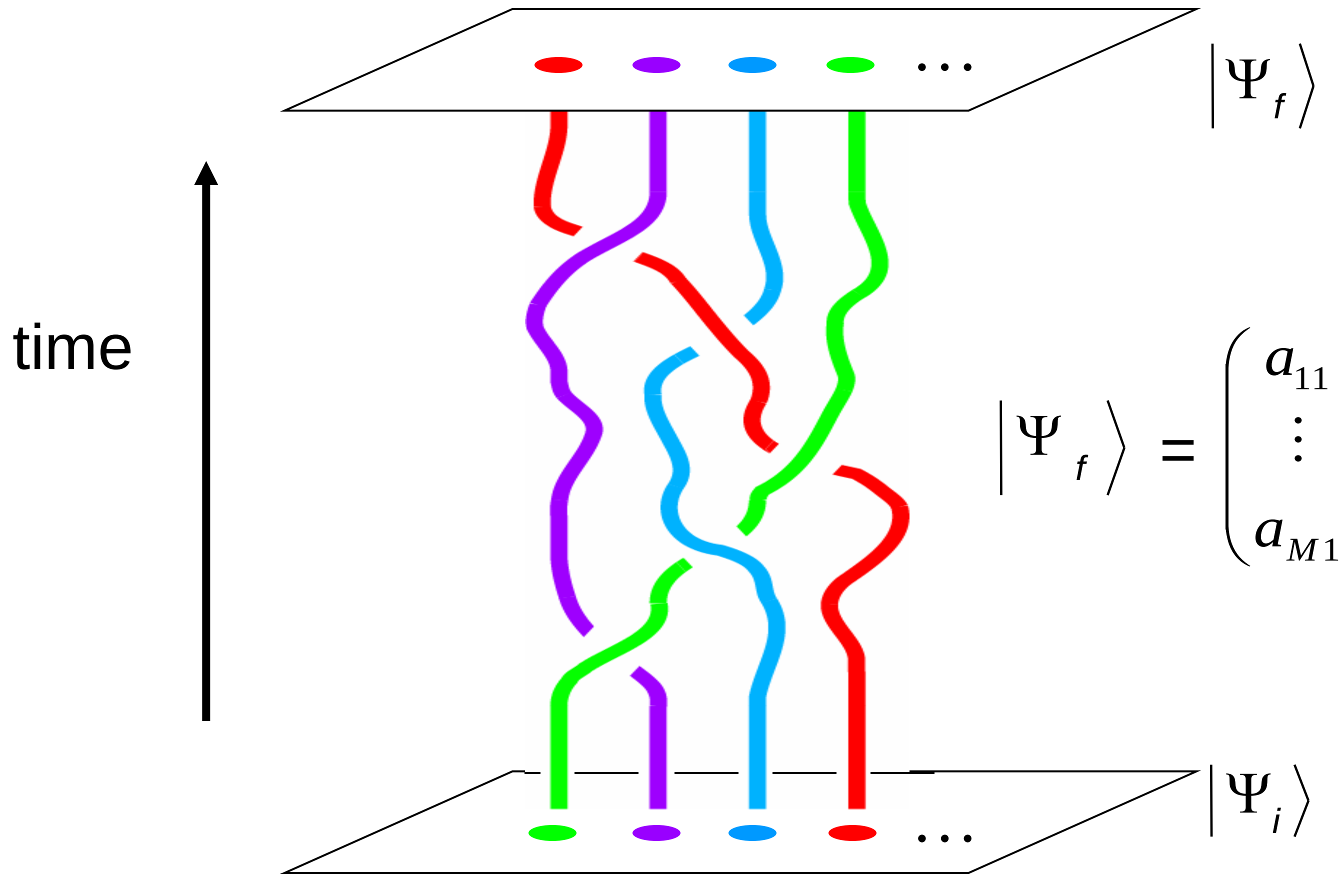
Particle “world-lines” form **braids** in 2+1 (=3) dimensions

Many Non-Abelian Anyons



$$|\Psi_f\rangle = \begin{pmatrix} a_{11} & \cdots & a_{1M} \\ \vdots & \ddots & \vdots \\ a_{M1} & \cdots & a_{MM} \end{pmatrix} |\Psi_i\rangle$$

Many Non-Abelian Anyons



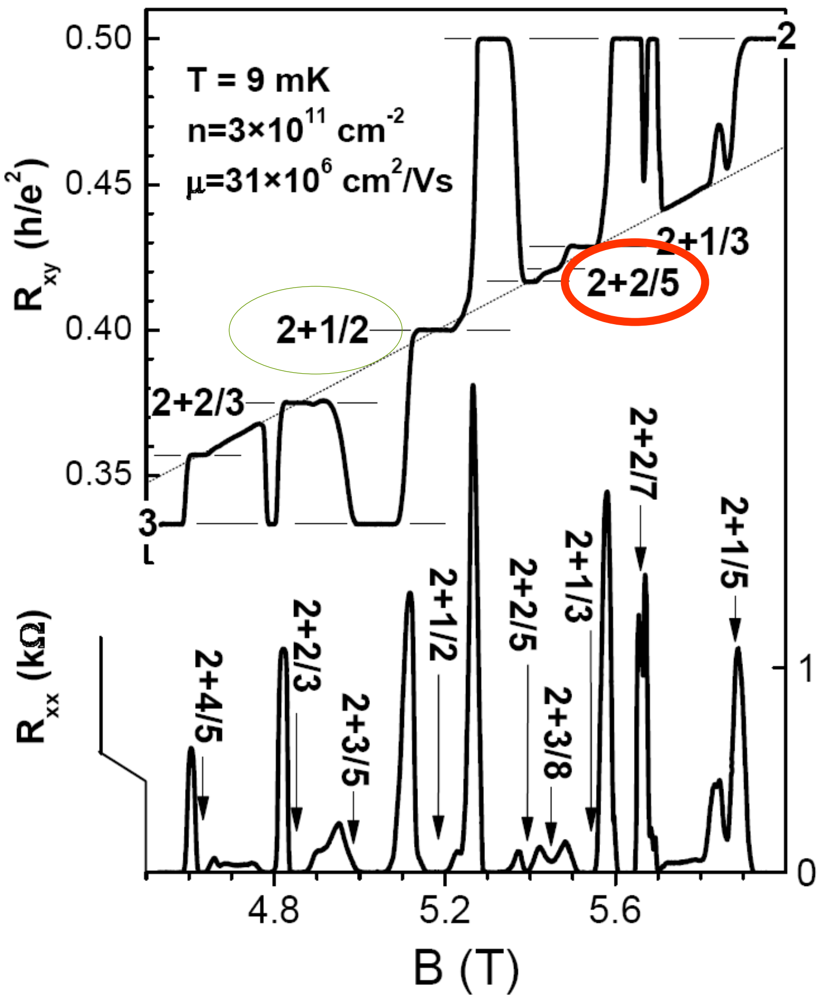
$$|\Psi_f\rangle = \begin{pmatrix} a_{11} & \cdots & a_{1M} \\ \vdots & \ddots & \vdots \\ a_{M1} & \cdots & a_{MM} \end{pmatrix} |\Psi_i\rangle$$

Matrix depends only on the topology of the braid swept out by anyon world lines!

Robust quantum computation?

Possible Non-Abelian FQH States

J.S. Xia et al., PRL (2004).



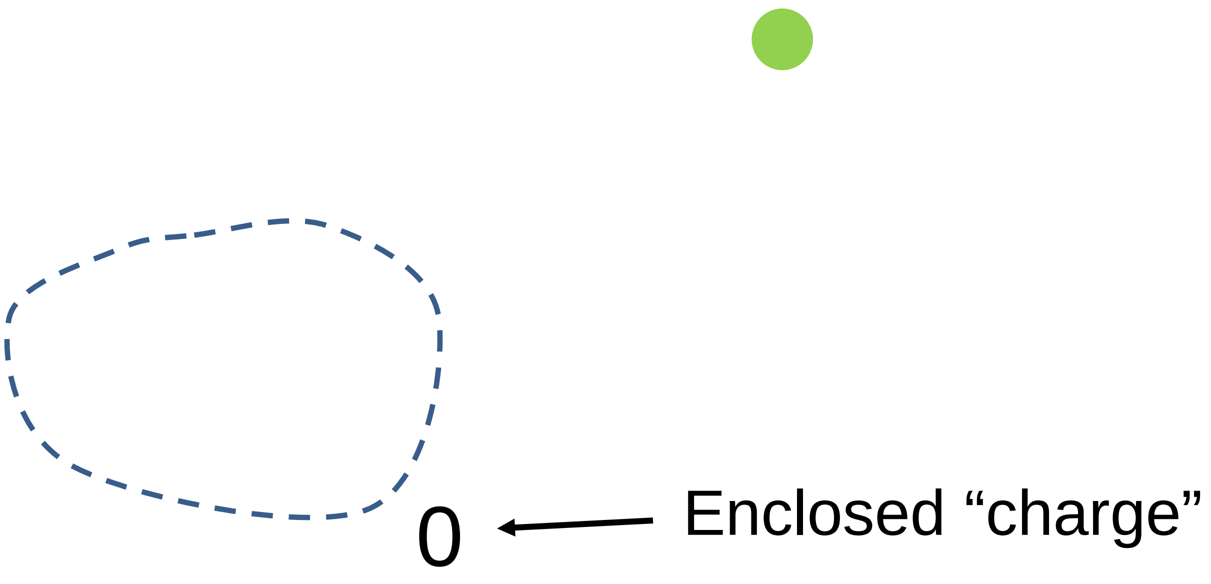
ν = 5/2: Probable Moore-Read Pfaffian state.

Charge *e/4* quasiparticles are **Majorana fermions**.
Moore & Read '91

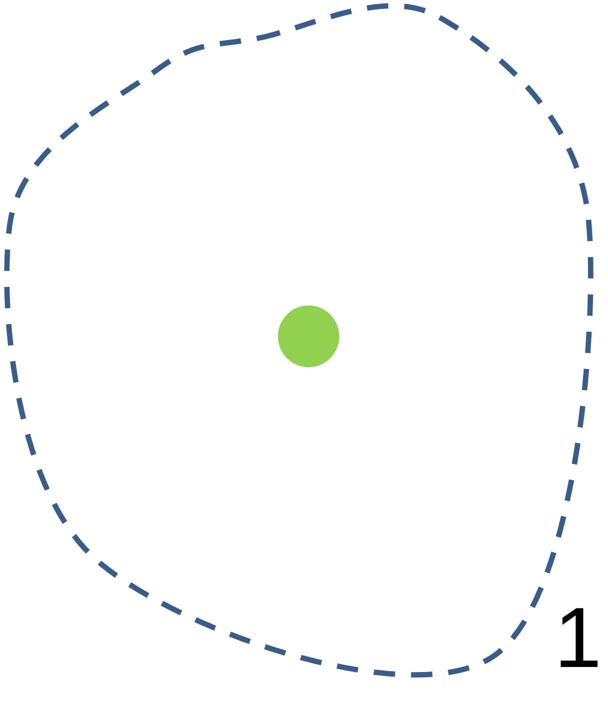
ν = 12/5: Possible Read-Rezayi "Parafermion" state. Read & Rezayi, '99

Charge *e/5* quasiparticles are **Fibonacci anyons.**
Slingerland & Bais '01

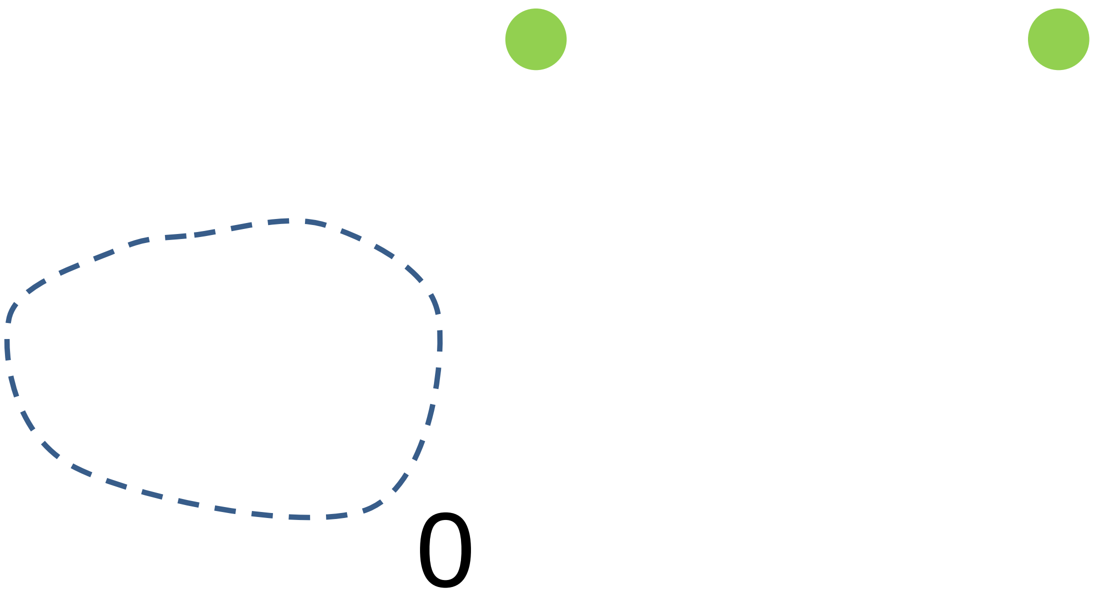
Universal for Quantum Computation!
Freedman, Larsen & Wang '02



0
Enclosed “charge”



1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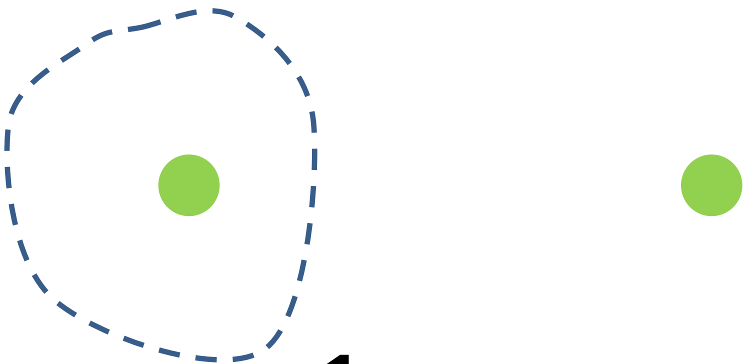
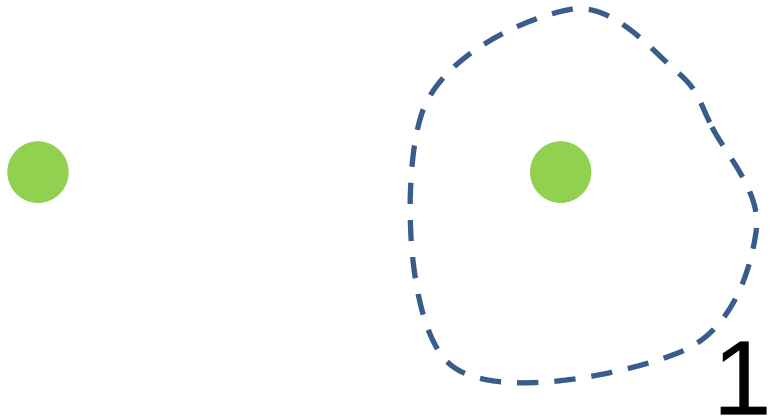
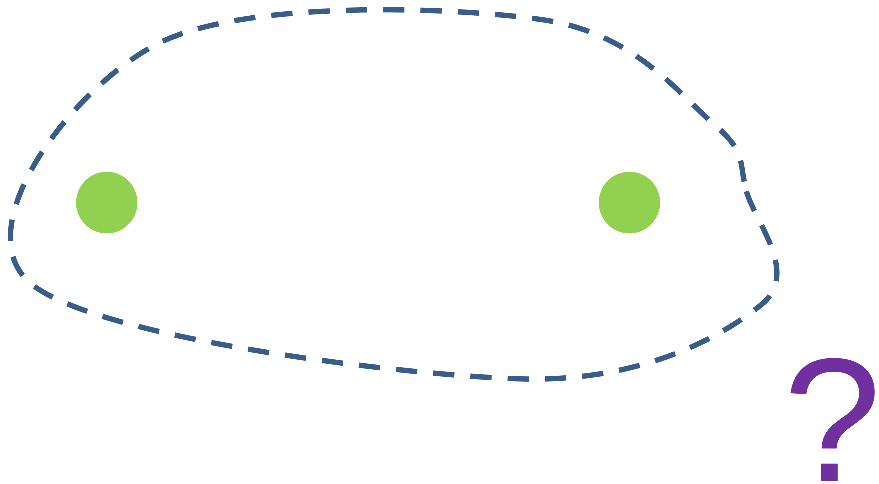
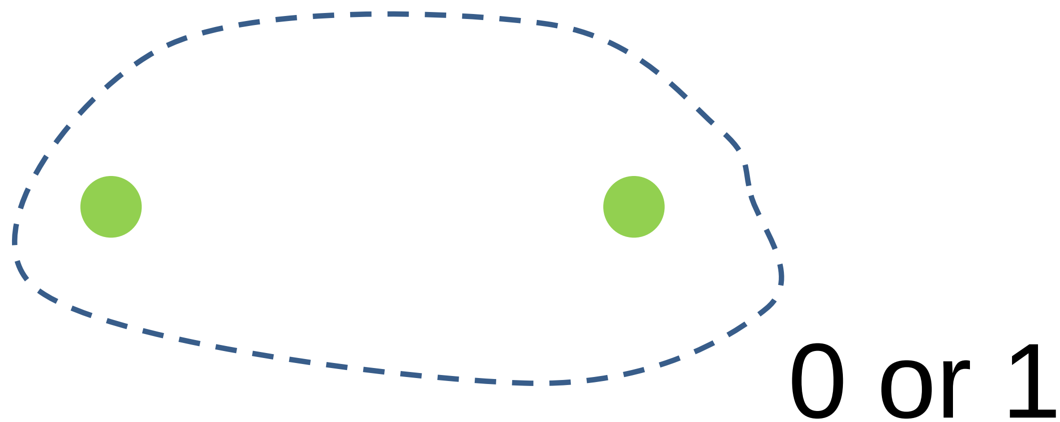
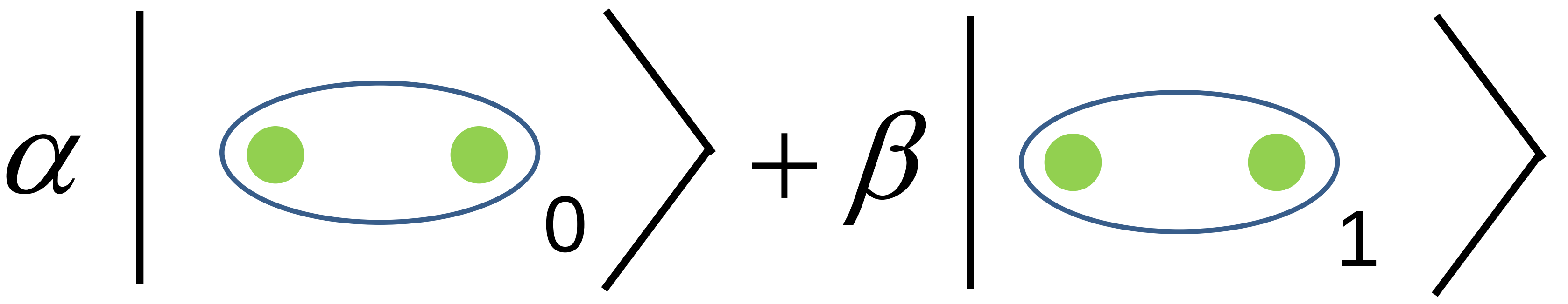


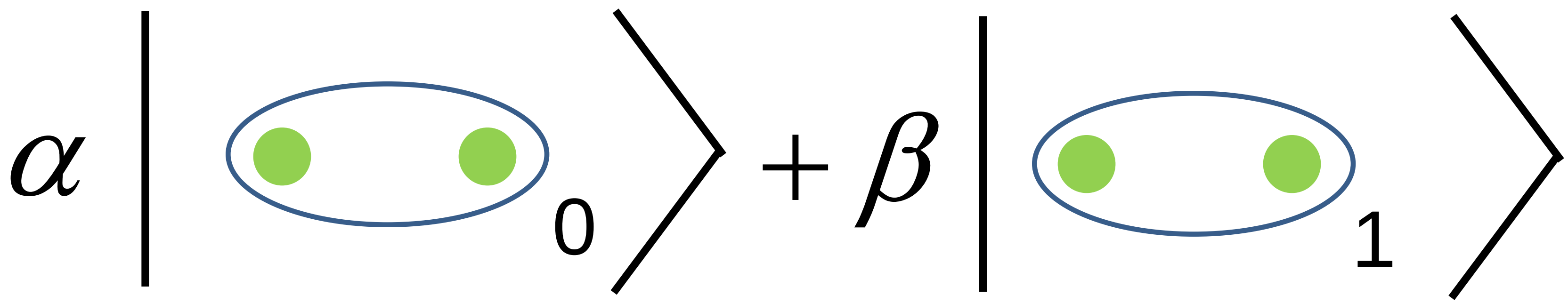
1


1

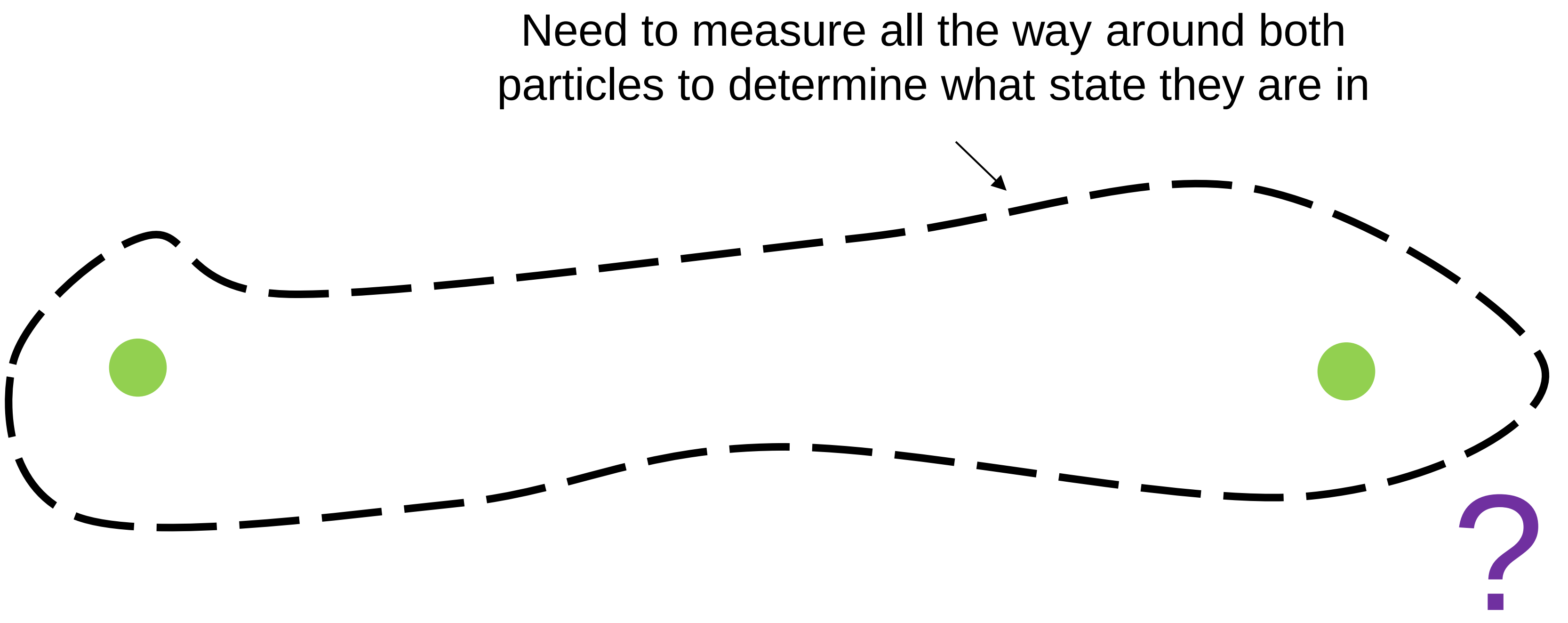



0 or 1

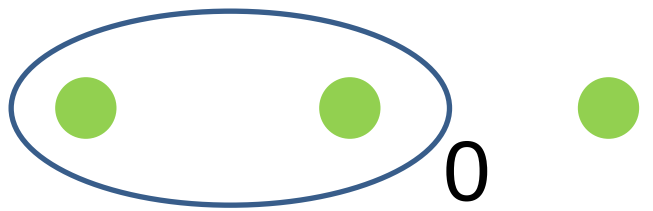

α
0
+ β
1

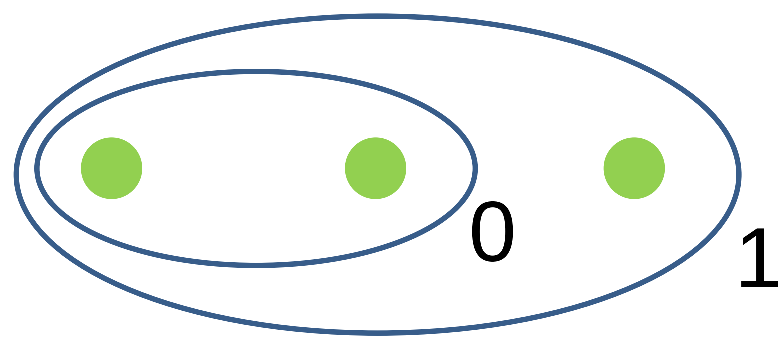


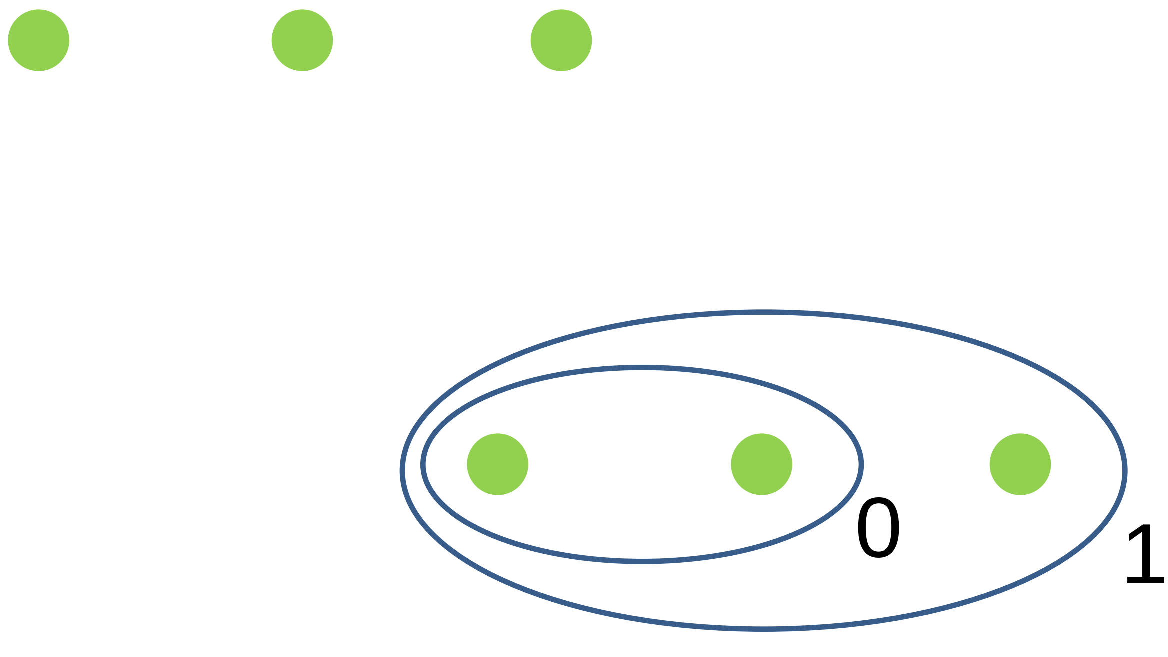
2 dimensional Hilbert space

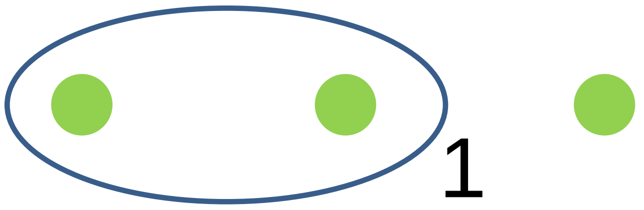


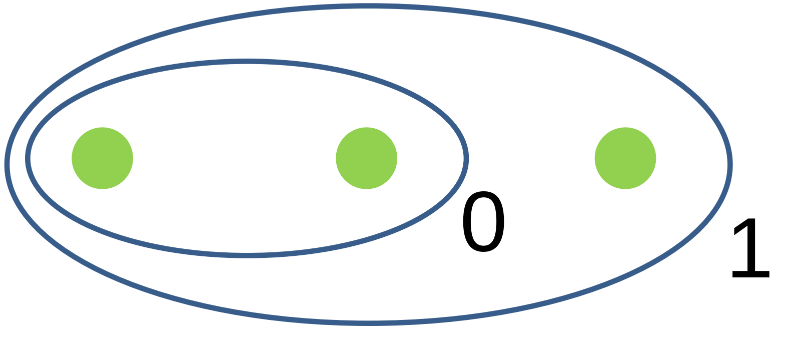
Quantum states are protected from environment if particles are kept far apart

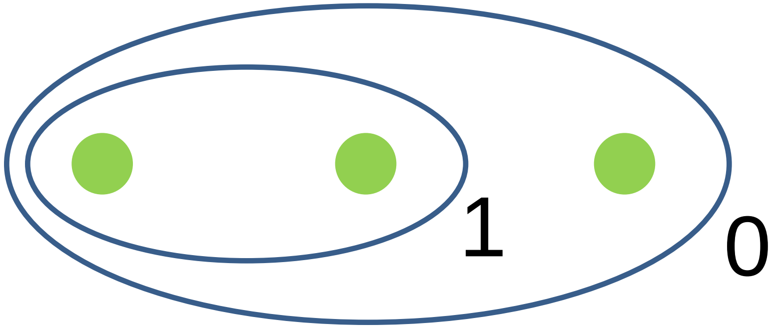

0

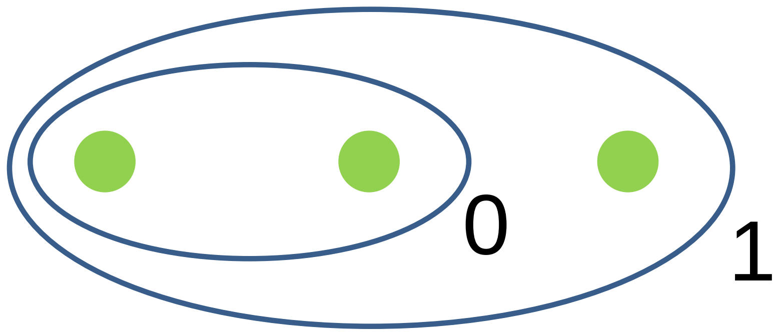

0
1

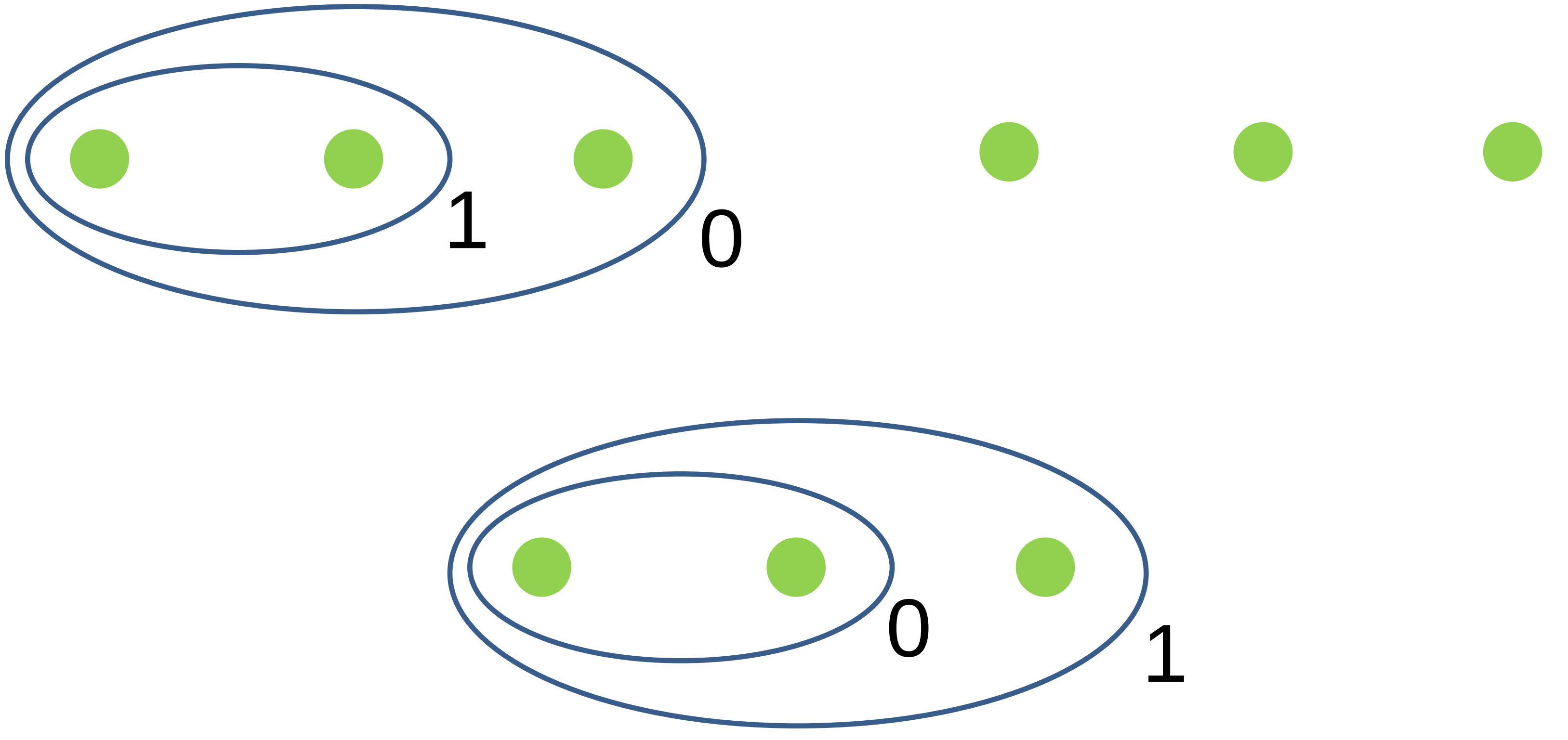

0
1

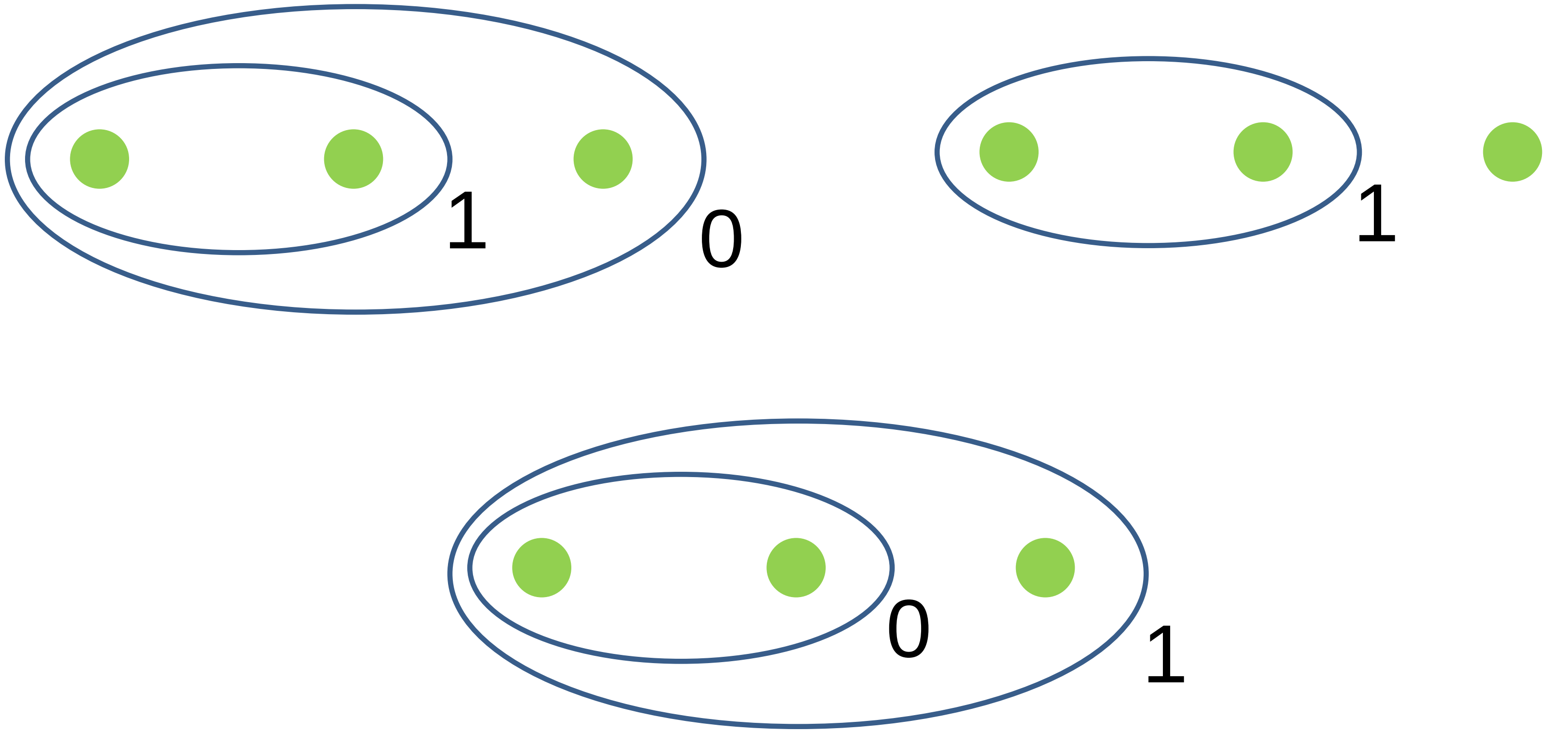

1

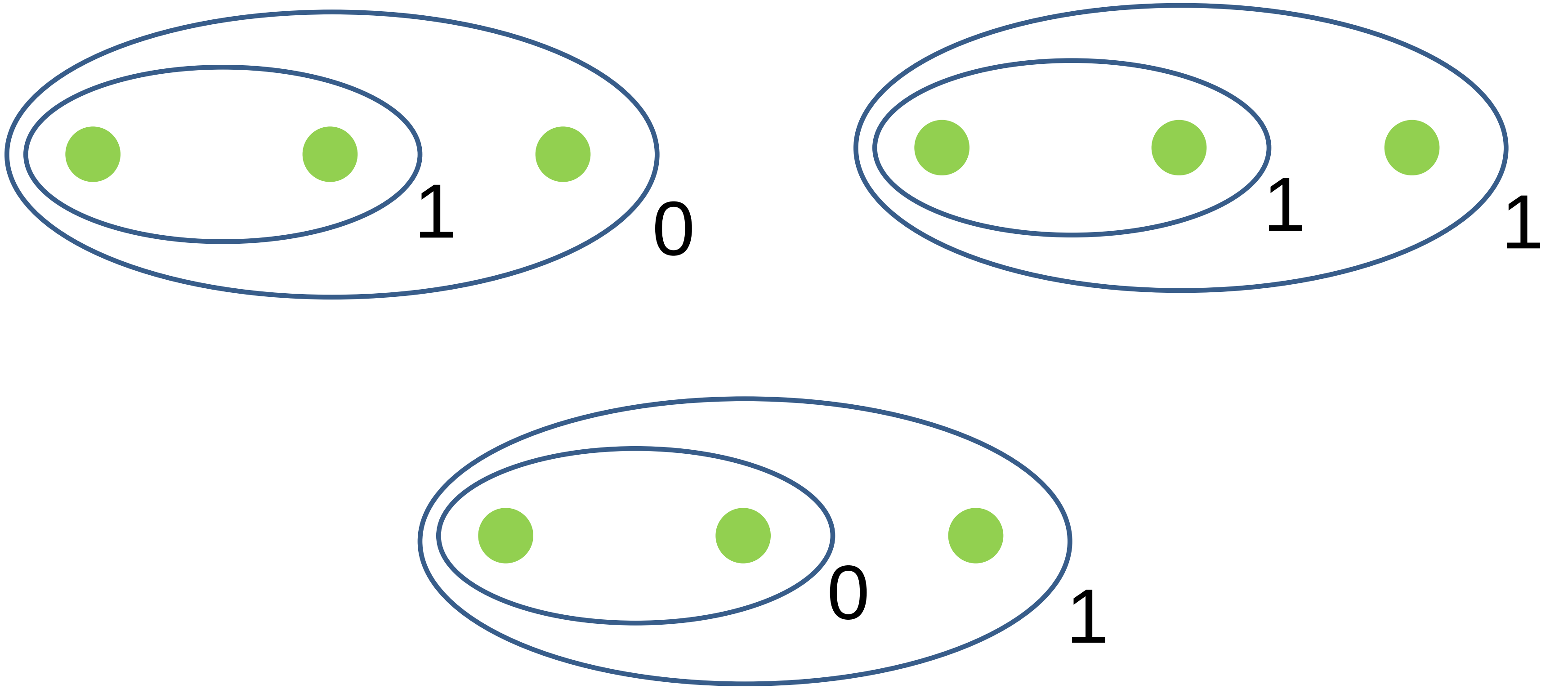

0
1

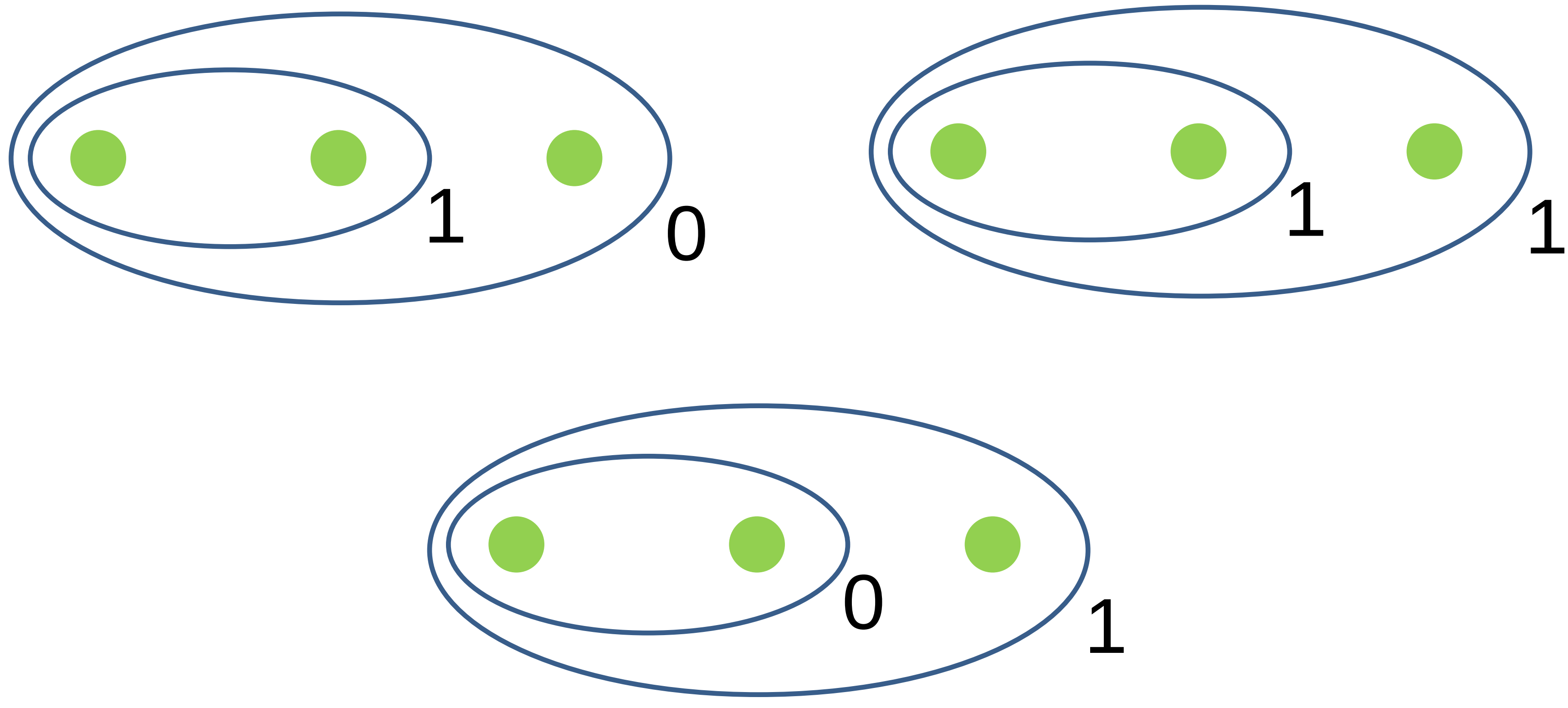

1
0


0
1

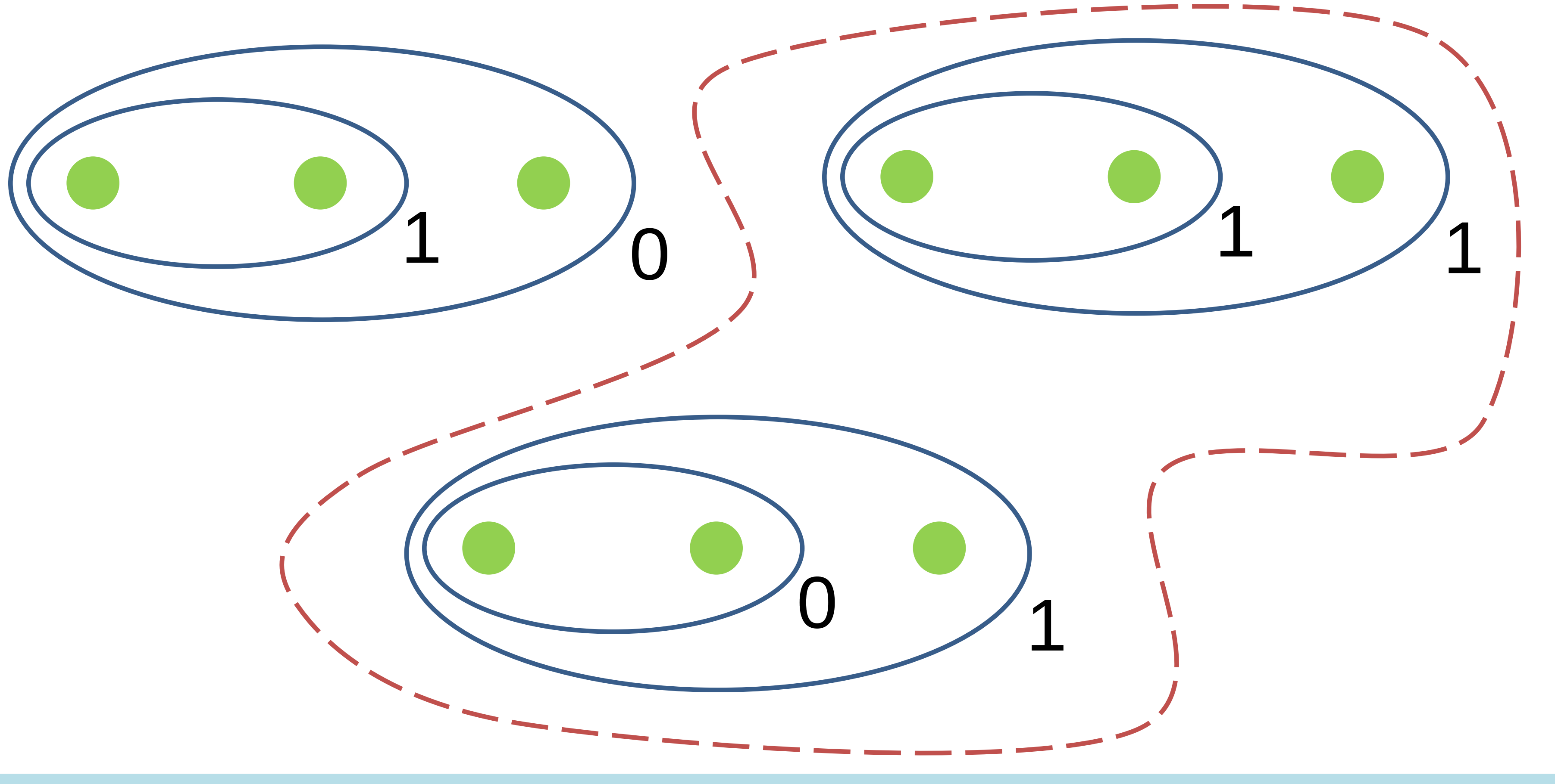

1
0
0
1


1
0
1
0
1

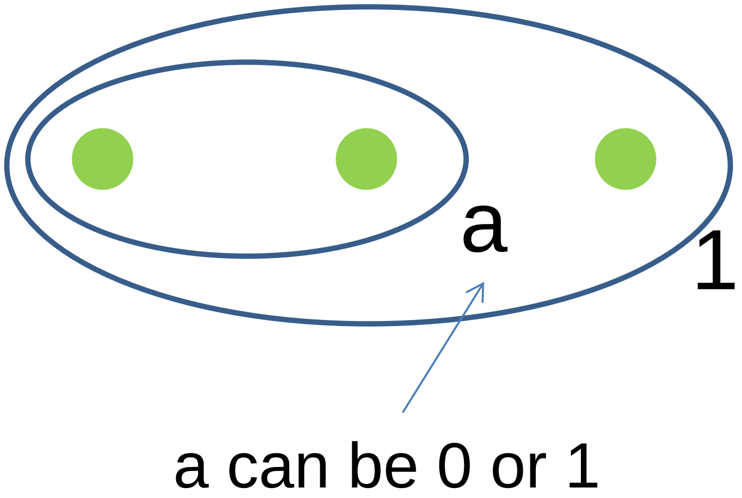

1
0
1
1
0
1

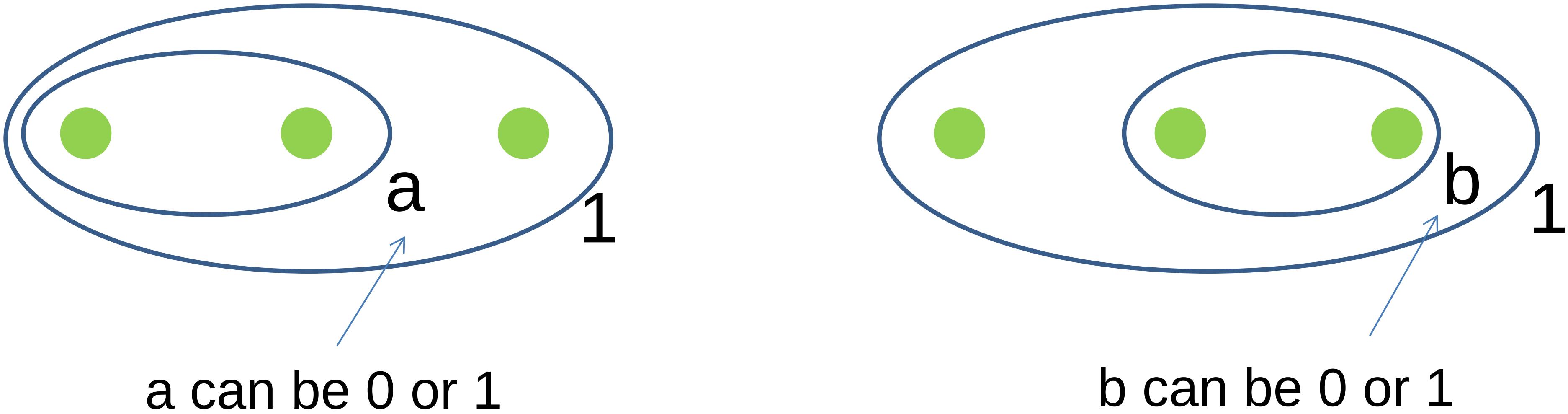


3 dimensional Hilbert space



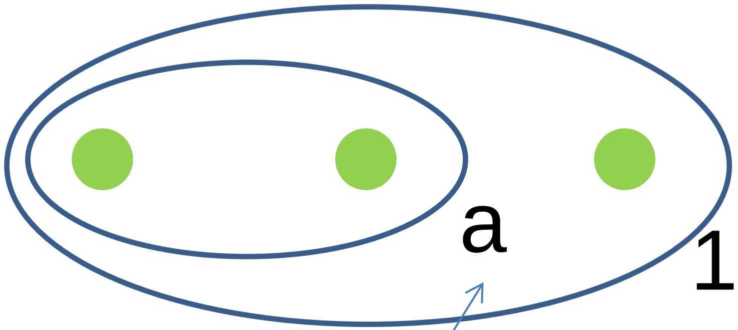
3 dimensional Hilbert space


a
1
a can be 0 or 1

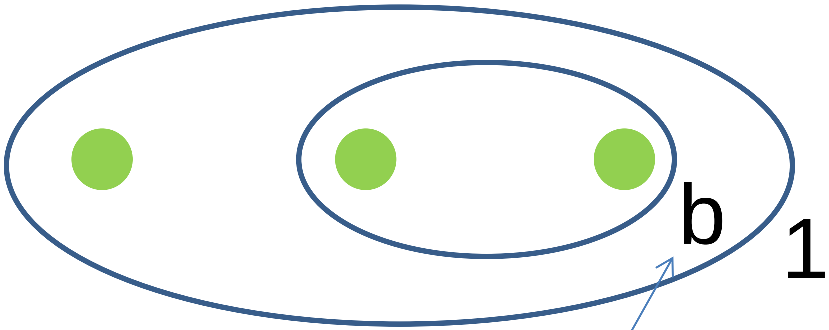


a
1
a can be 0 or 1
b
1
b can be 0 or 1

The F Matrix



$$= \sum_b F_{ab}$$

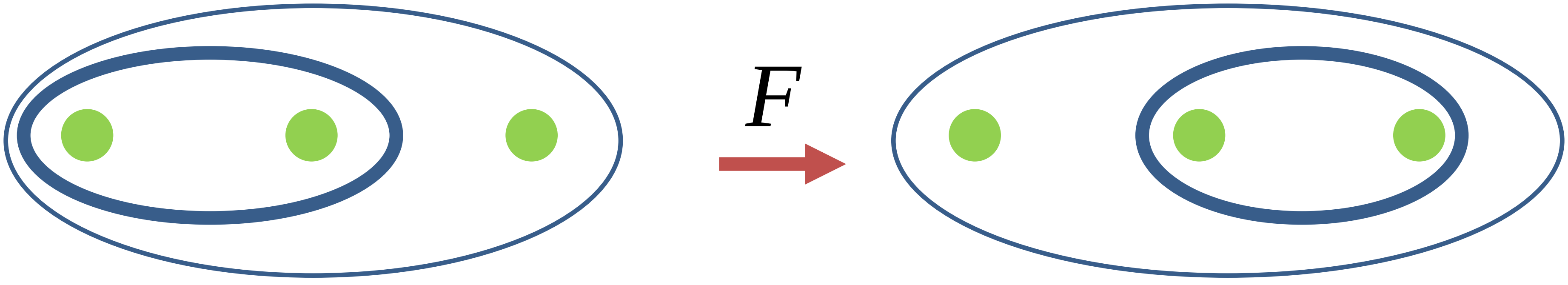


a can be 0 or 1

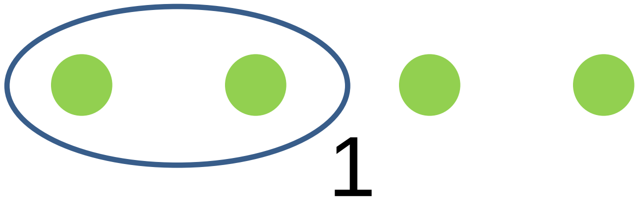
b can be 0 or 1

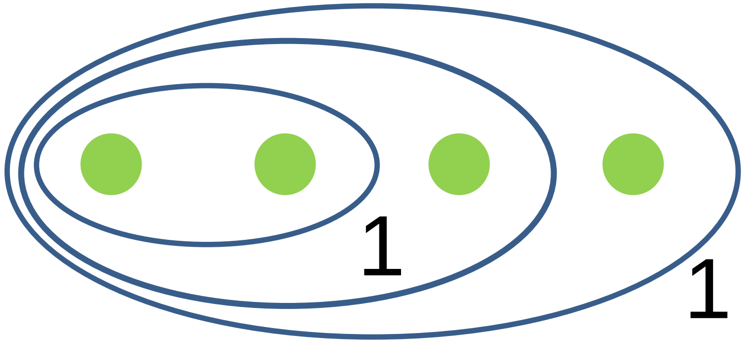
$$F = \begin{pmatrix} F_{00} & F_{01} \\ F_{10} & F_{11} \end{pmatrix}$$

The F Matrix

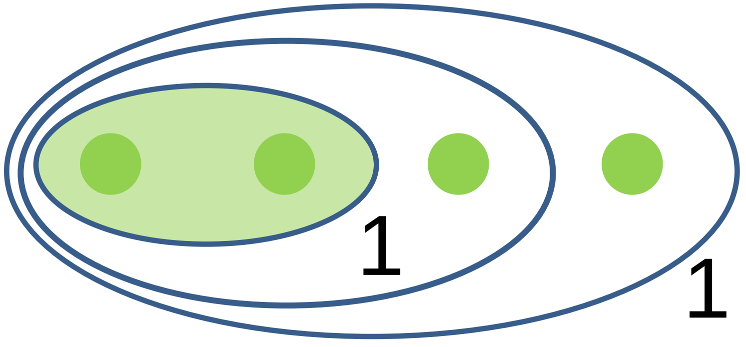


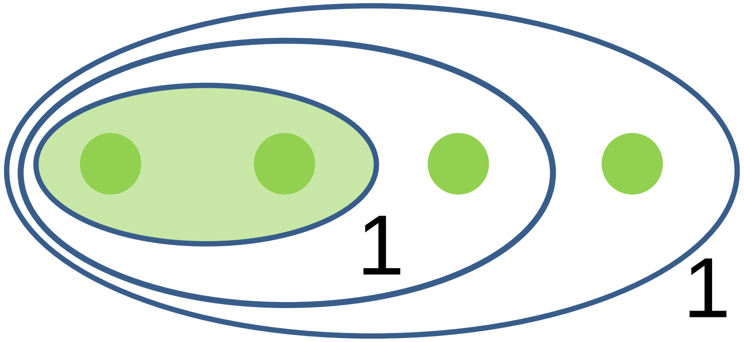
$$F = \begin{pmatrix} F_{00} & F_{01} \\ F_{10} & F_{11} \end{pmatrix}$$


1

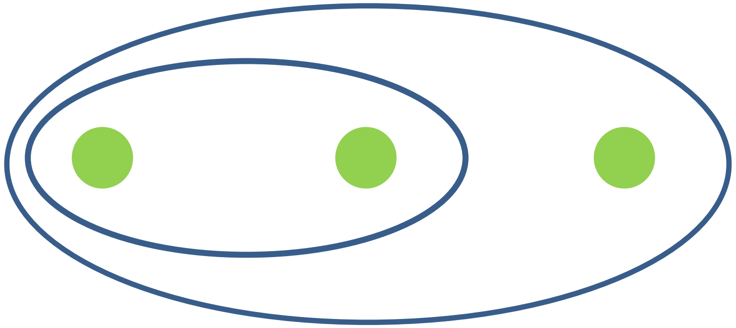


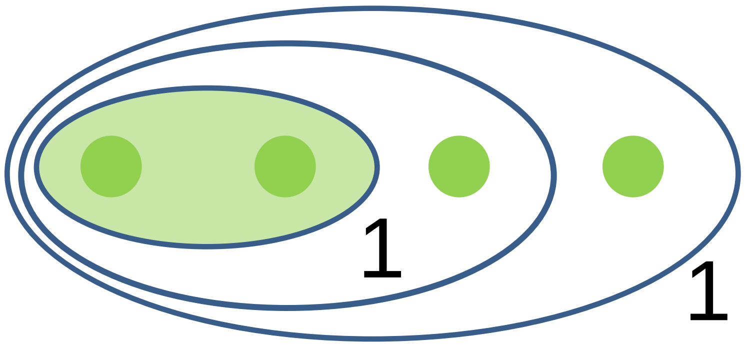
1
1


1
1

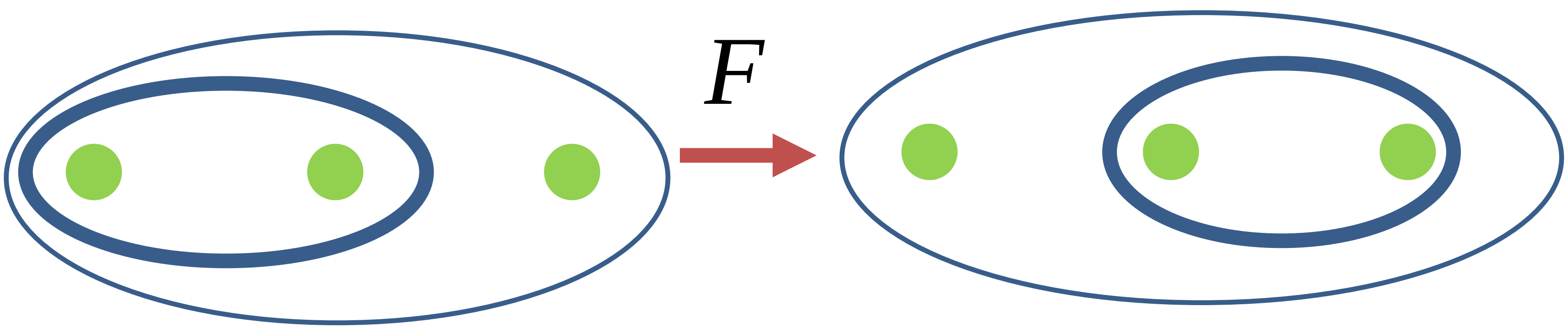


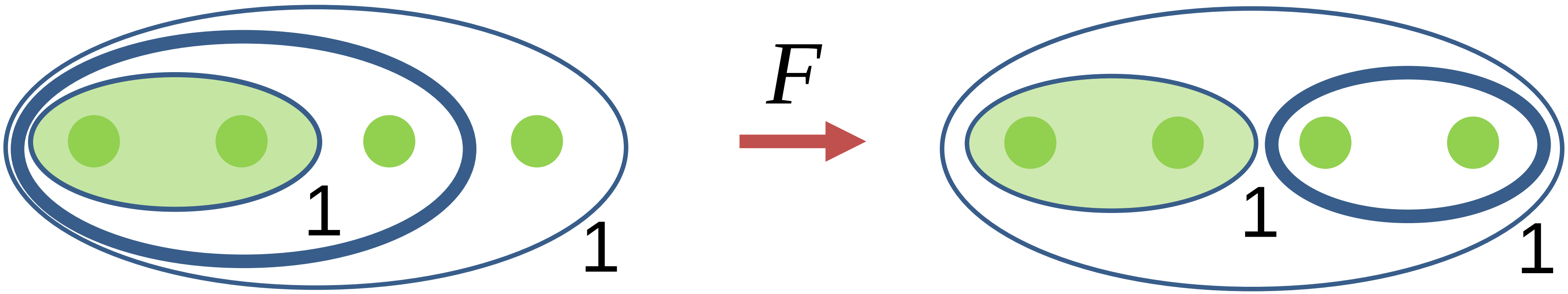
Equivalent to



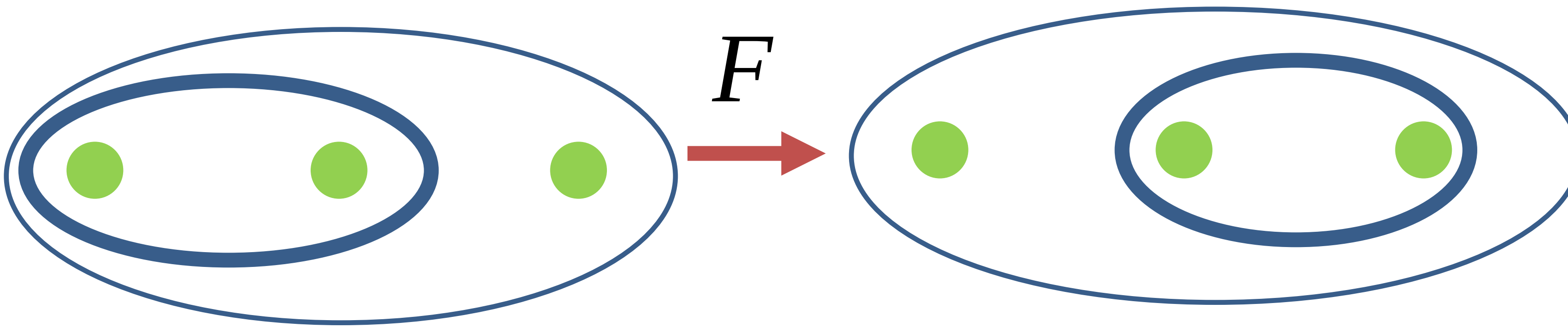


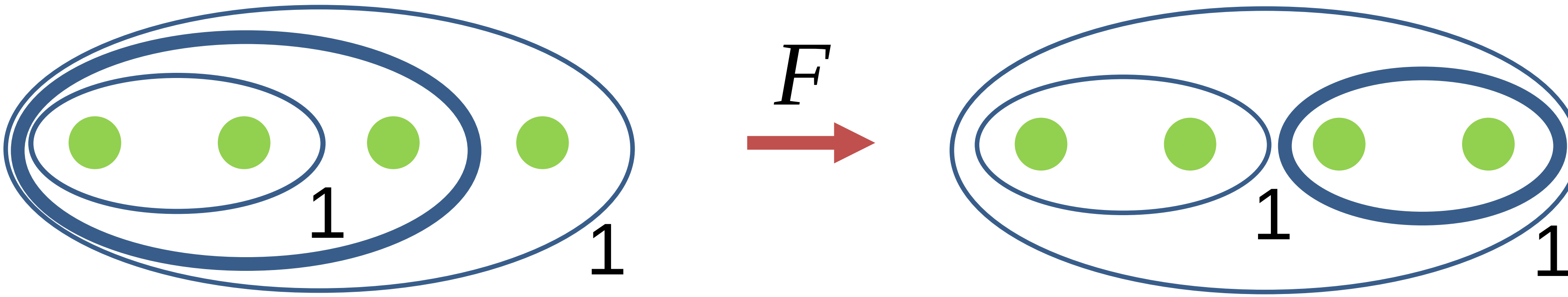
Equivalent to



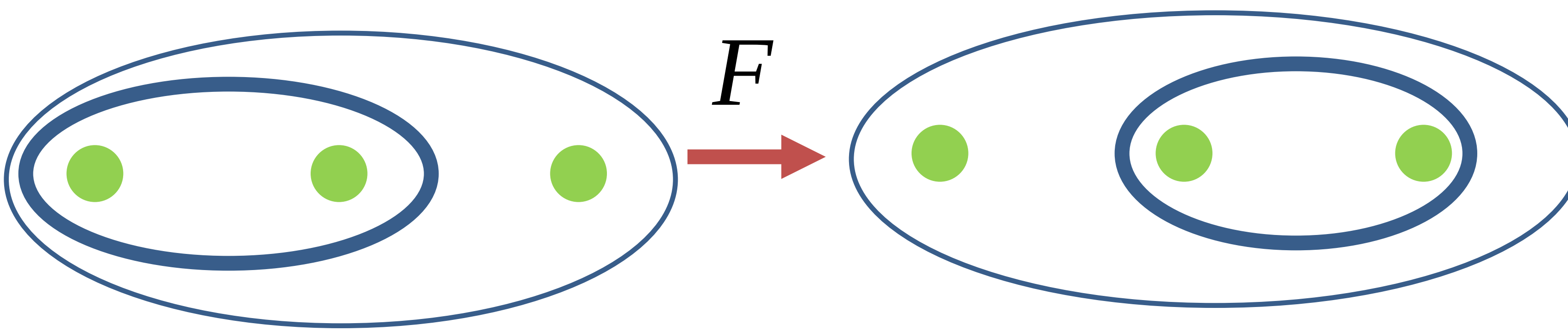


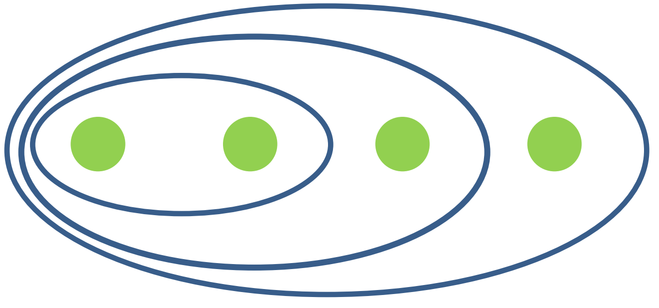
Equivalent to

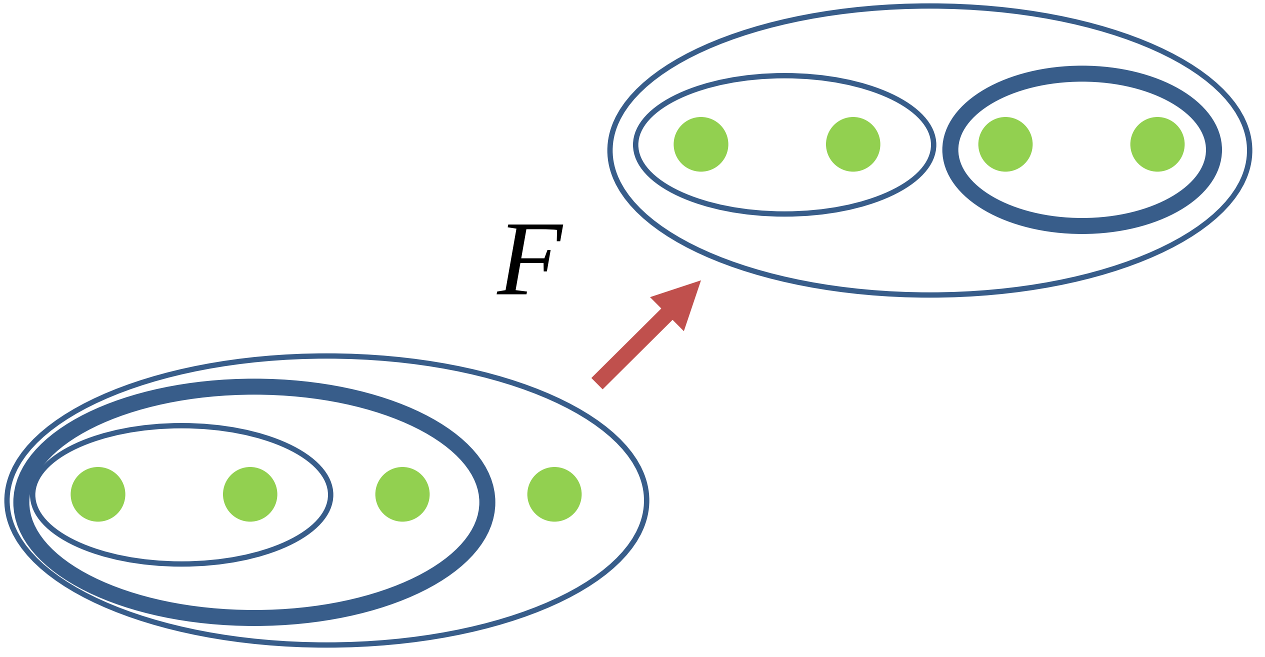


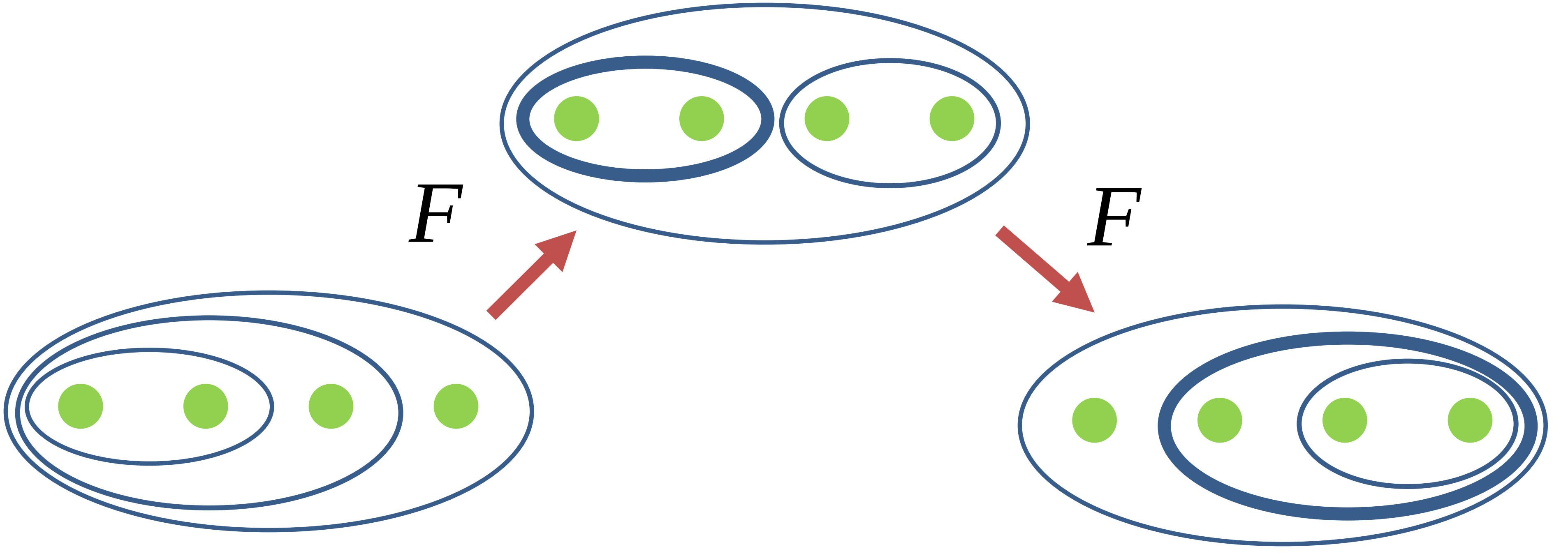


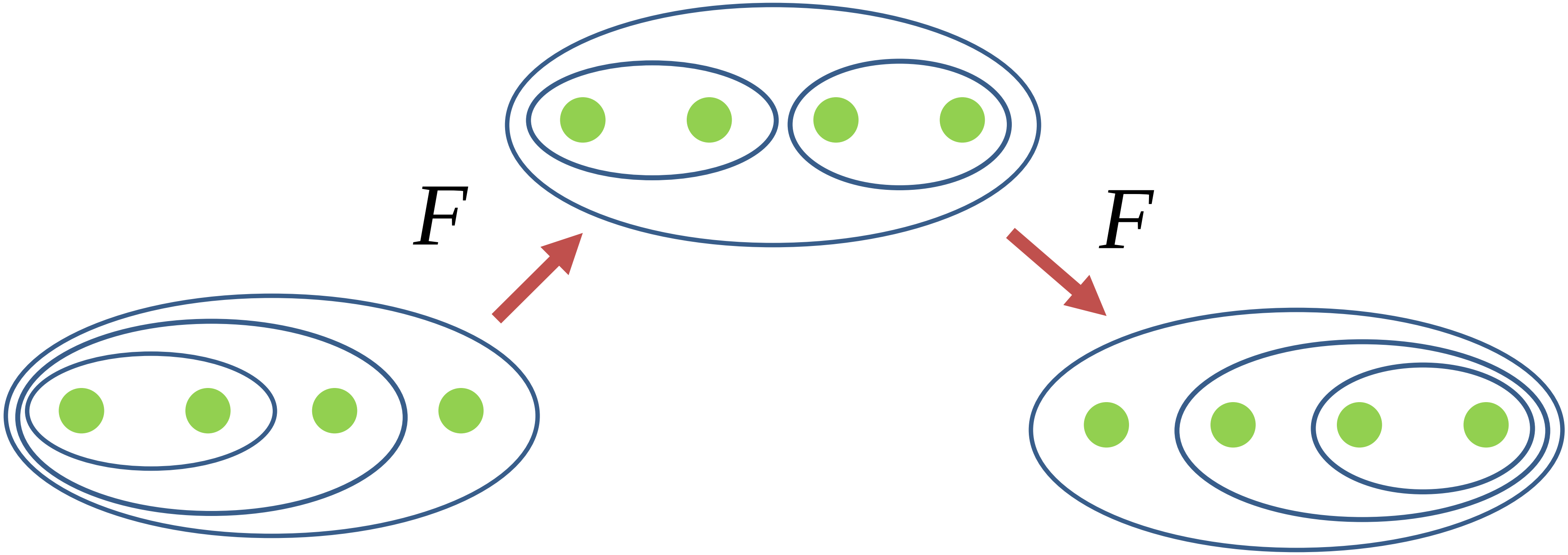
Equivalent to






F


F
F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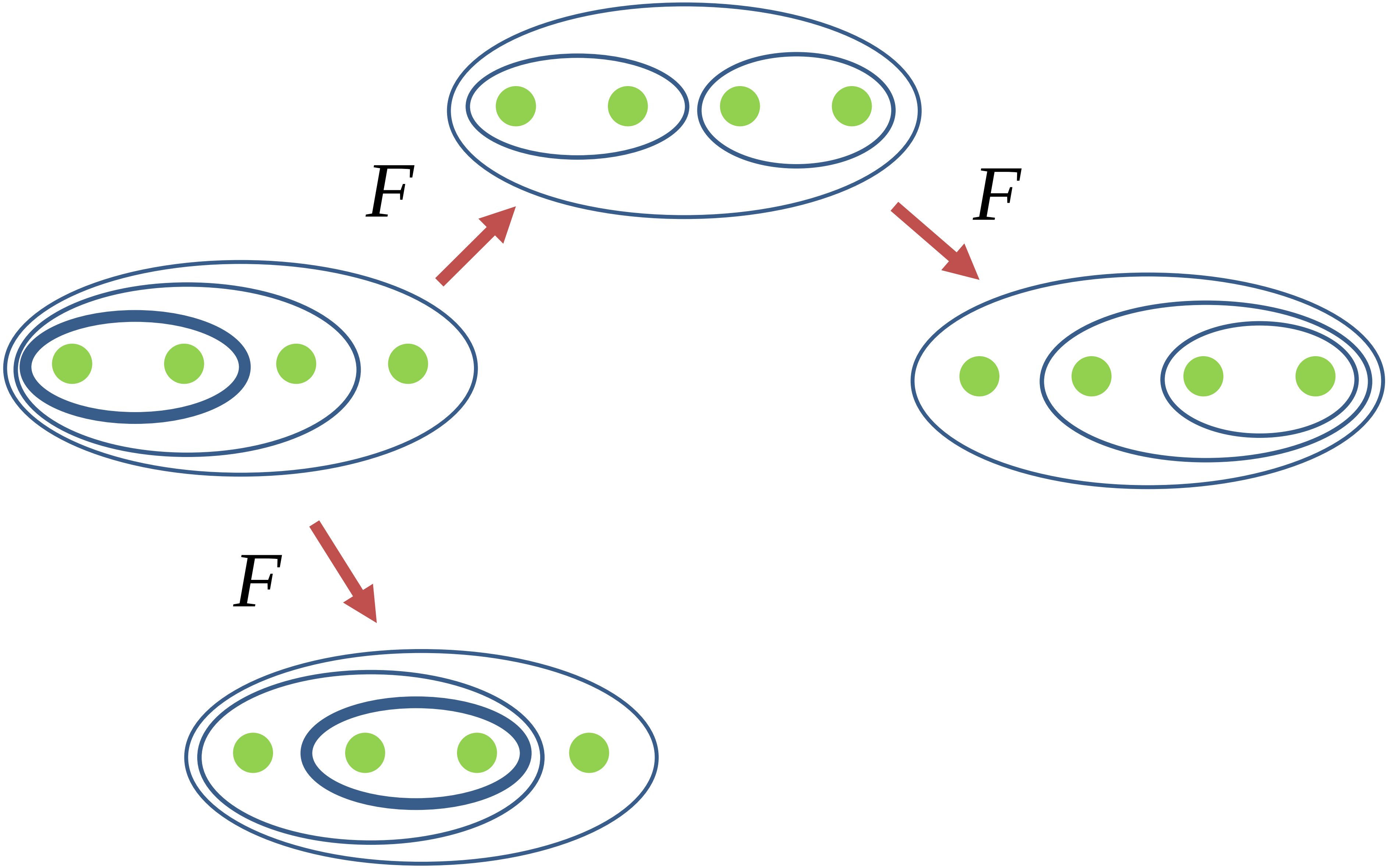
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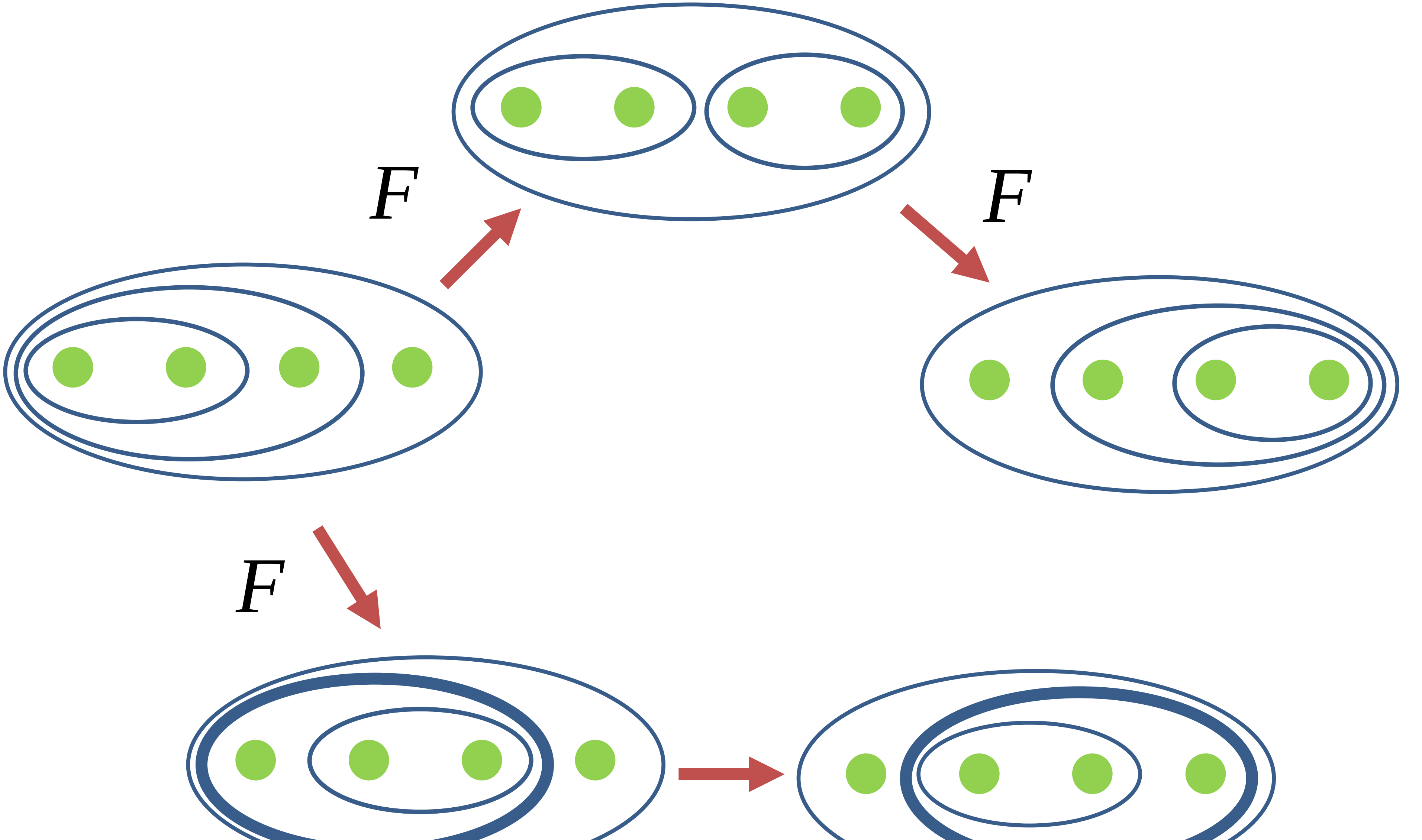
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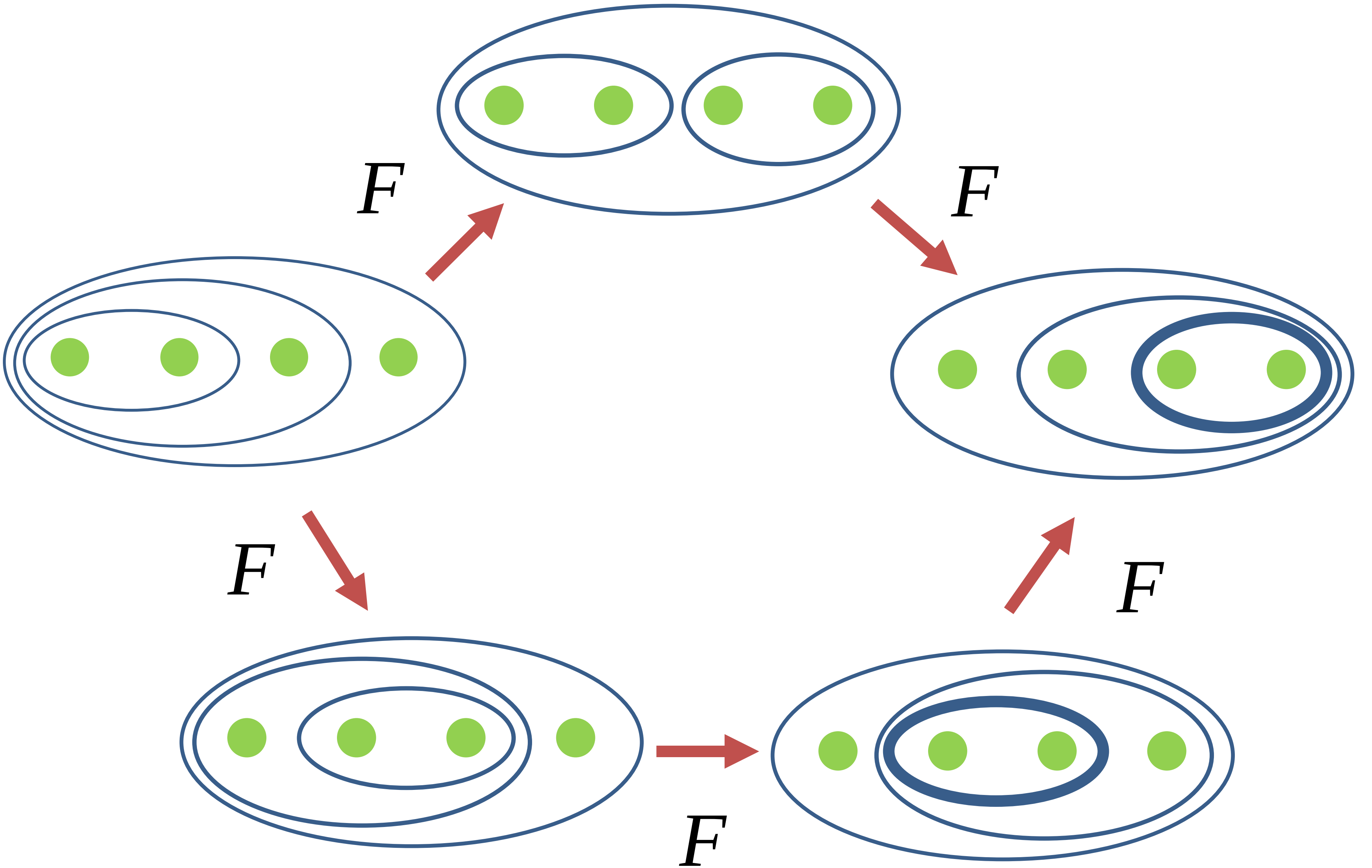
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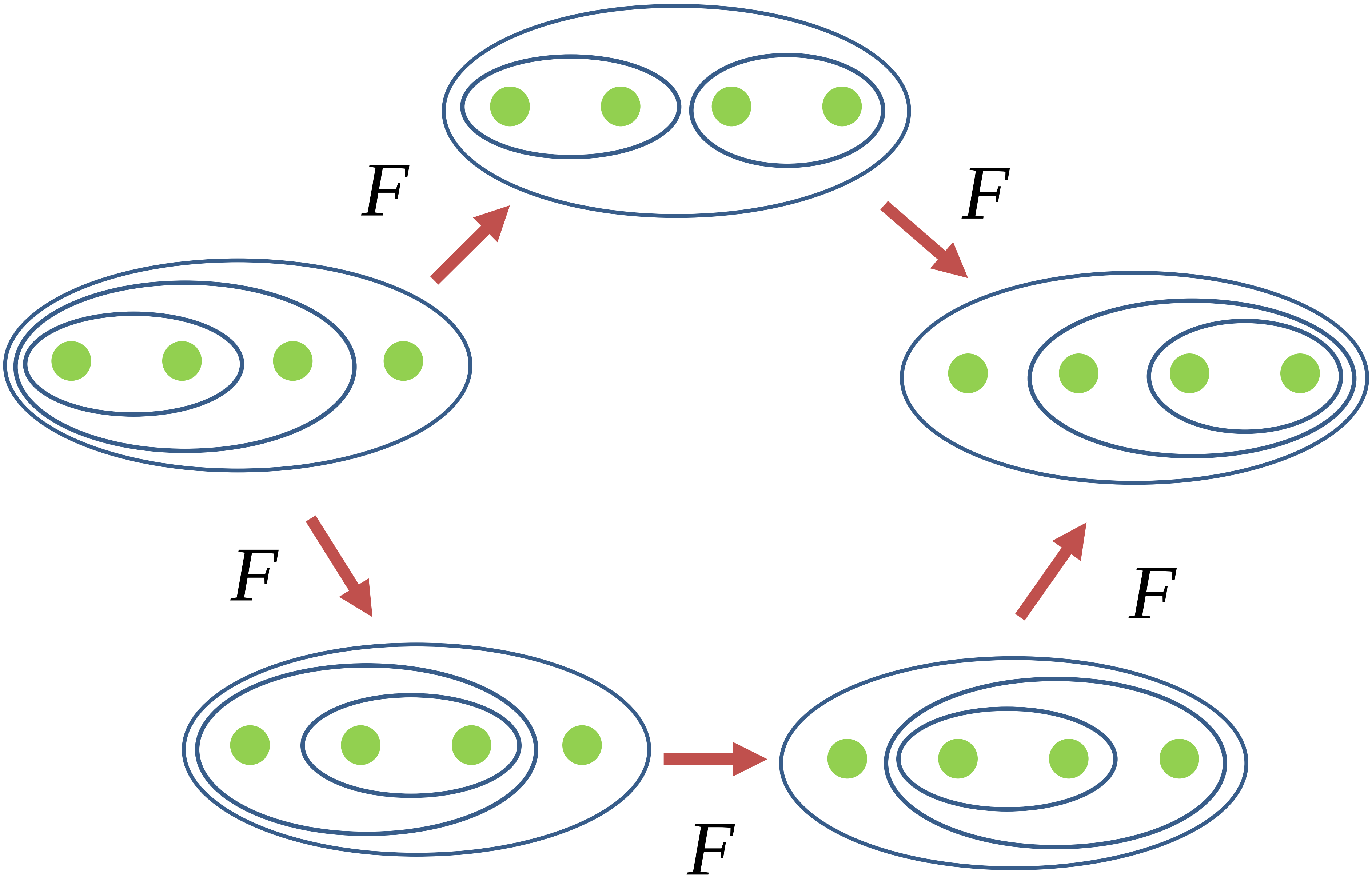
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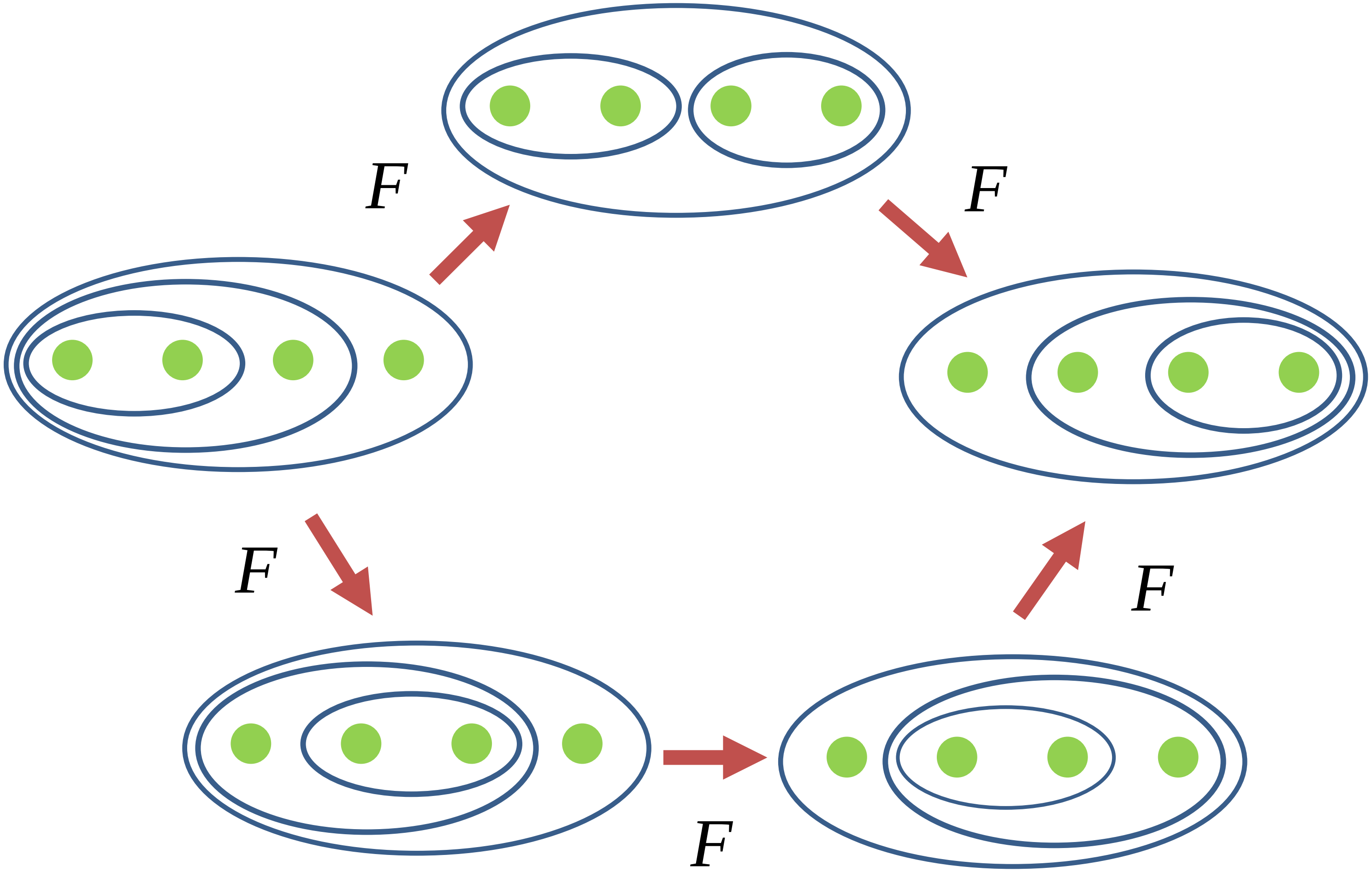
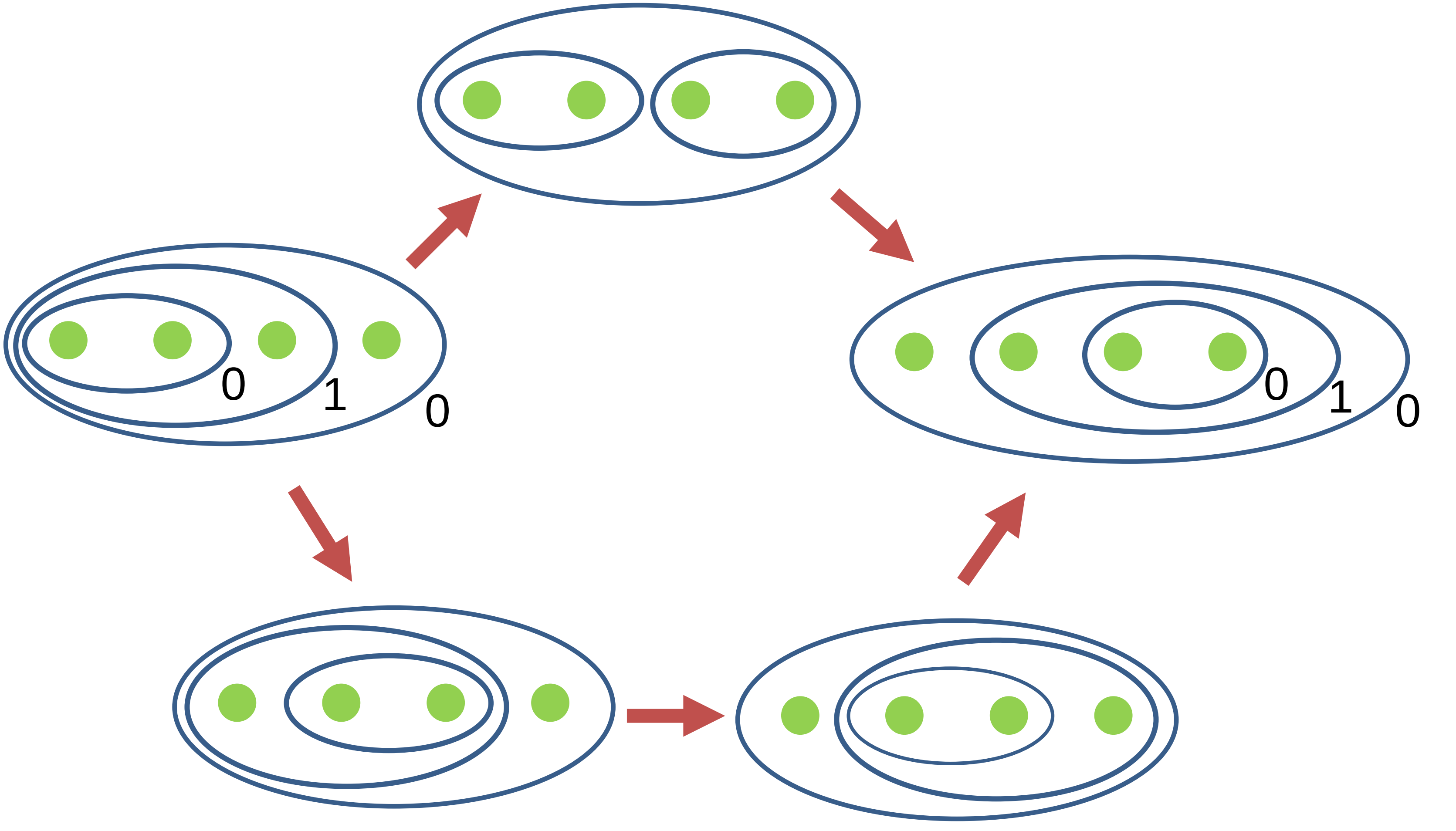
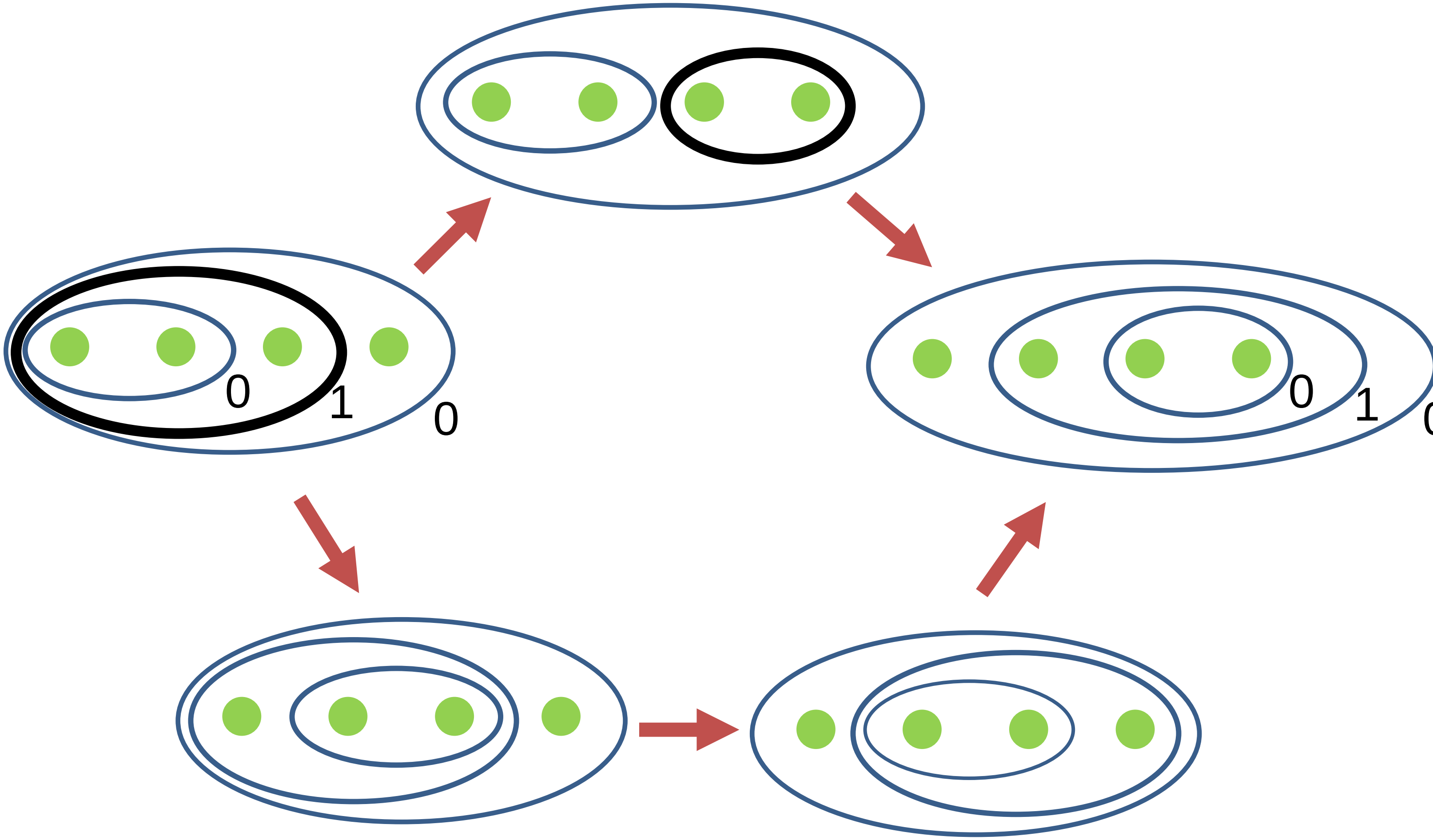
The Pentagon Equation



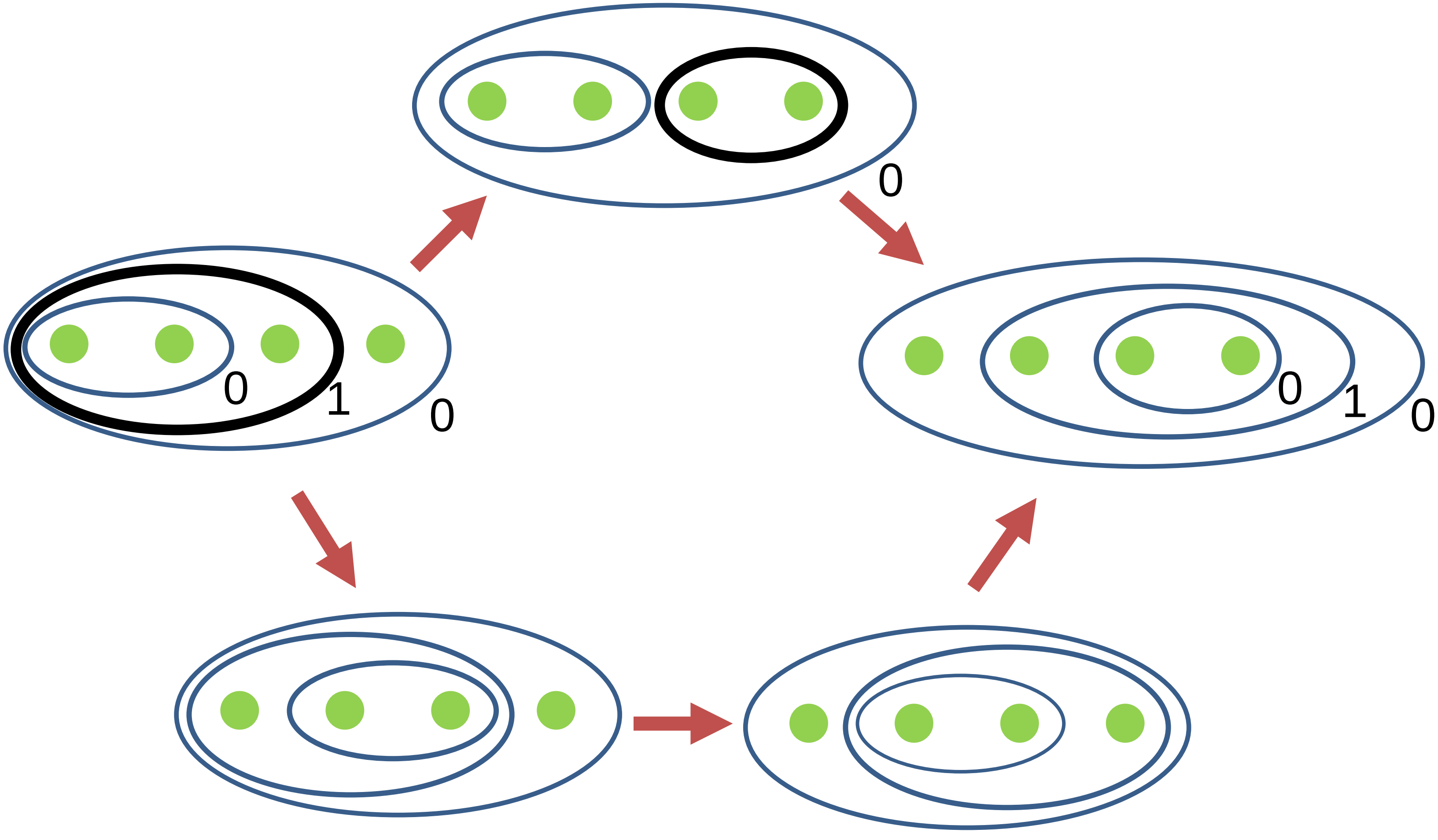
The Pentagon Equation



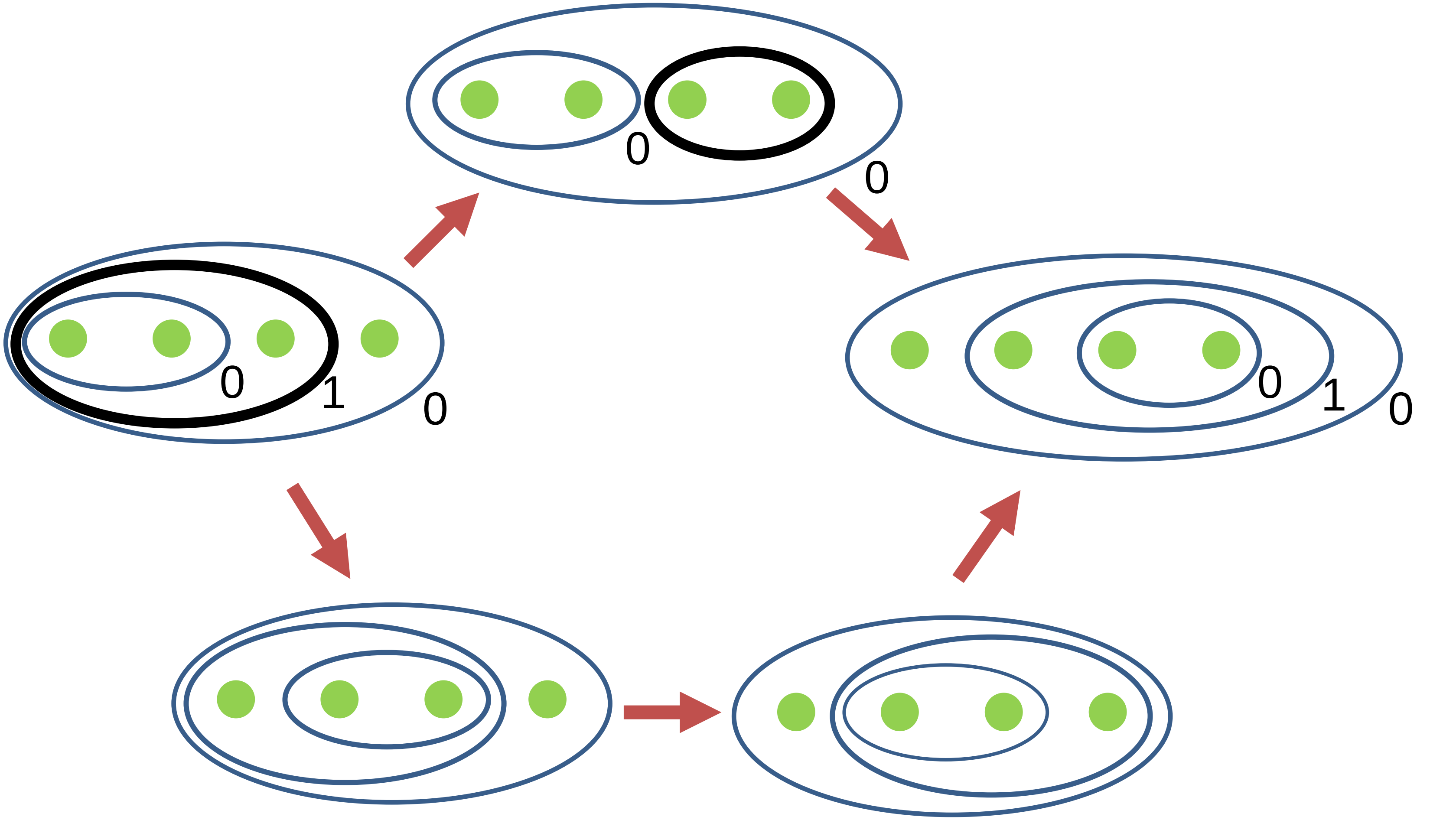
The Pentagon Equation



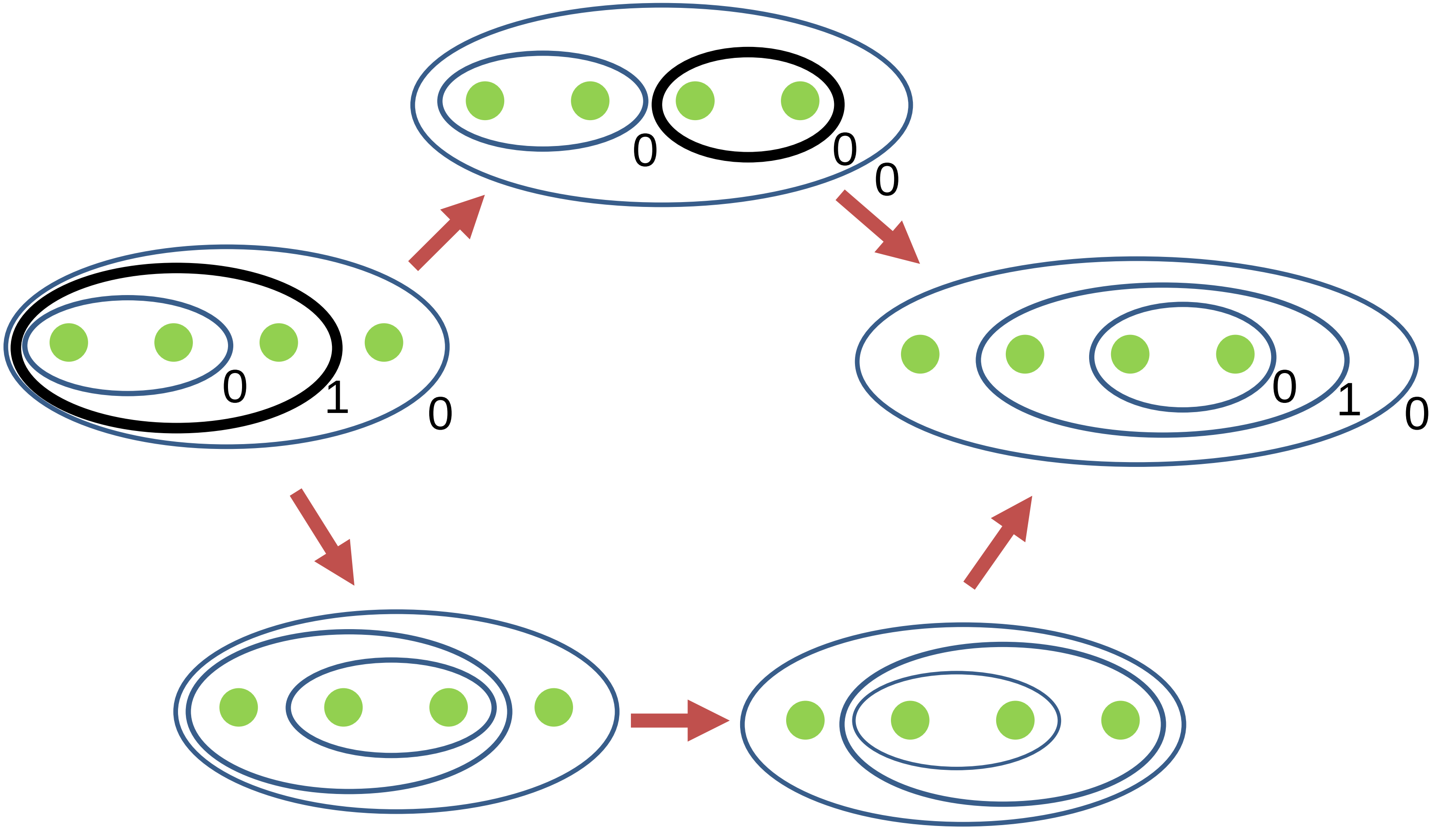
The Pentagon Equation



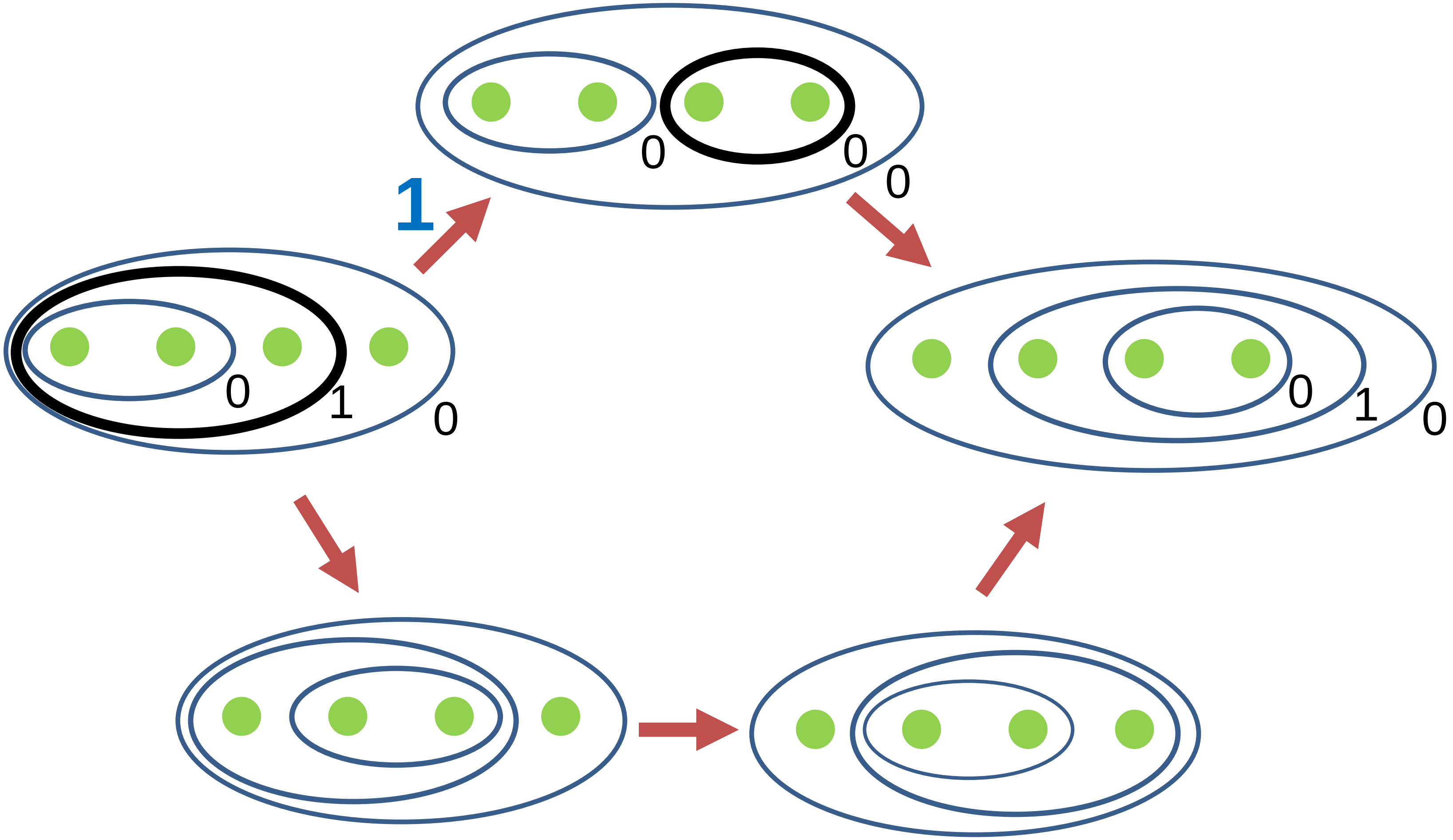
The Pentagon Equation



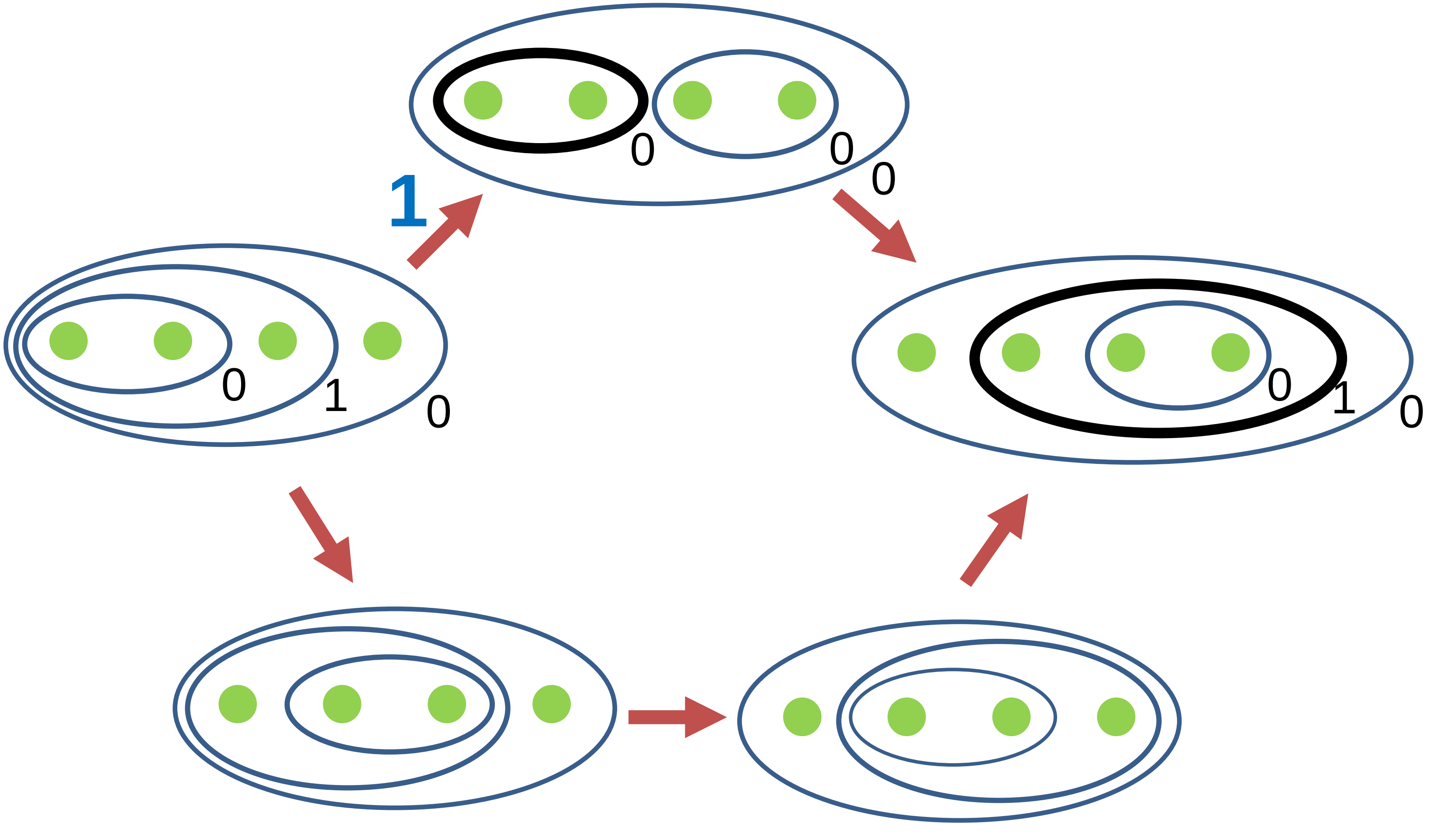
The Pentagon Equation



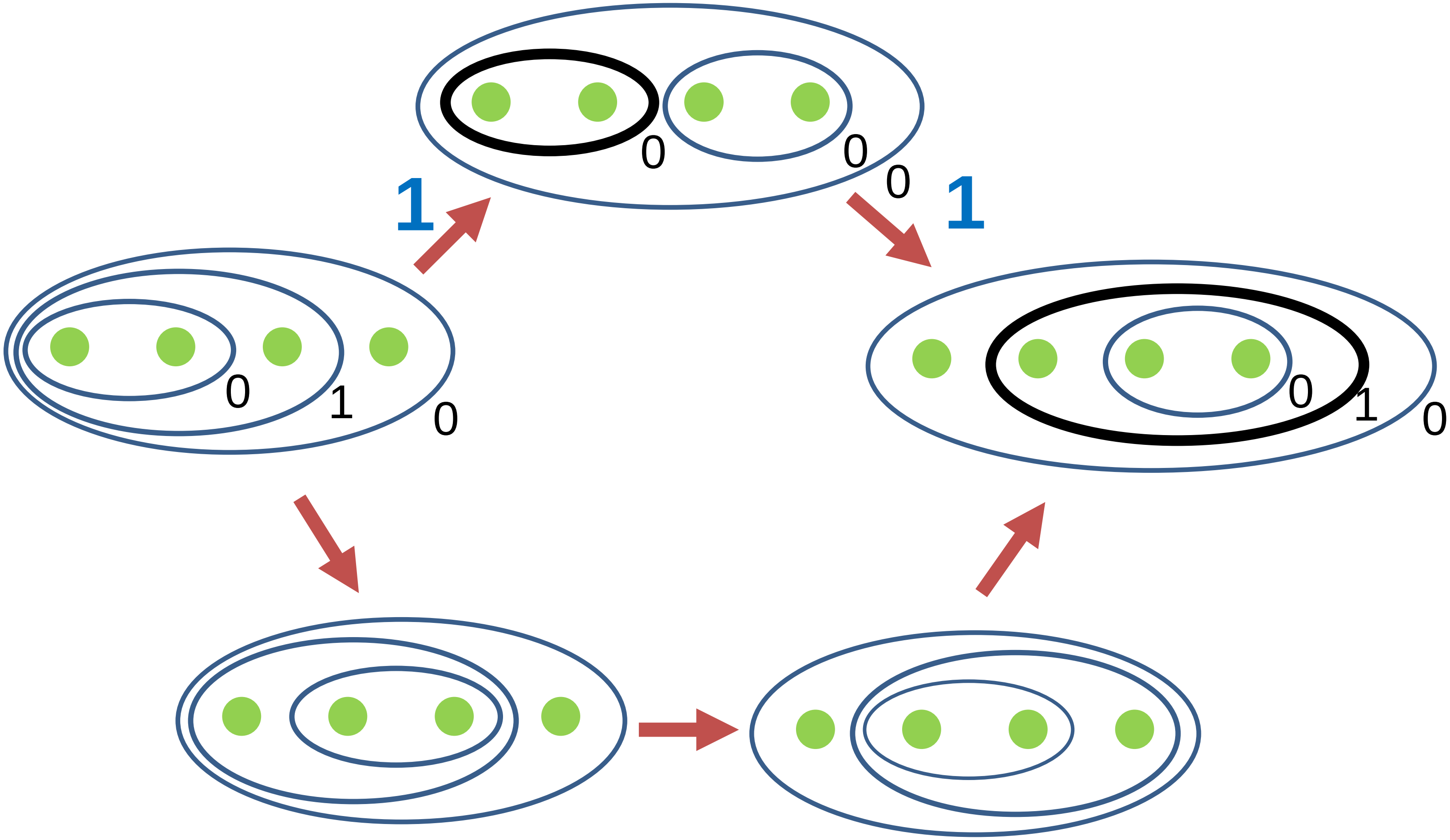
The Pentagon Equation



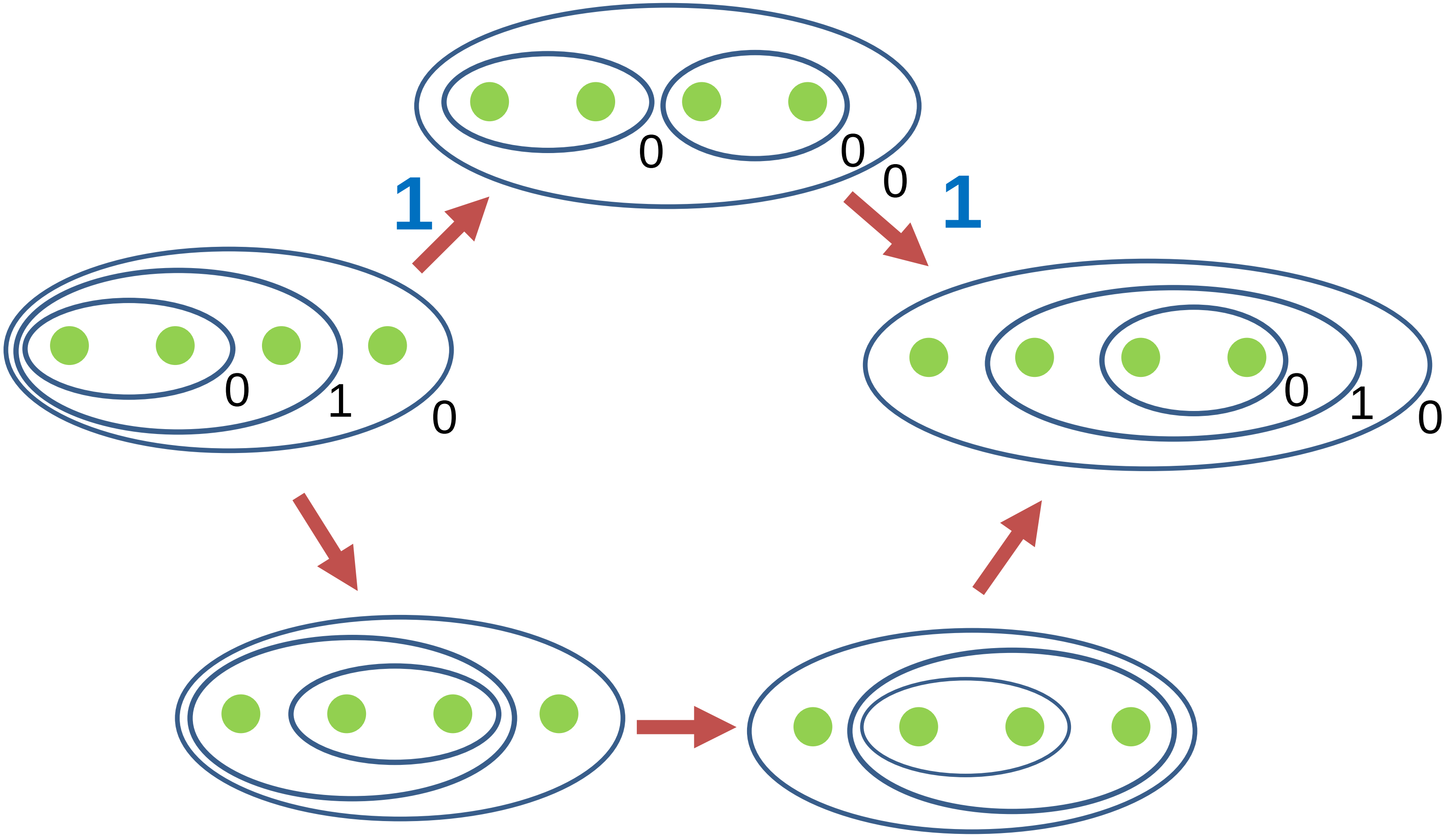
The Pentagon Equation



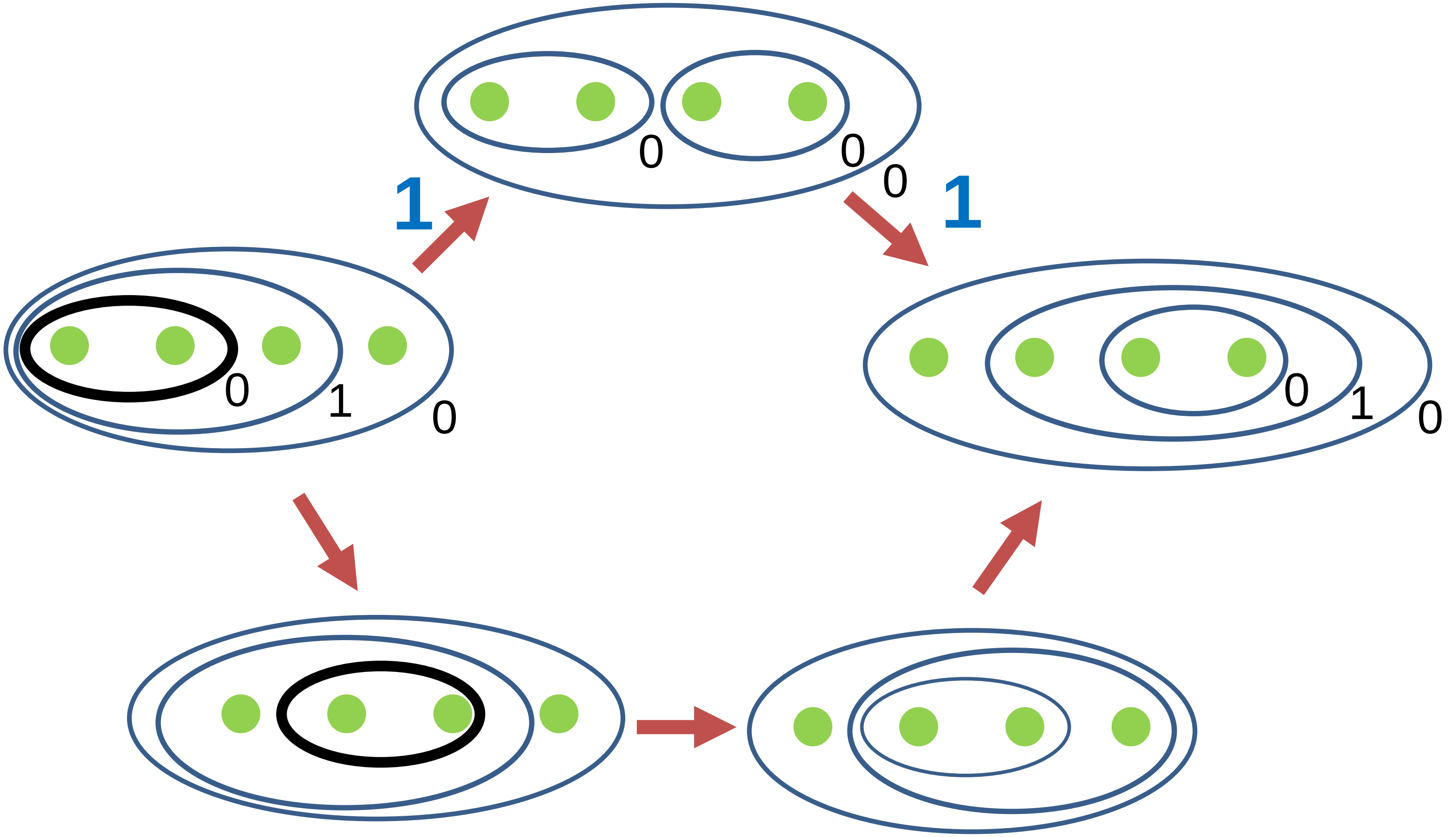
The Pentagon Equation



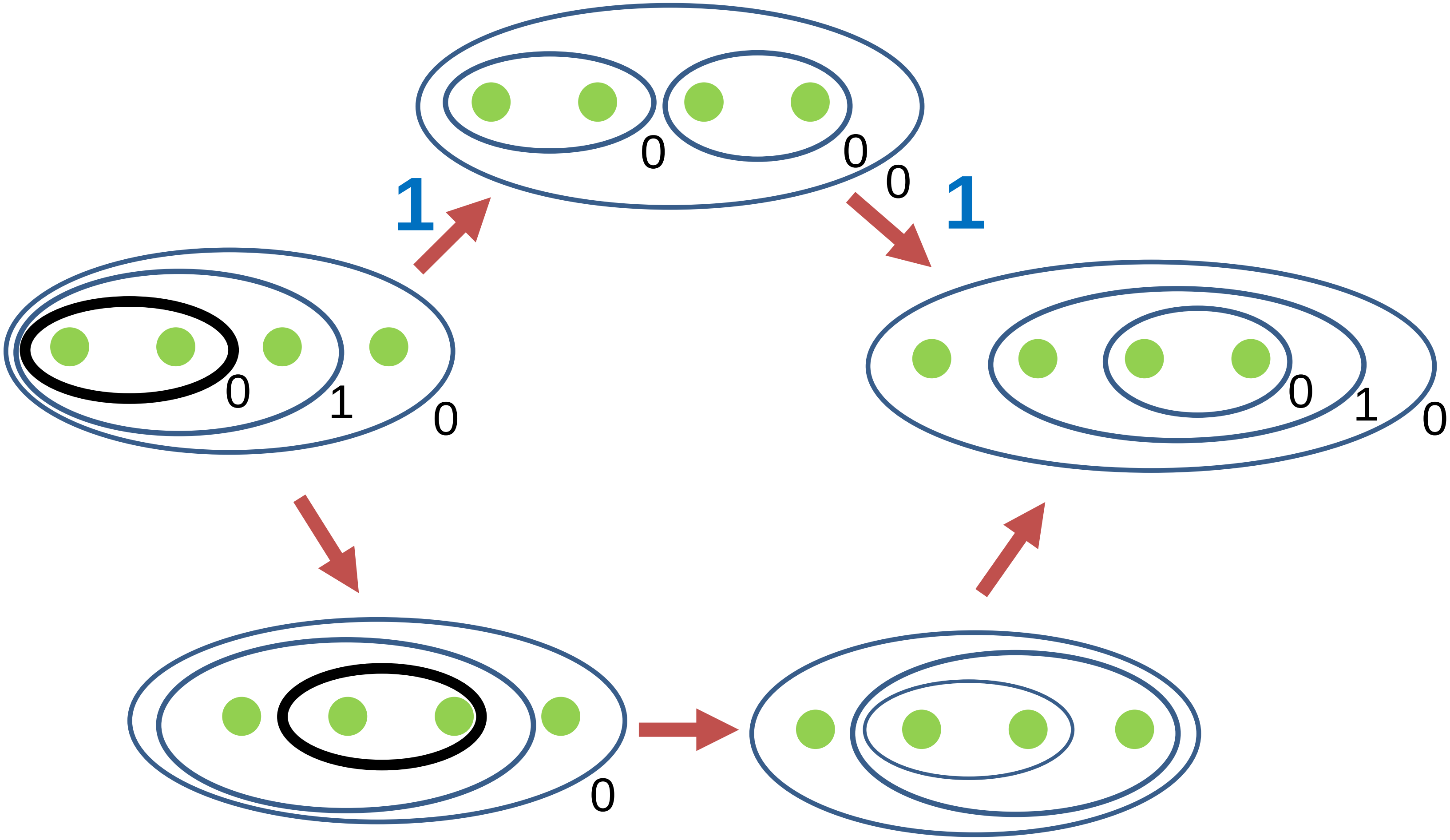
The Pentagon Equation



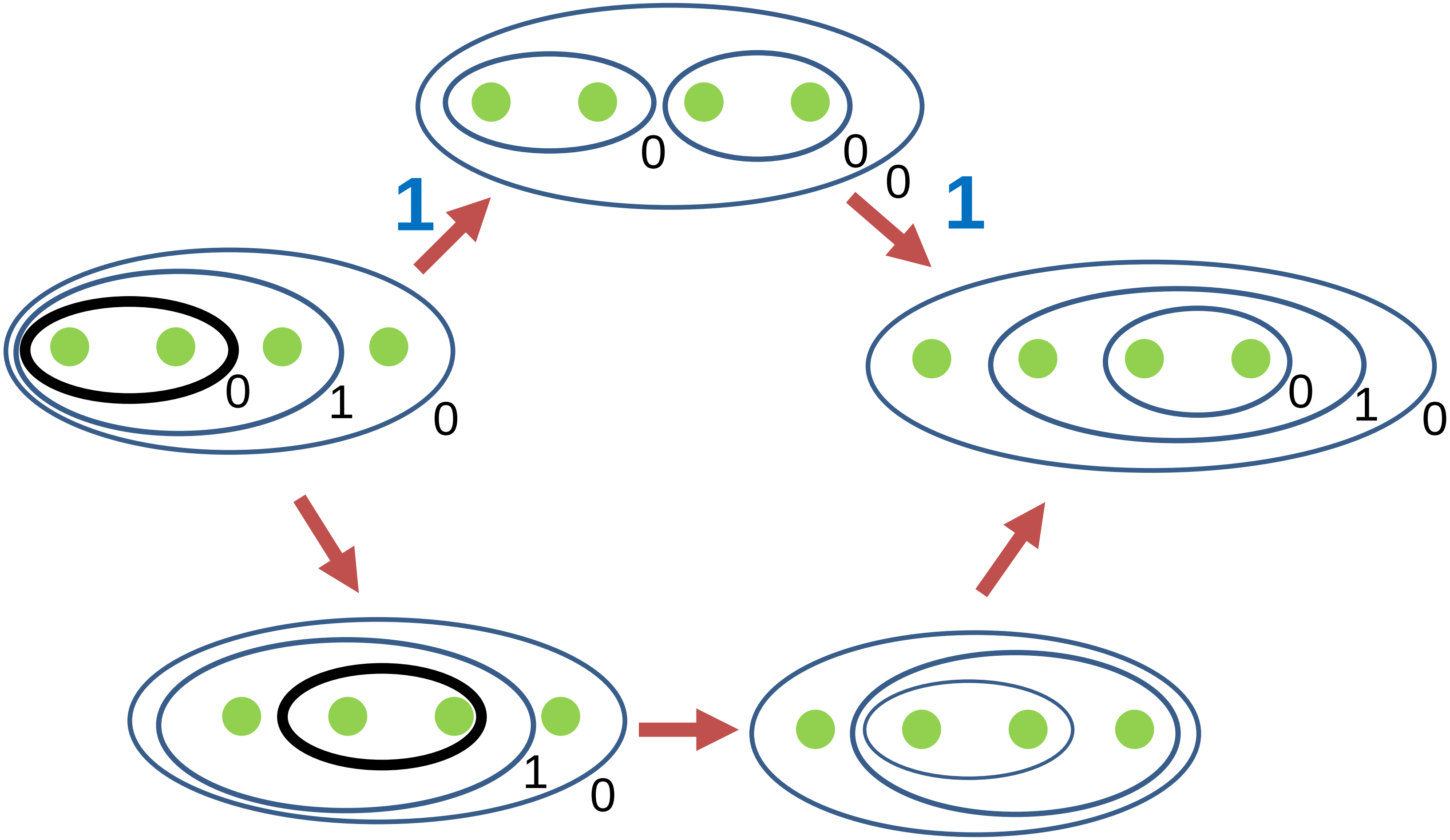
The Pentagon Equation



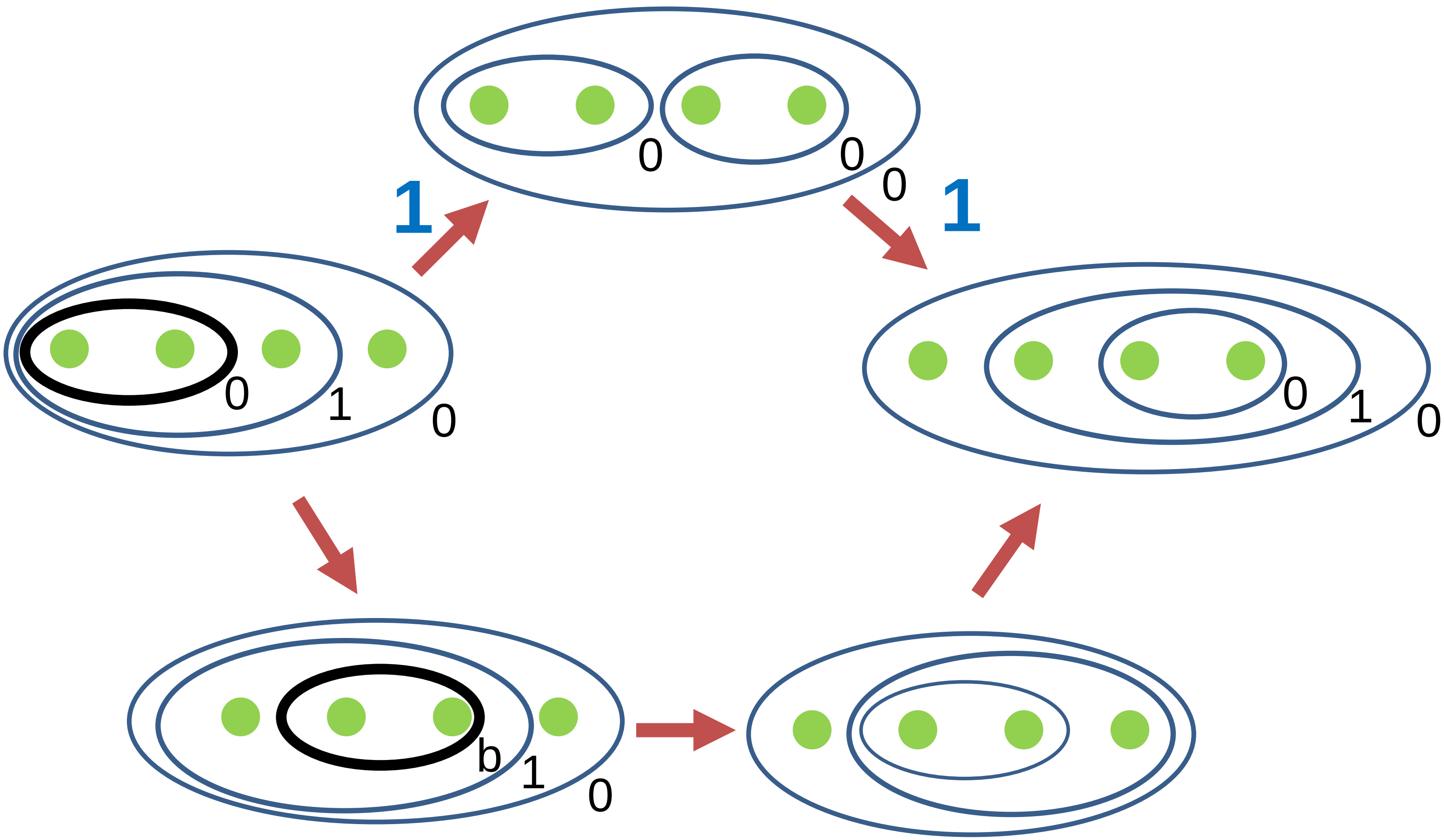
The Pentagon Equation



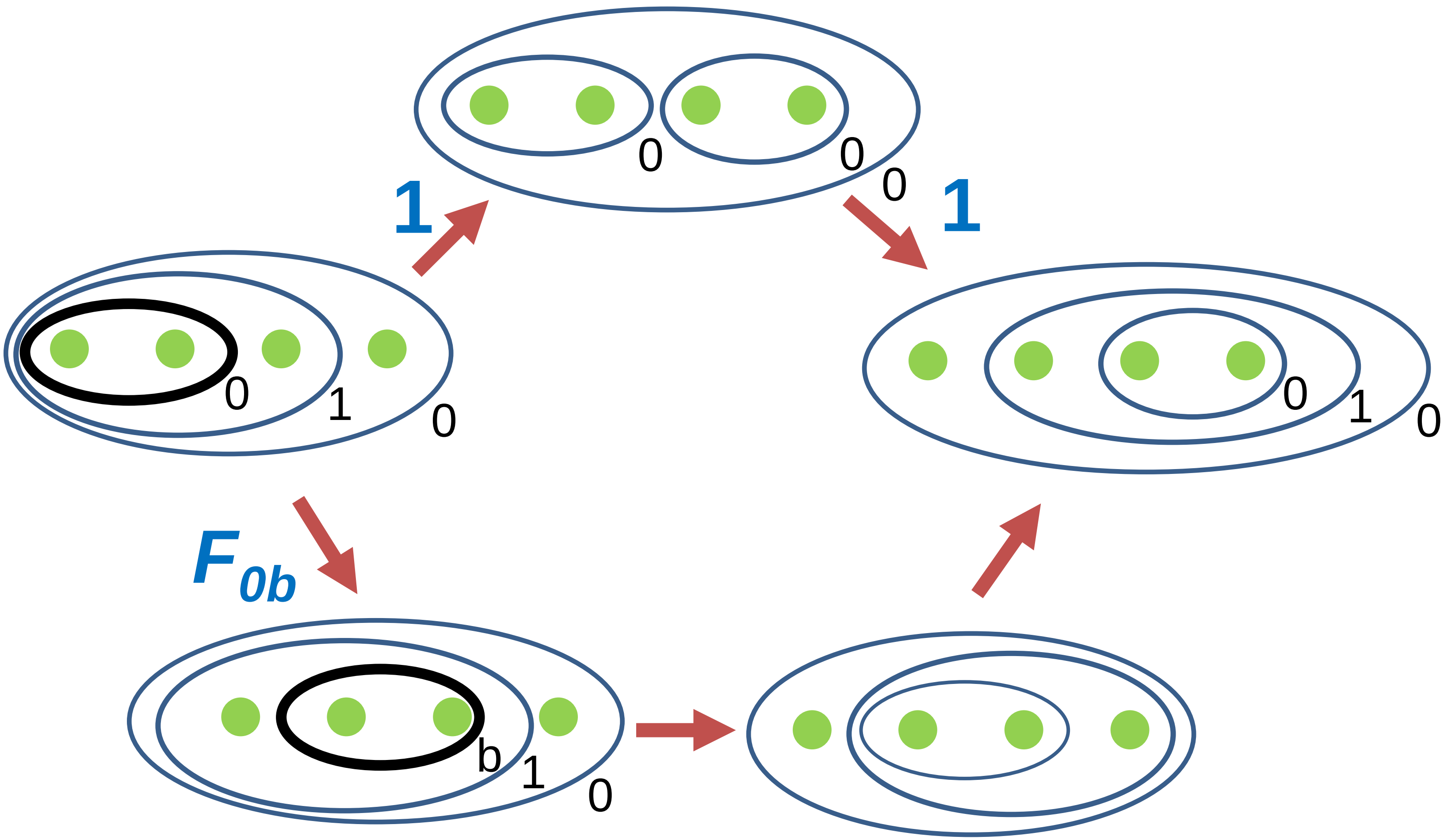
The Pentagon Equation



The Pentagon Equation



The Pentagon Equation



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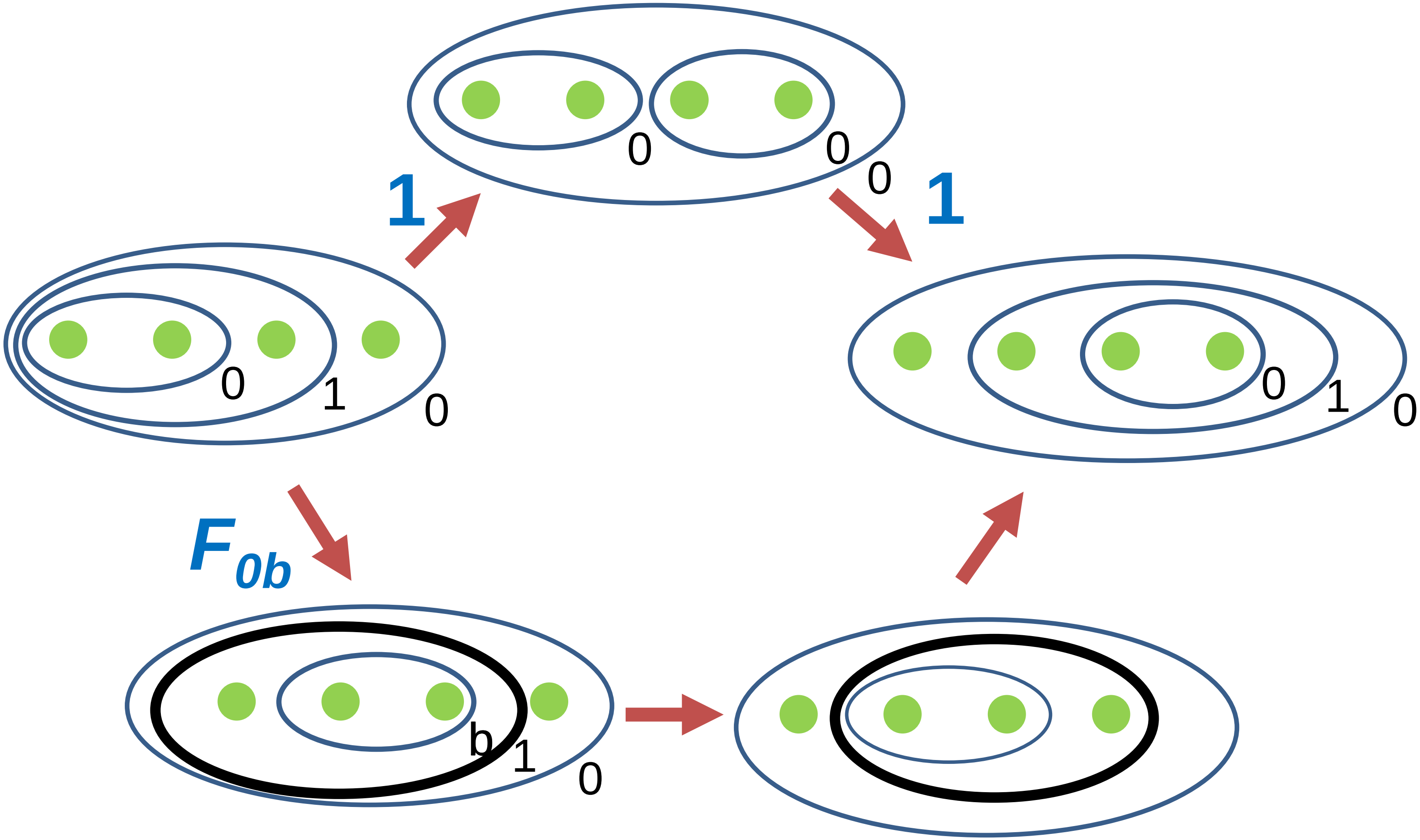

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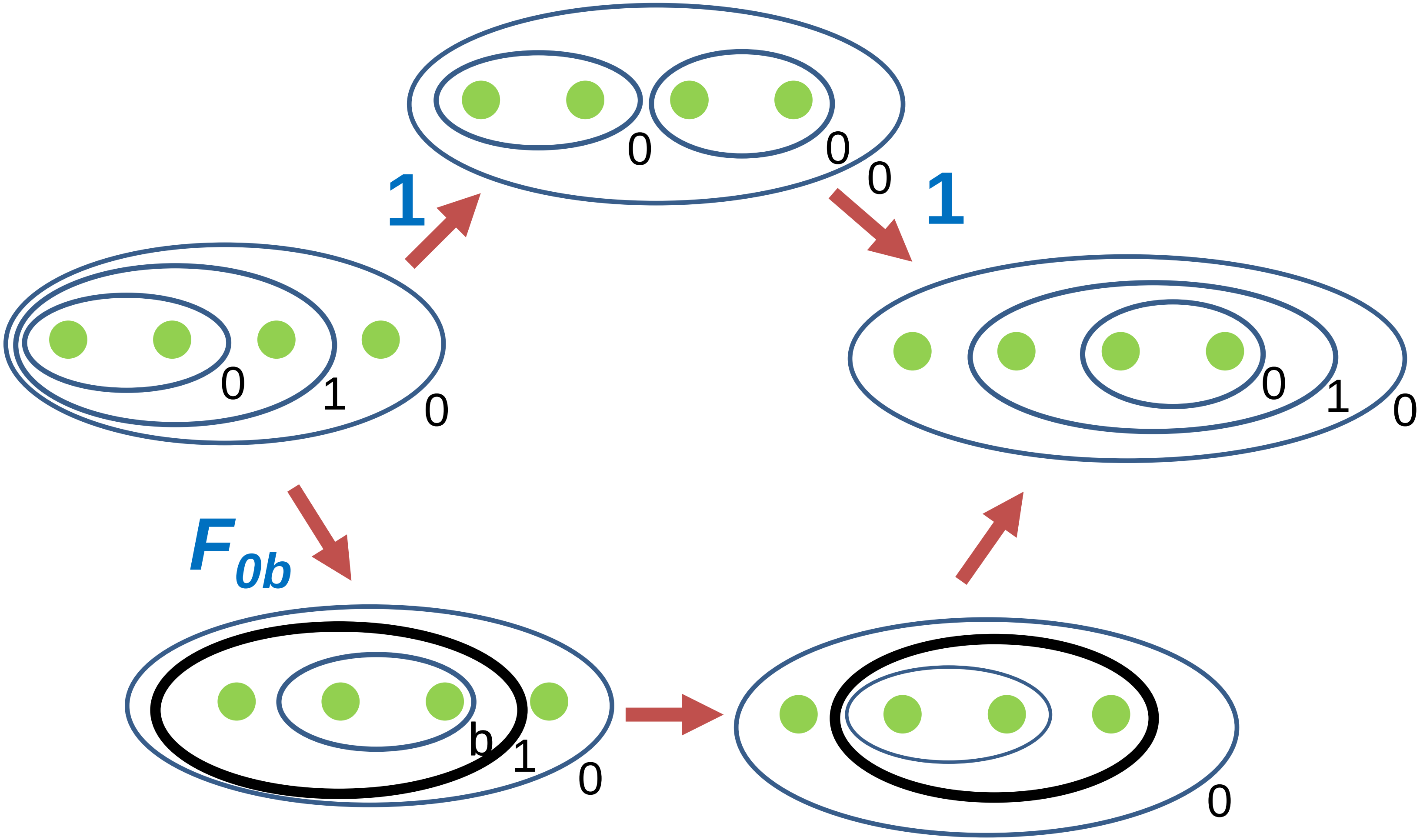

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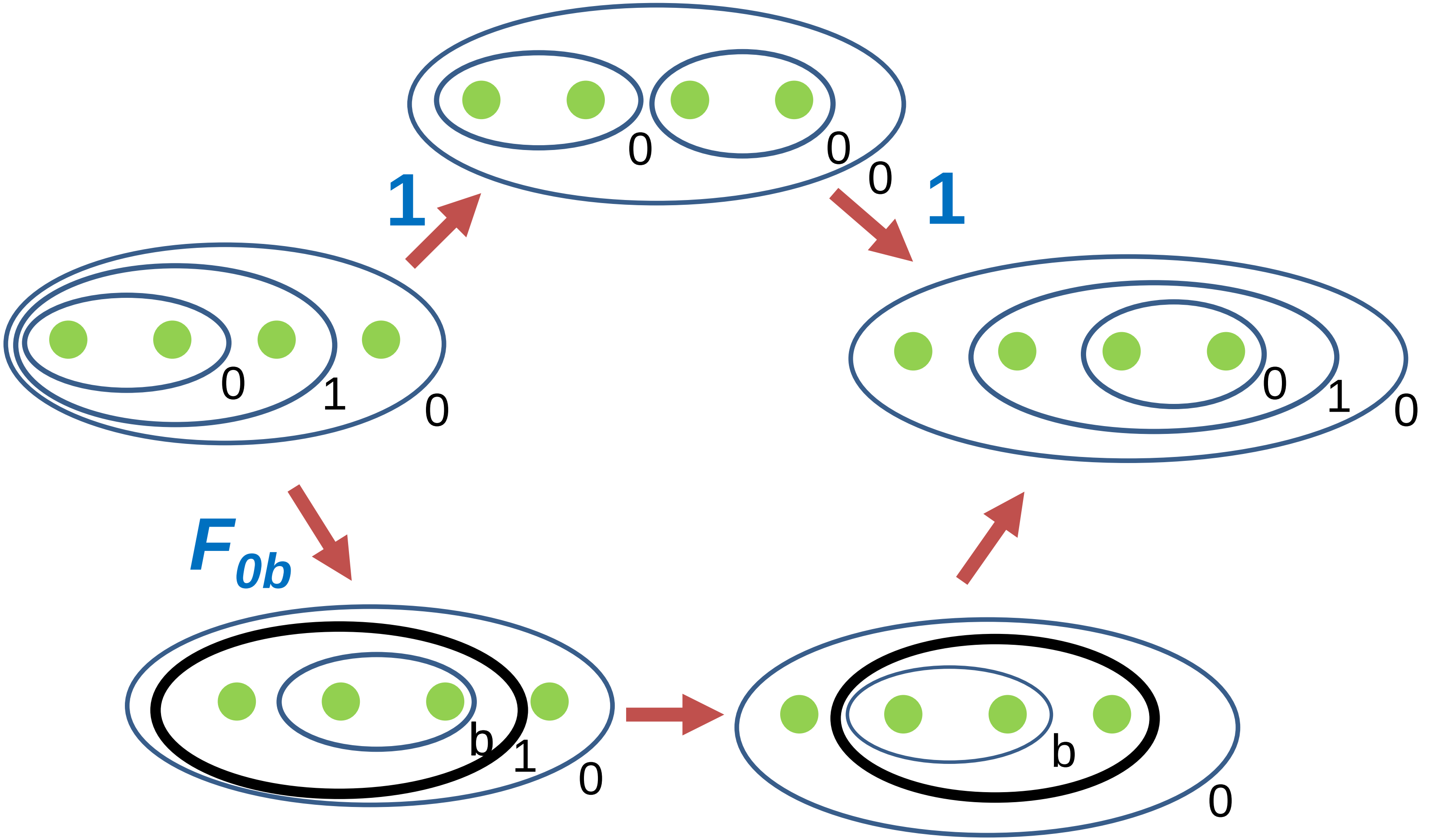

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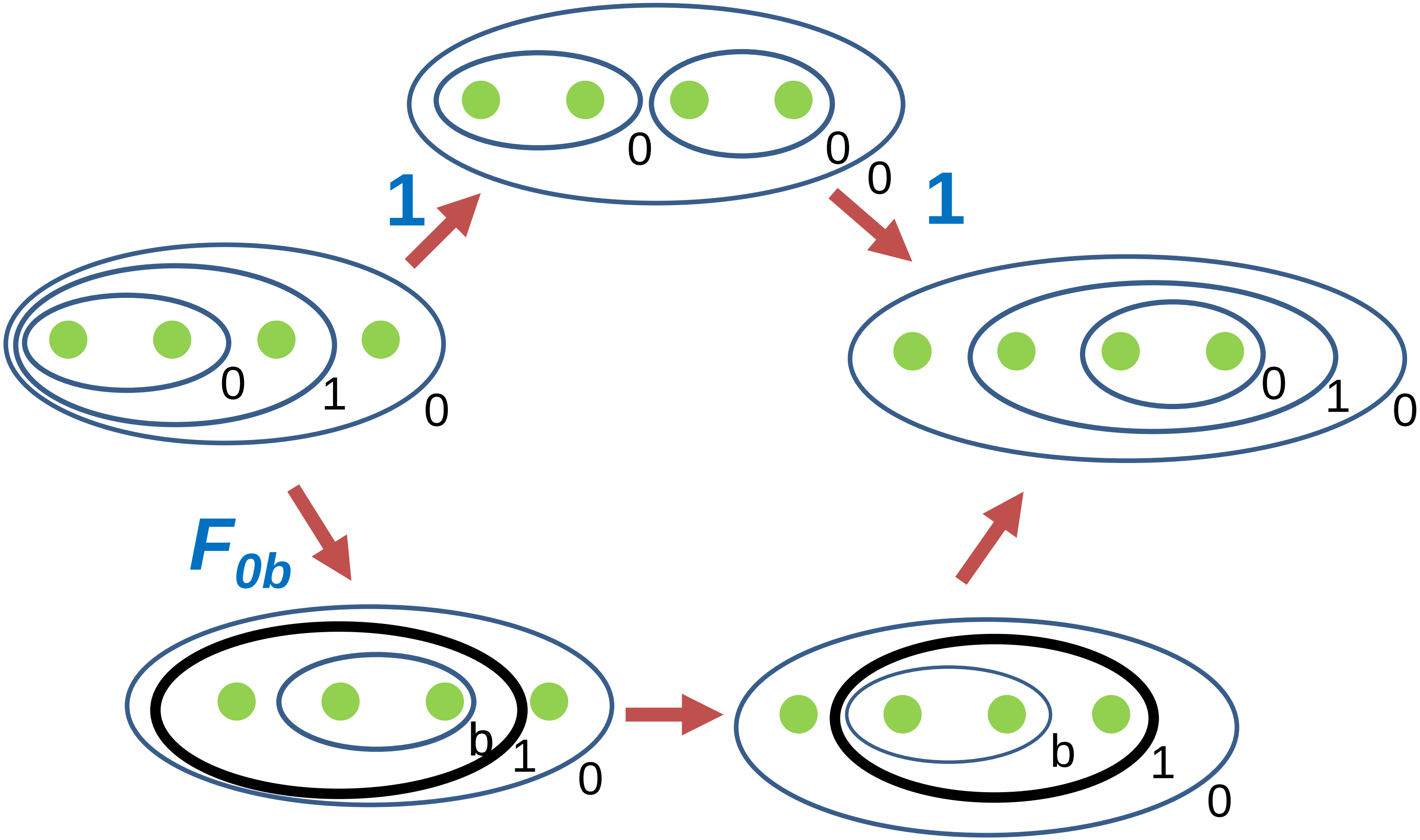

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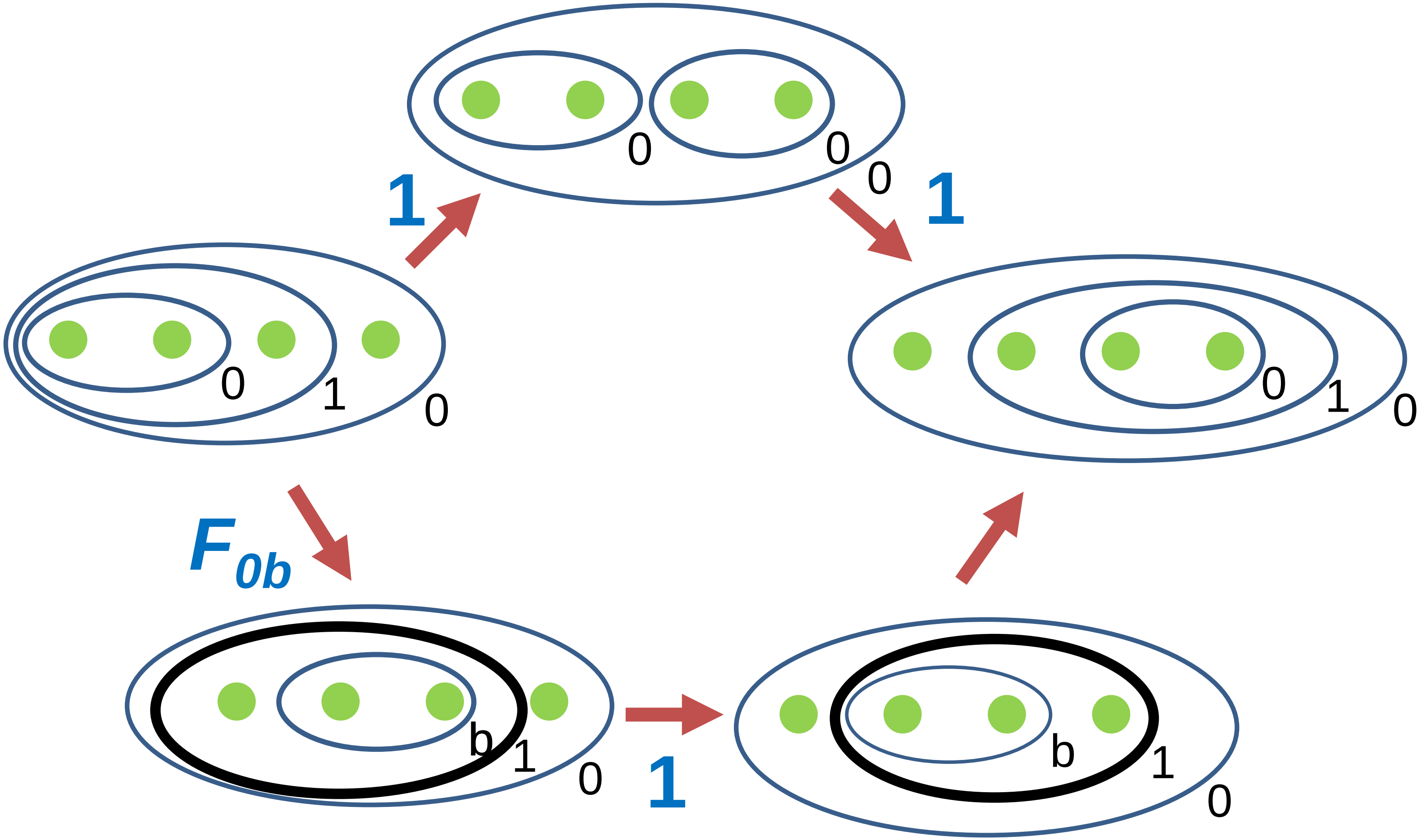
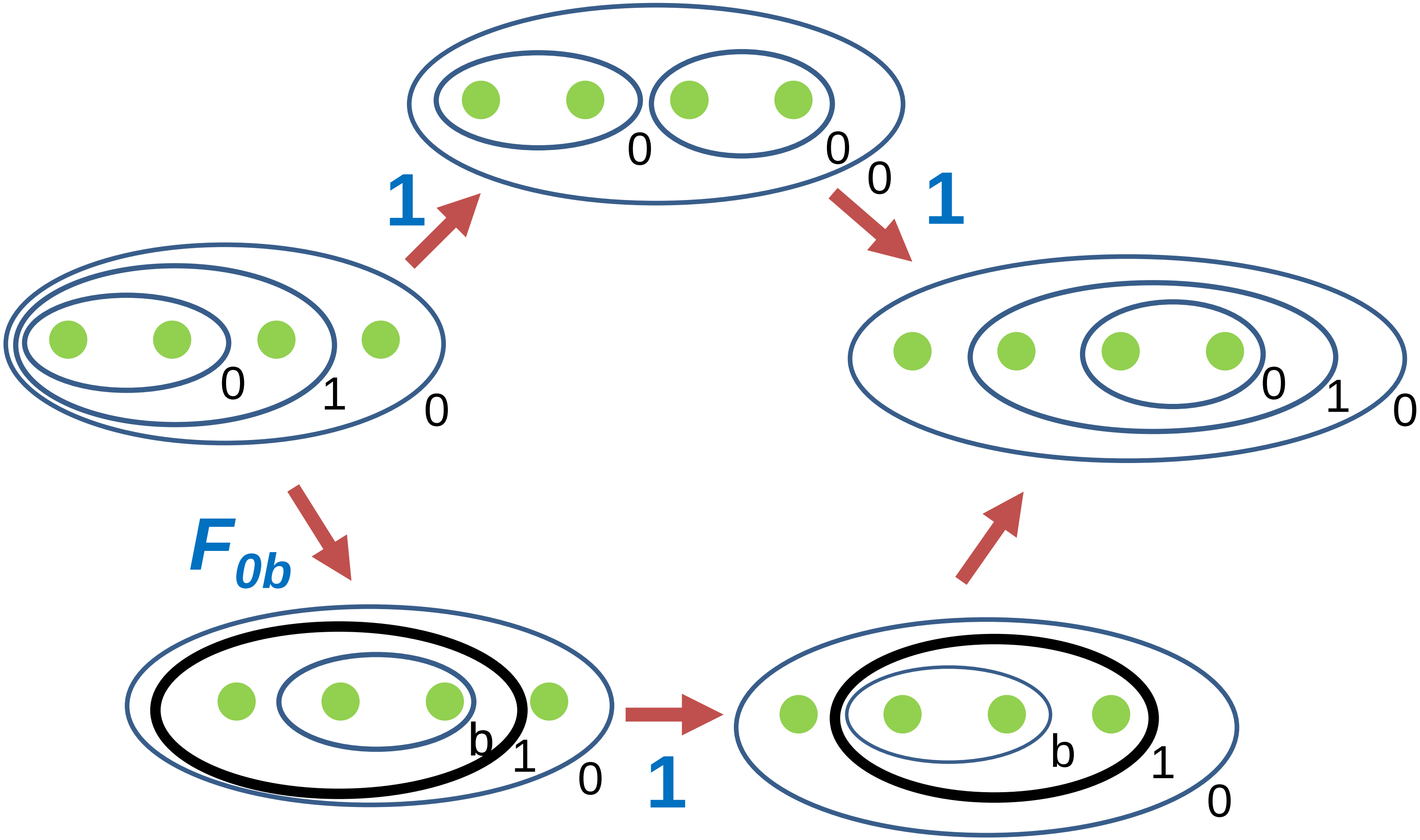
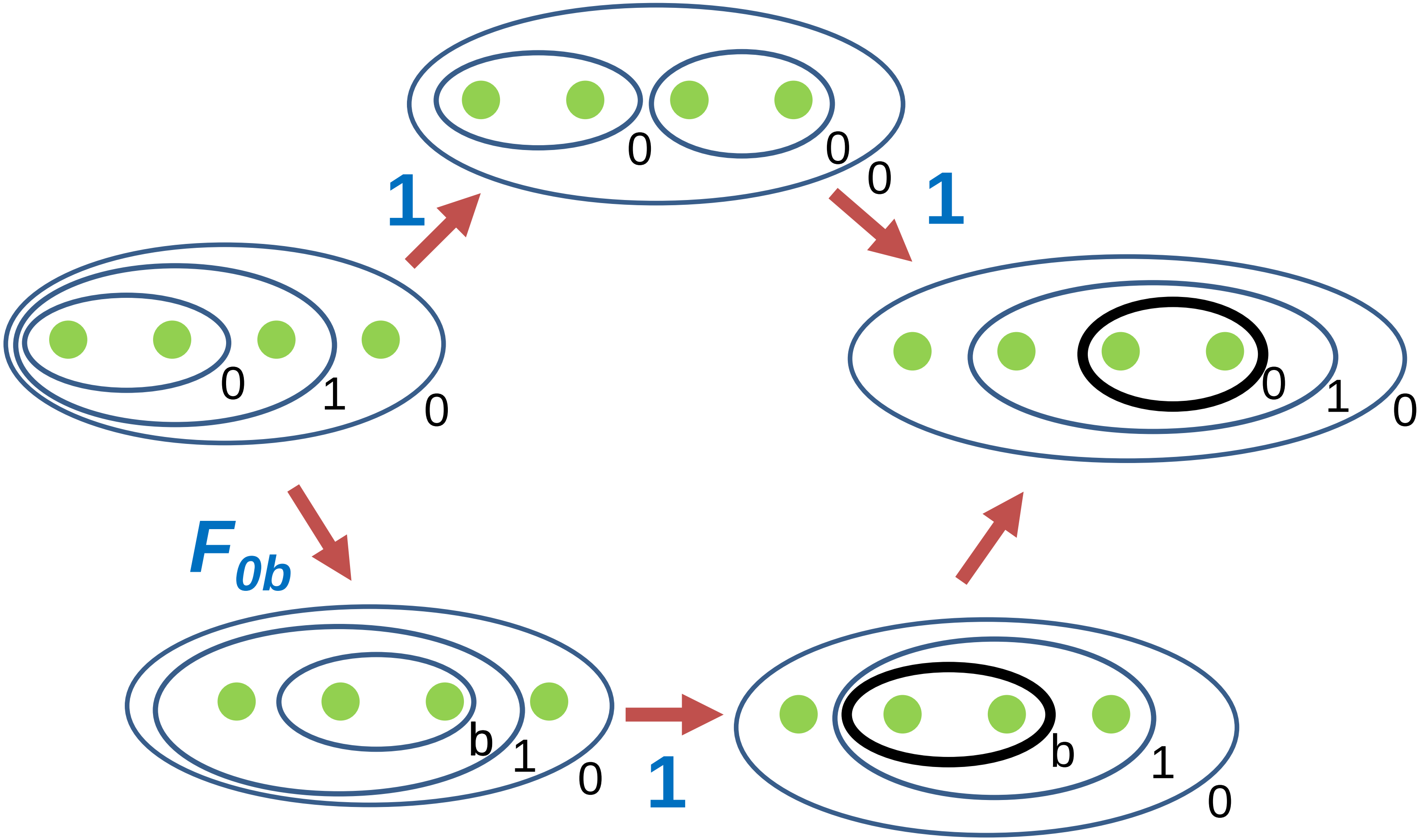


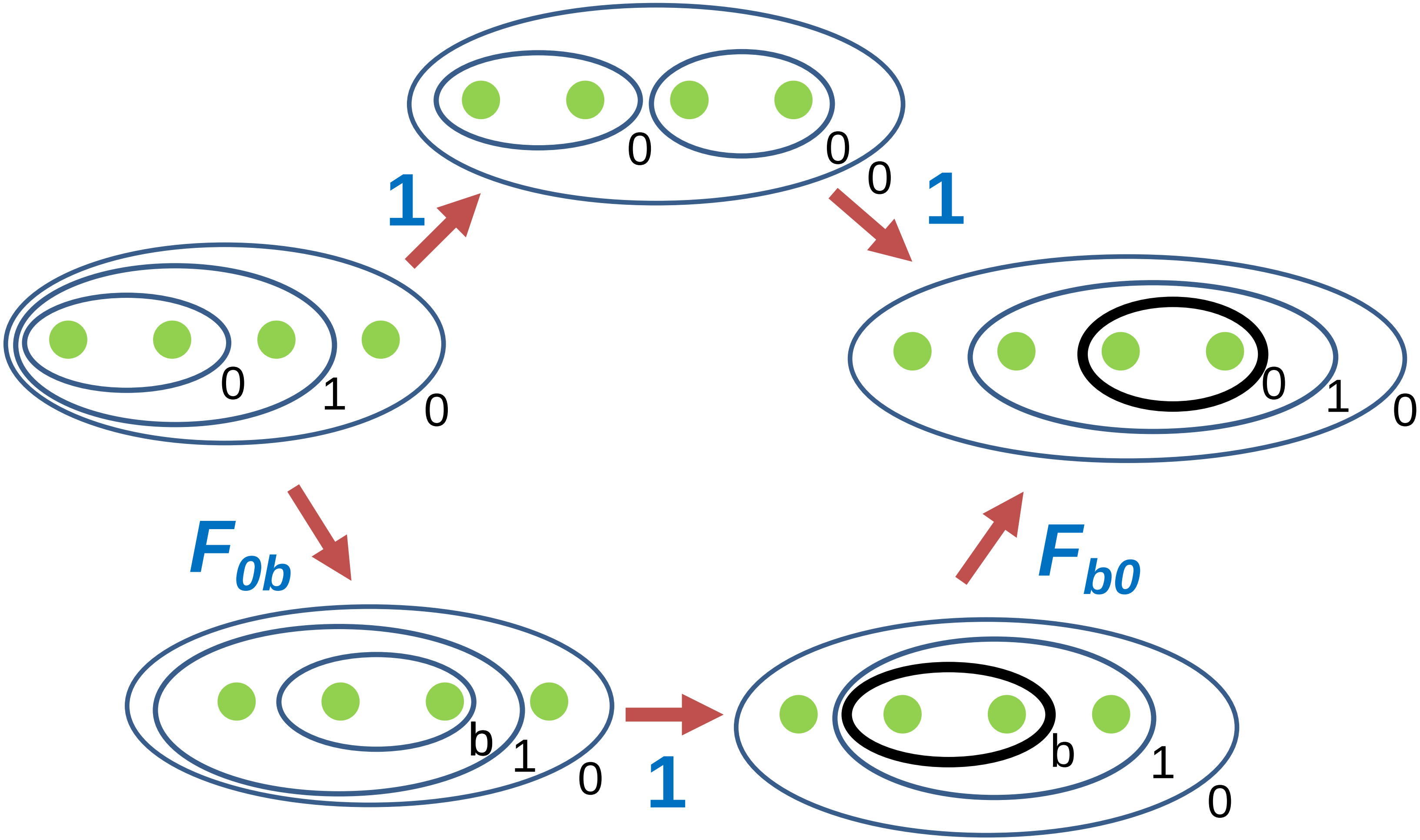
The Pentagon Equation



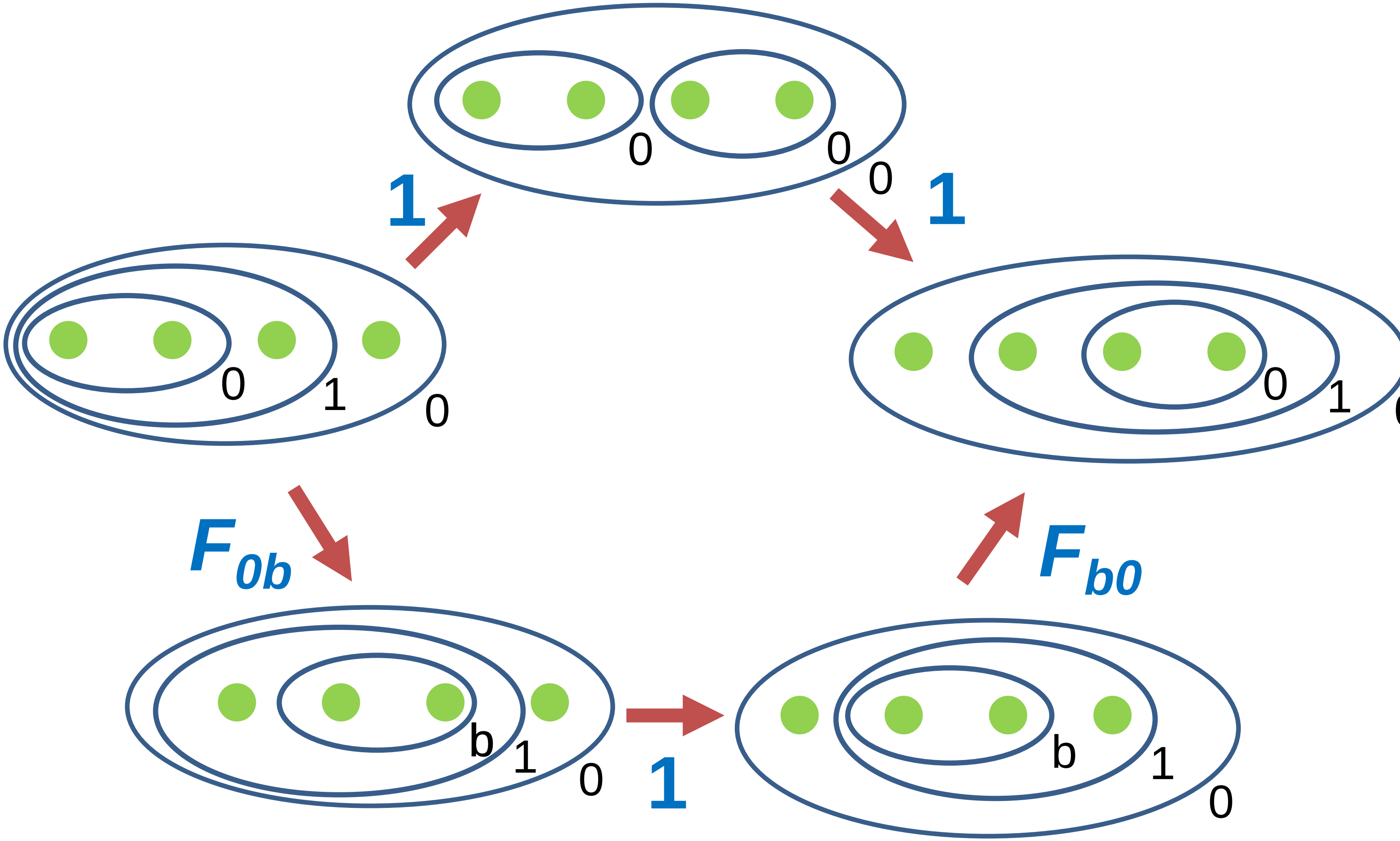
The Pentagon Equation



The Pentagon Equation

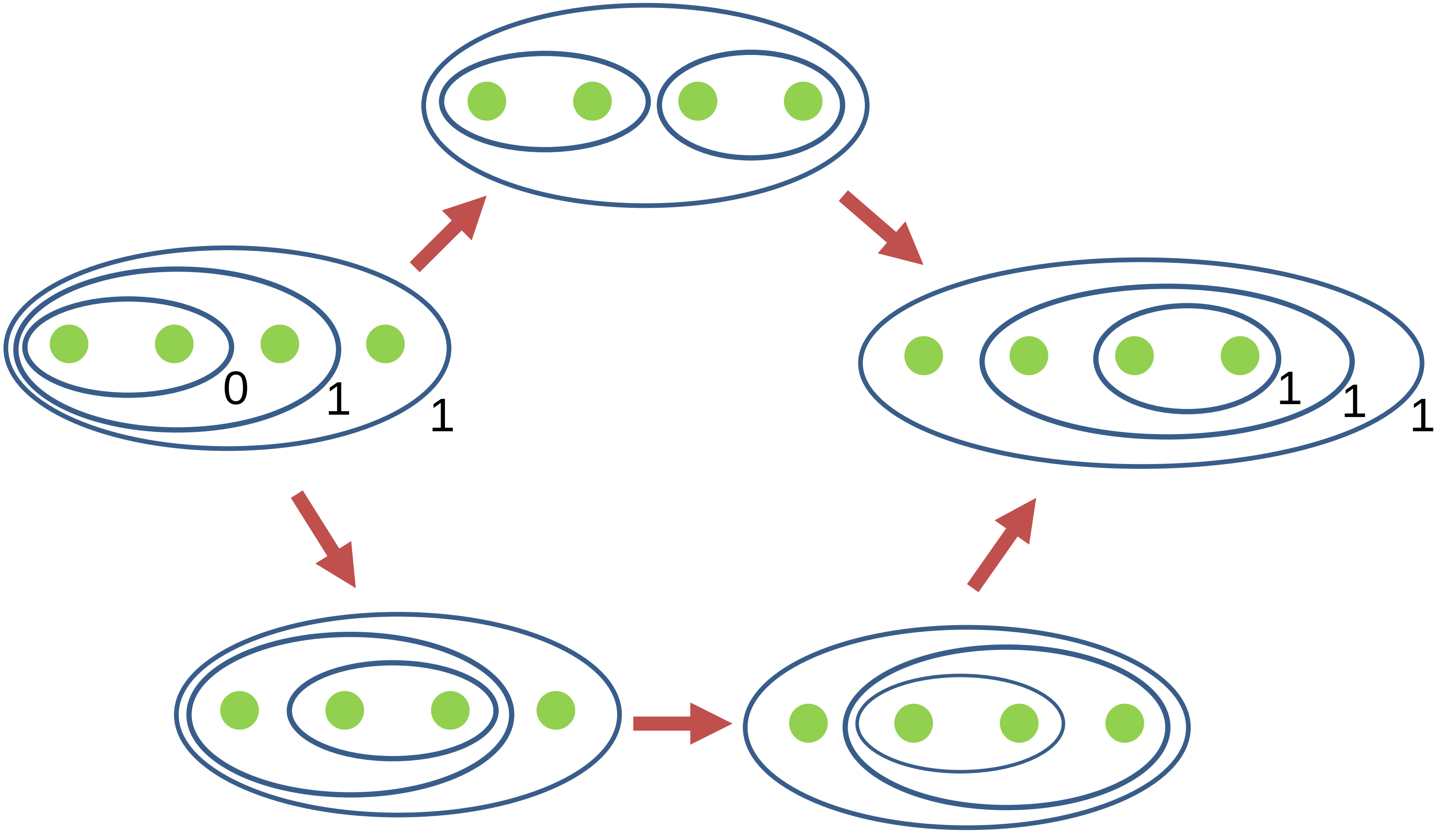


The Pentagon Equation

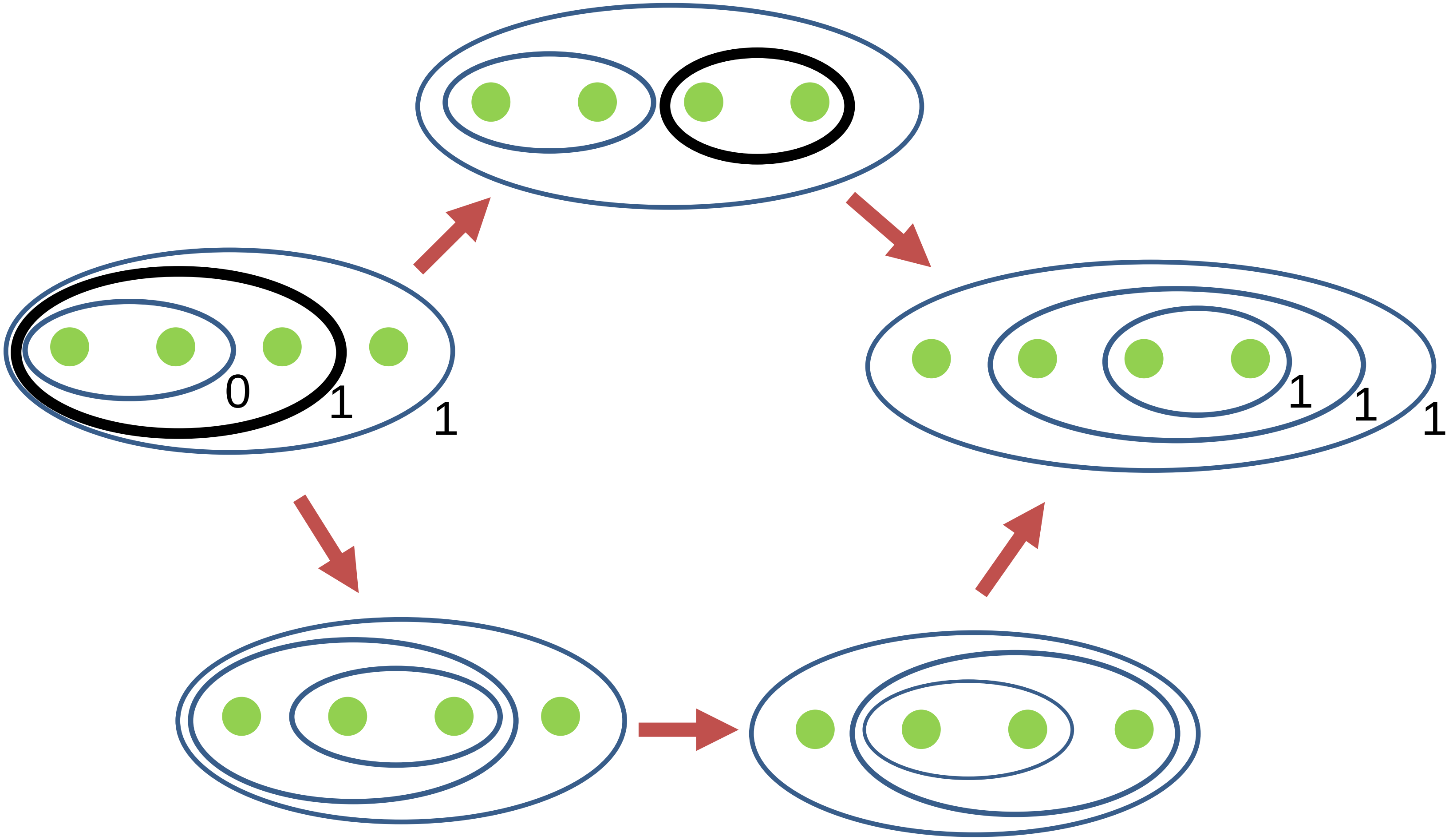


Top path → $$1 = \sum_b F_{0b} F_{b0} = F_{00} F_{00} + F_{01} F_{10}$$ ← Bottom path

The Pentagon Equation



The Pentagon Equation



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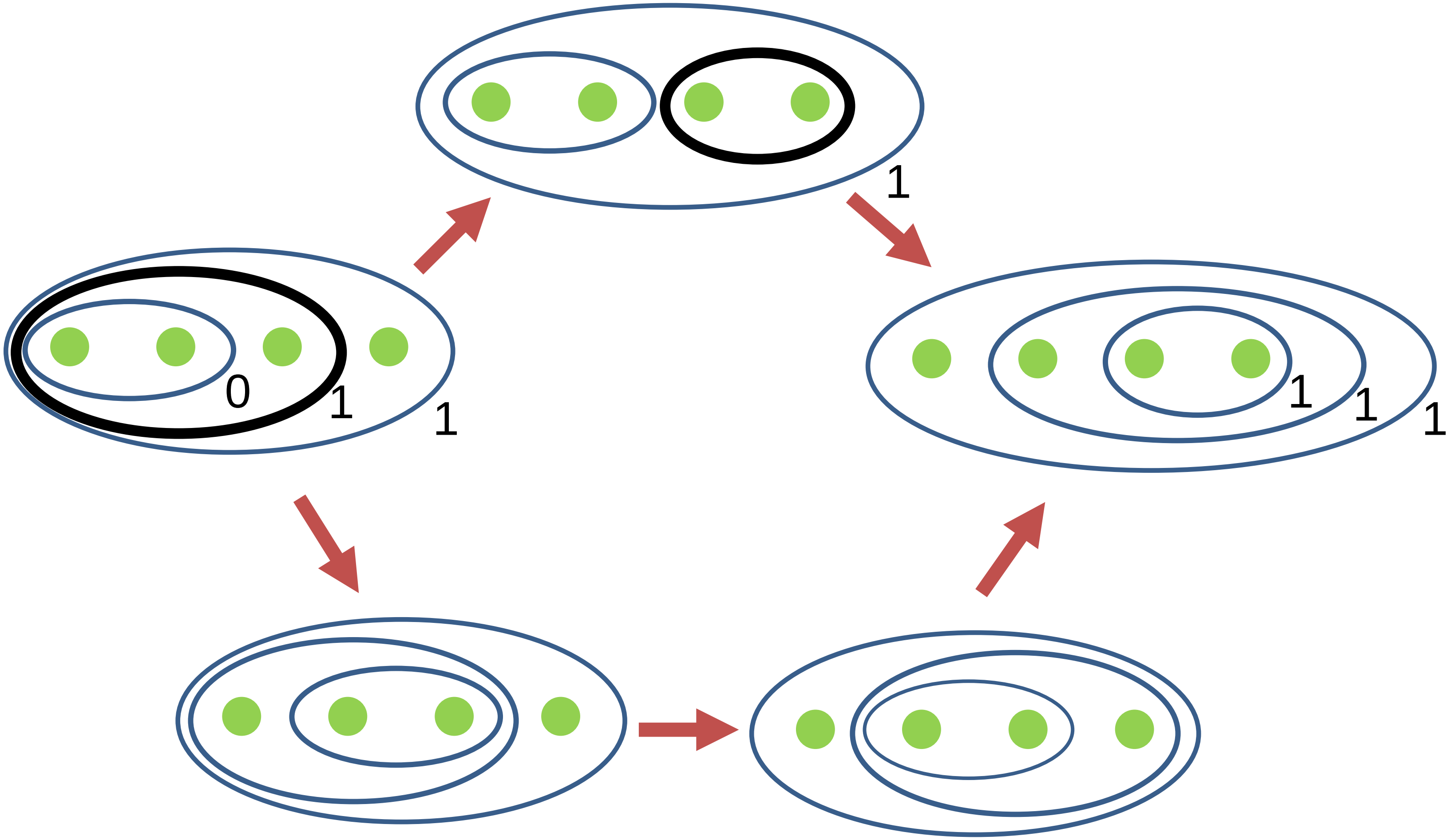
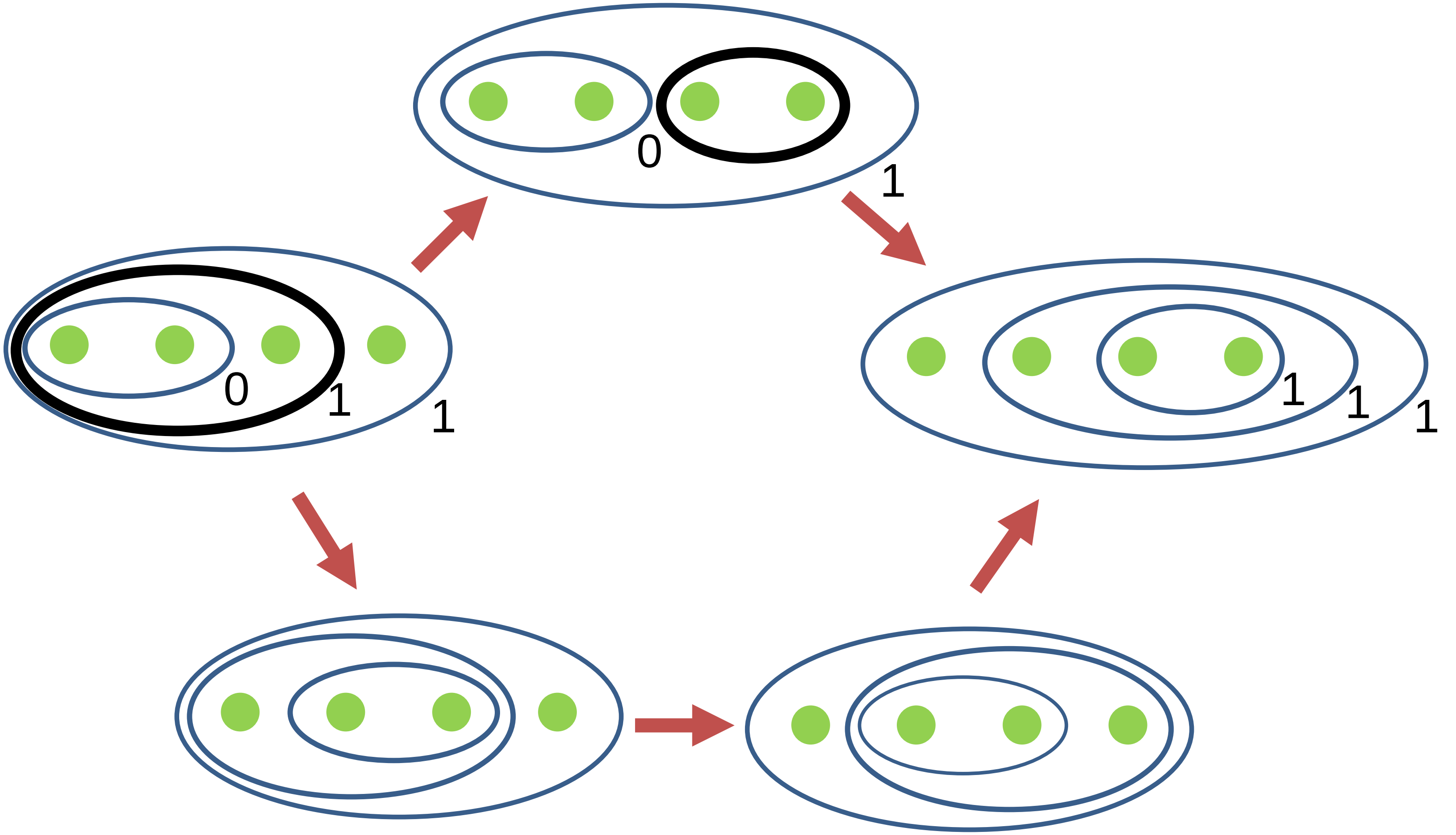
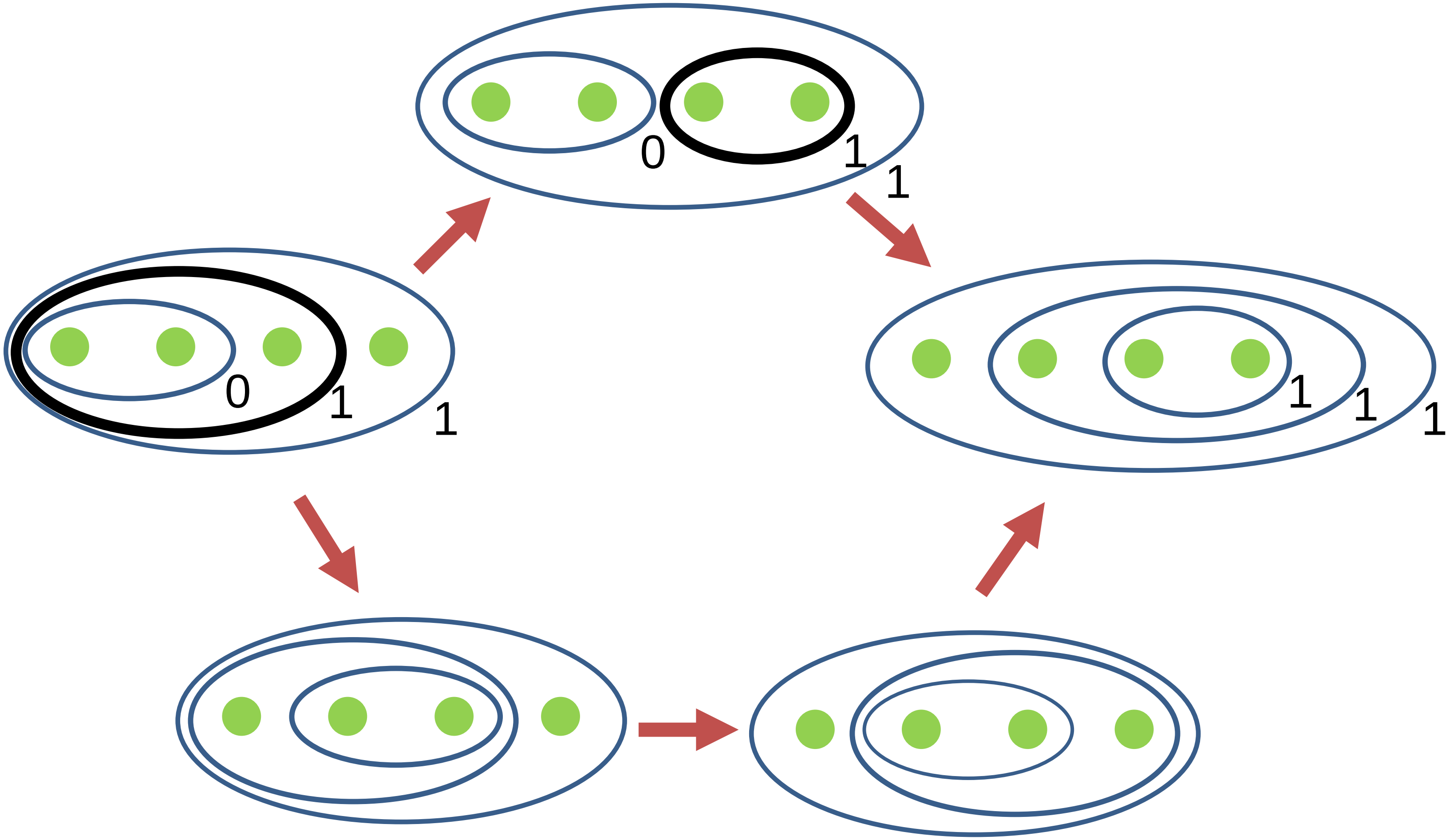


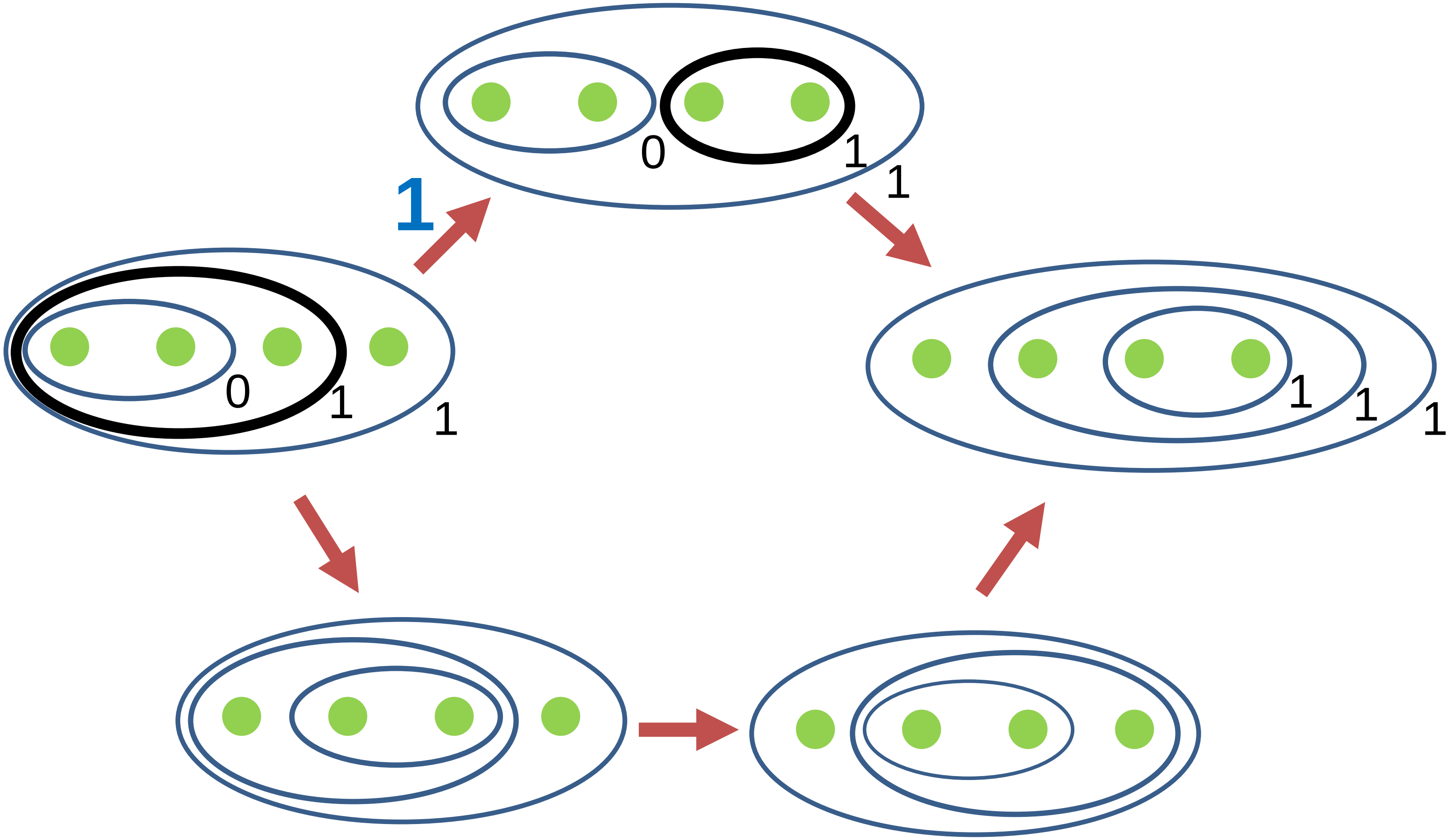
The Pentagon Equation



The Pentagon Equation



The Pentagon Equation



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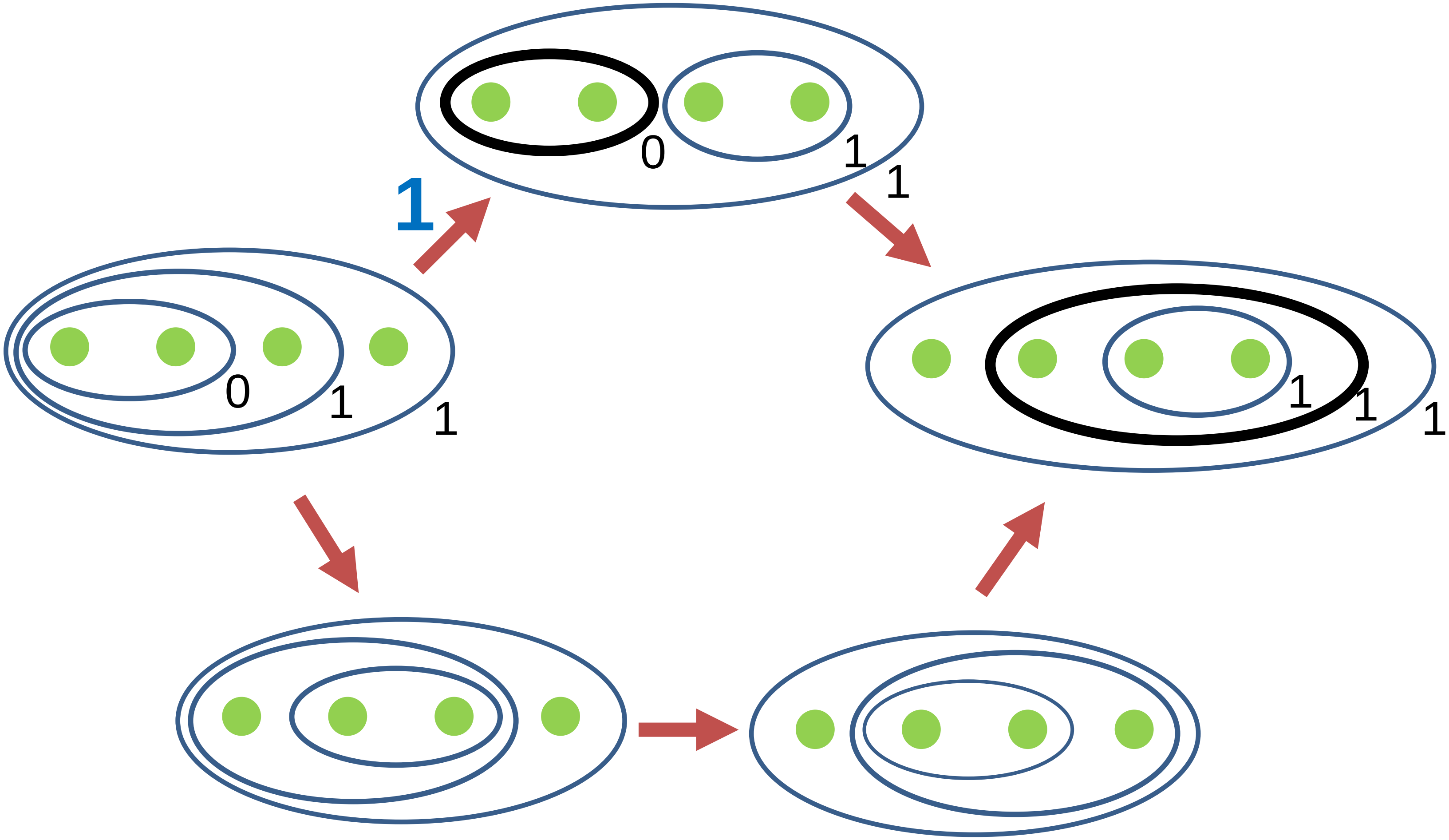

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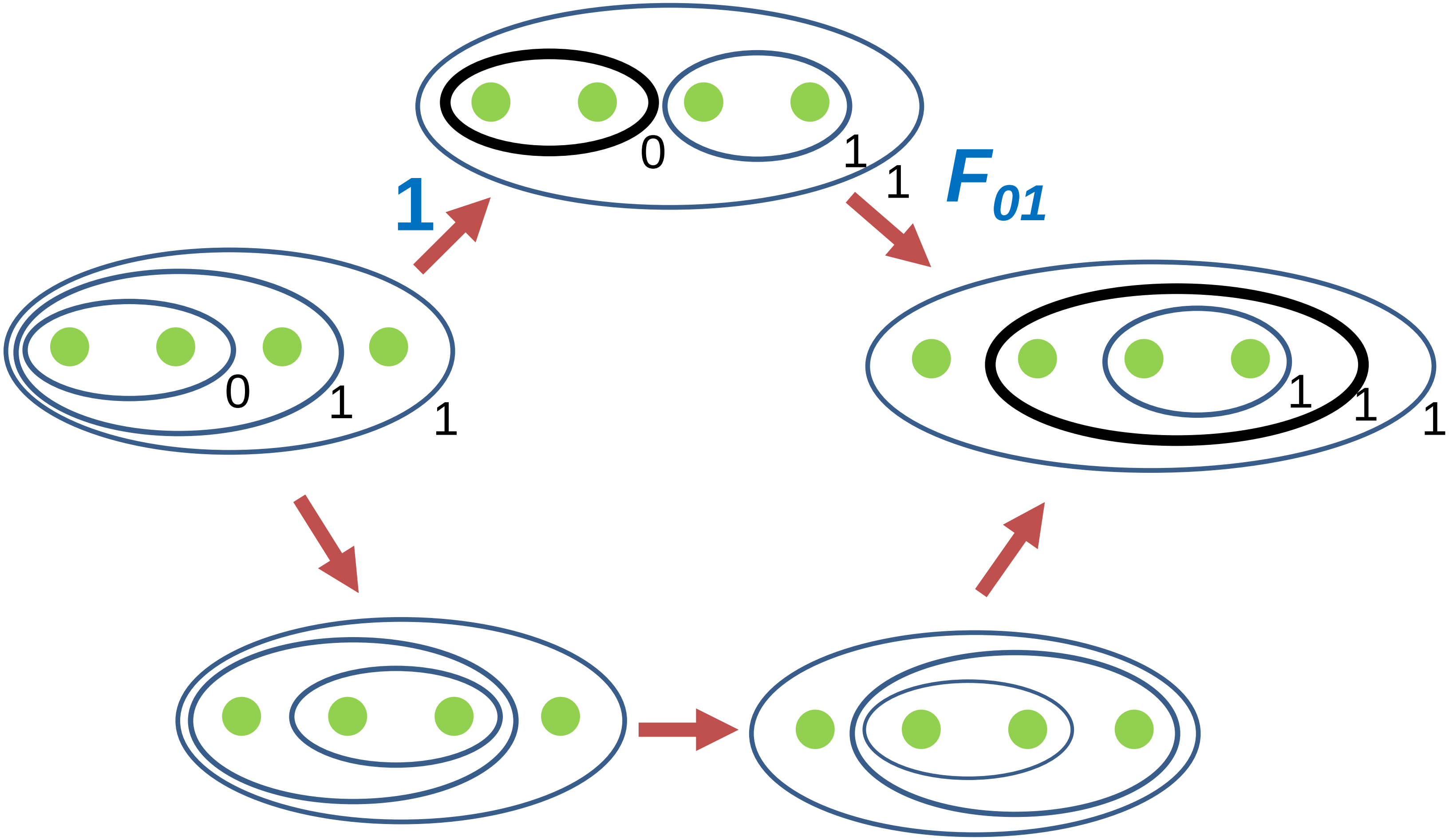
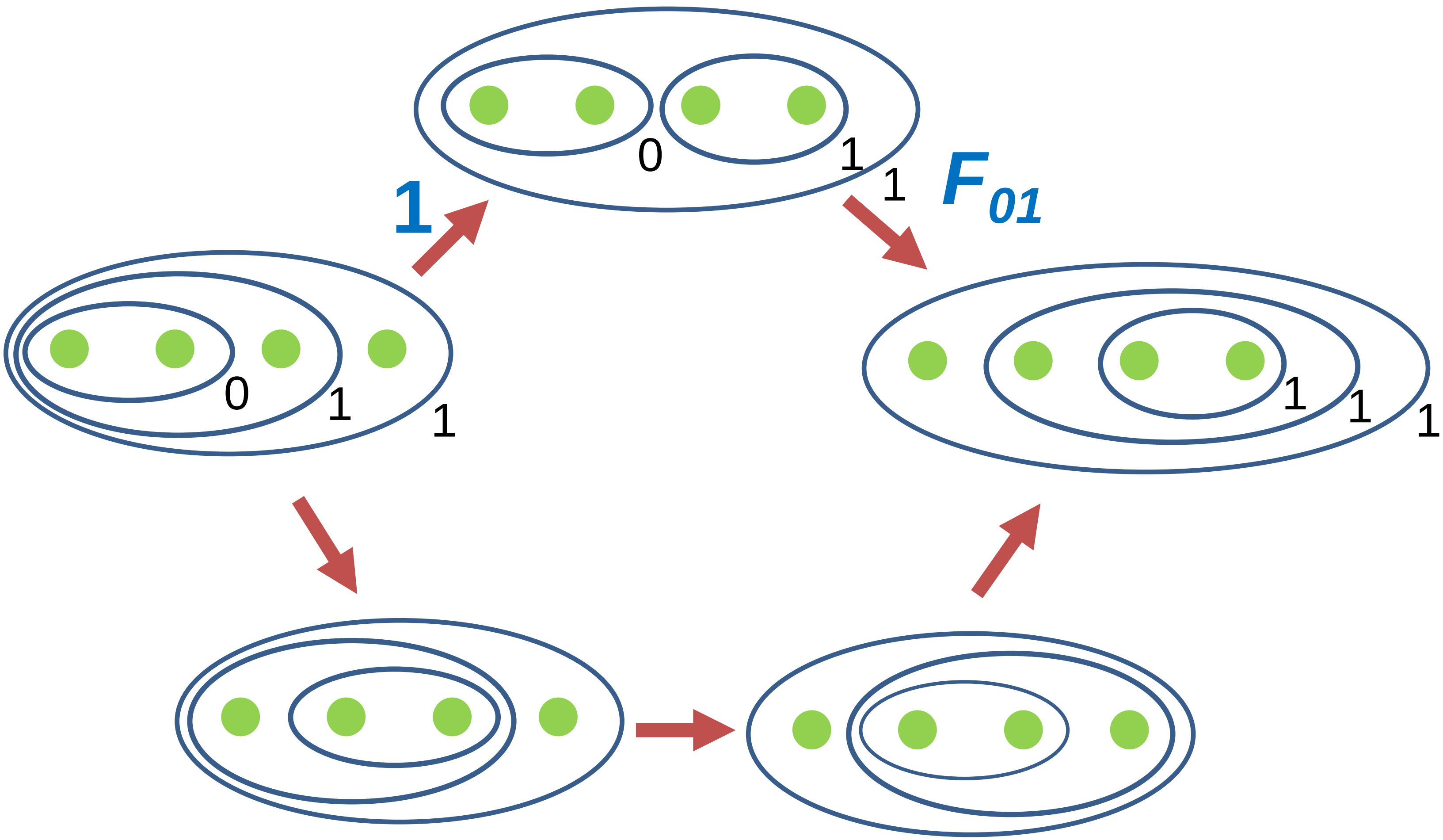
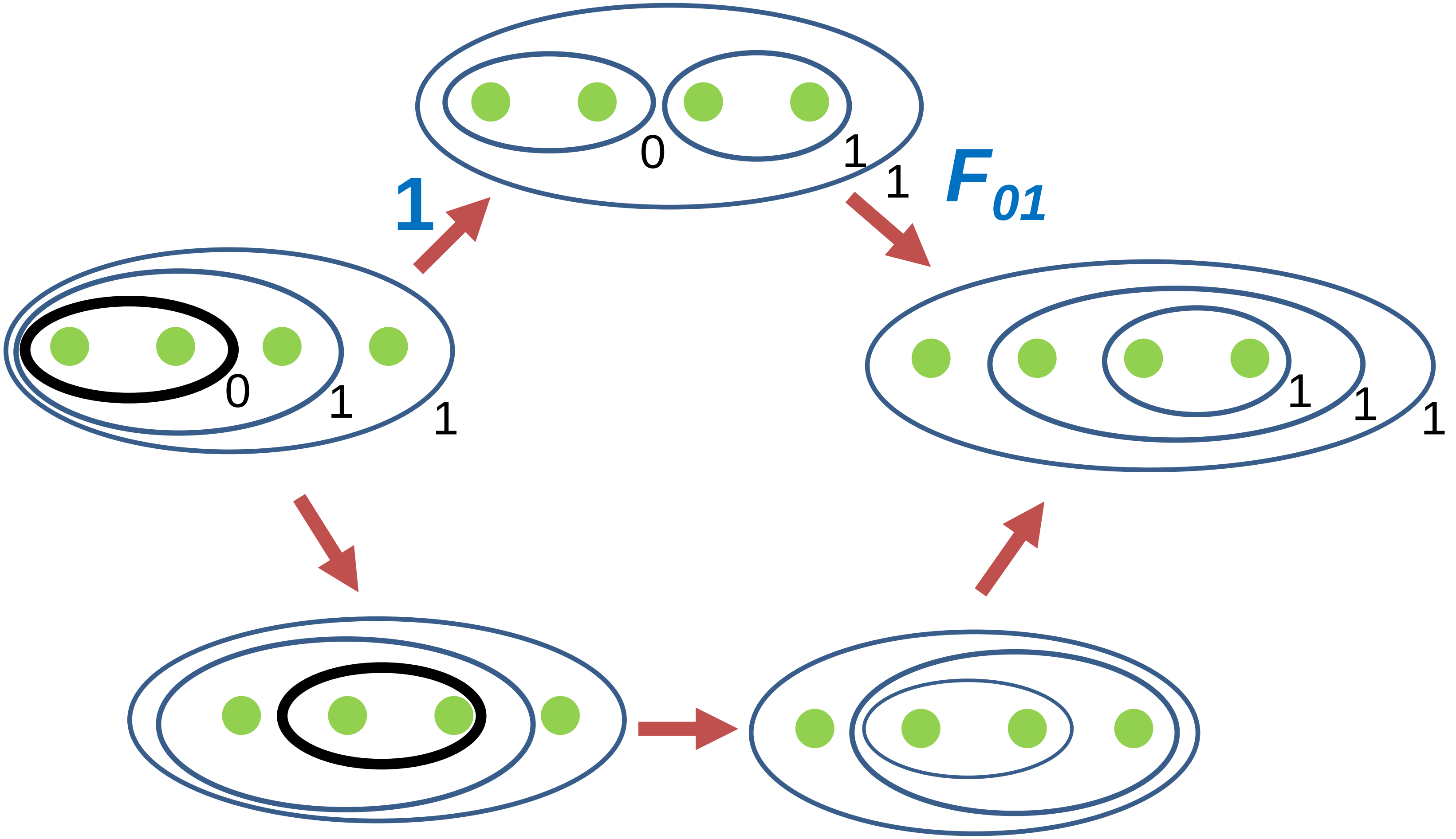


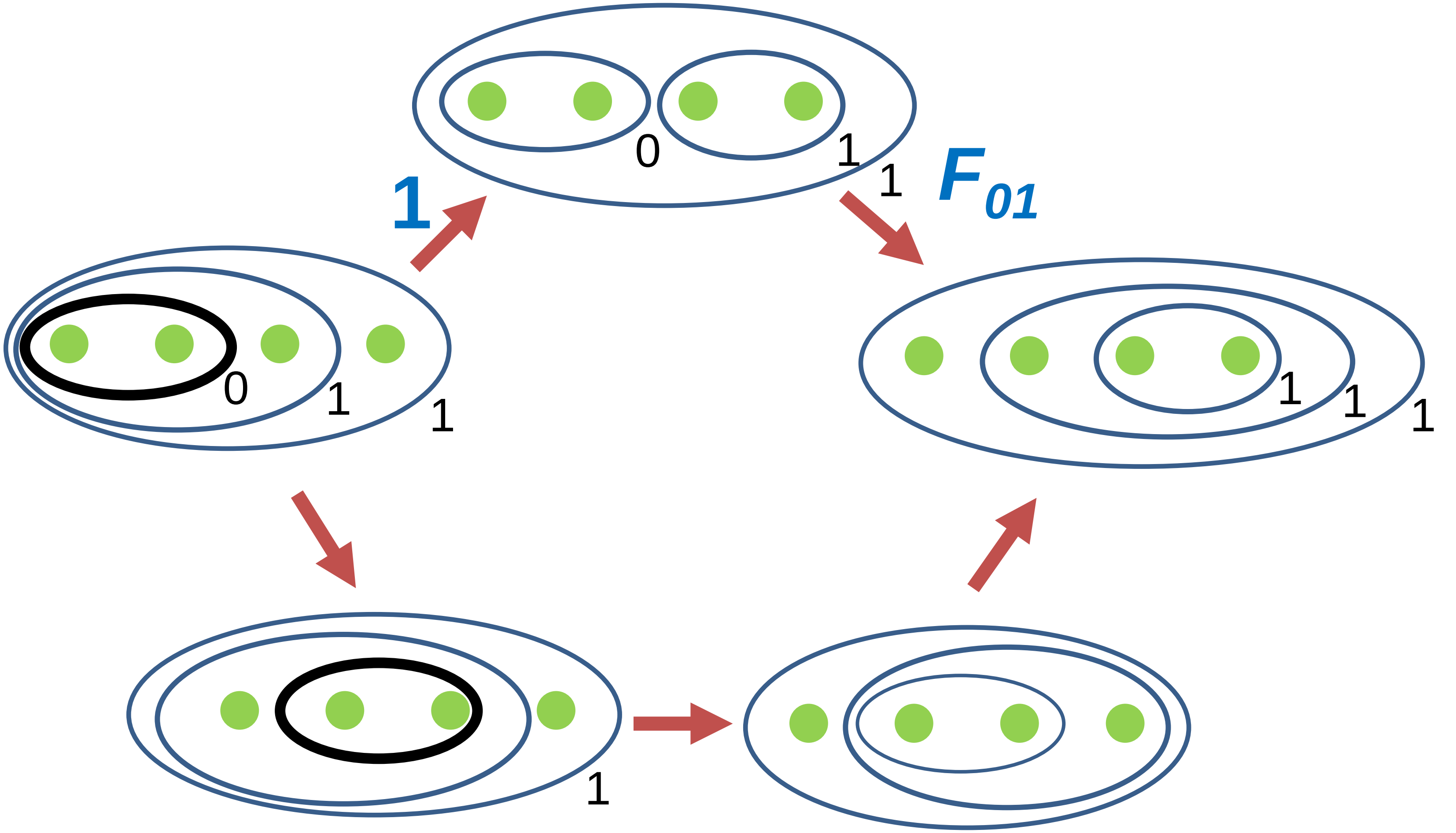
The Pentagon Equation



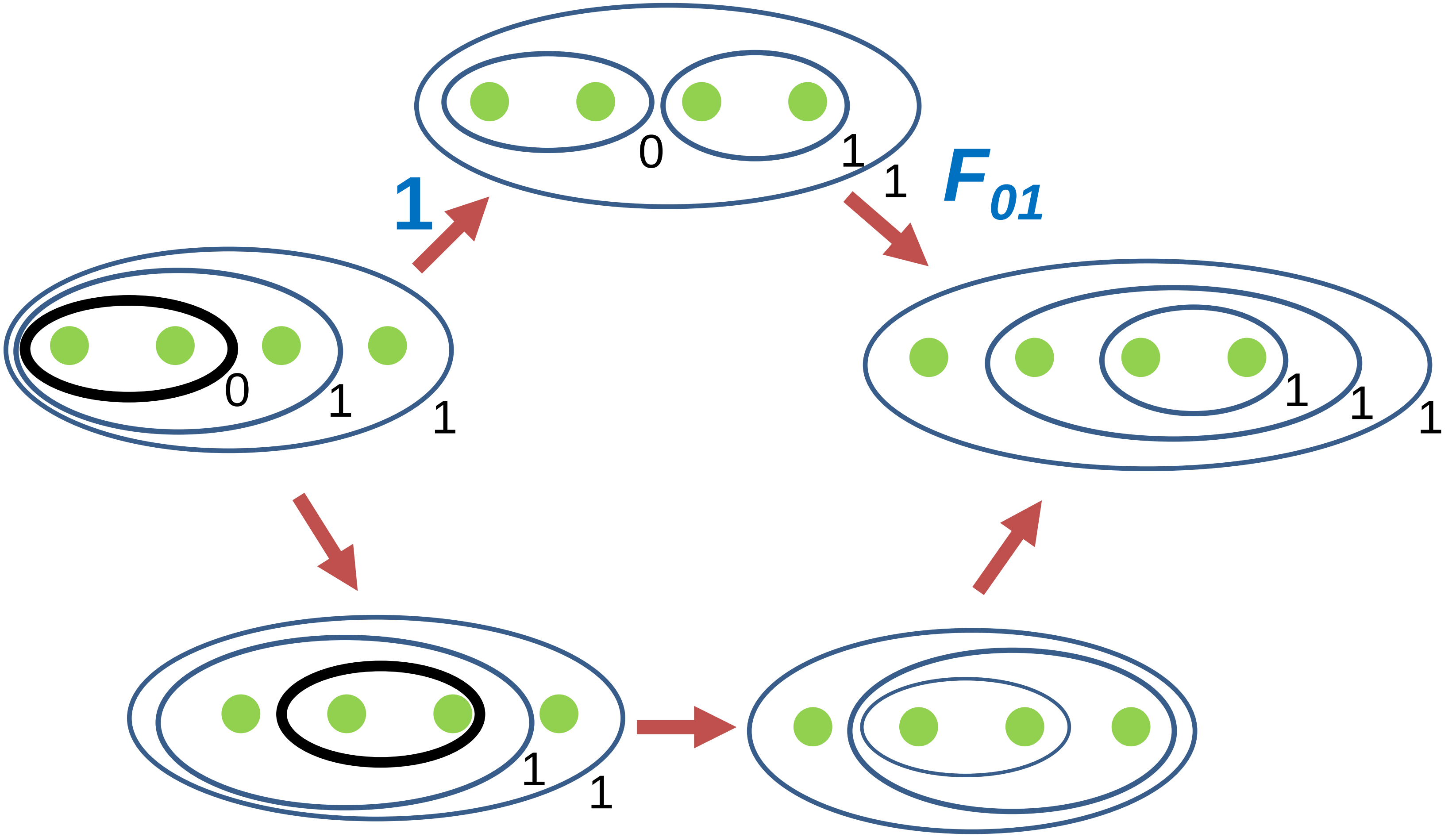
The Pentagon Equation



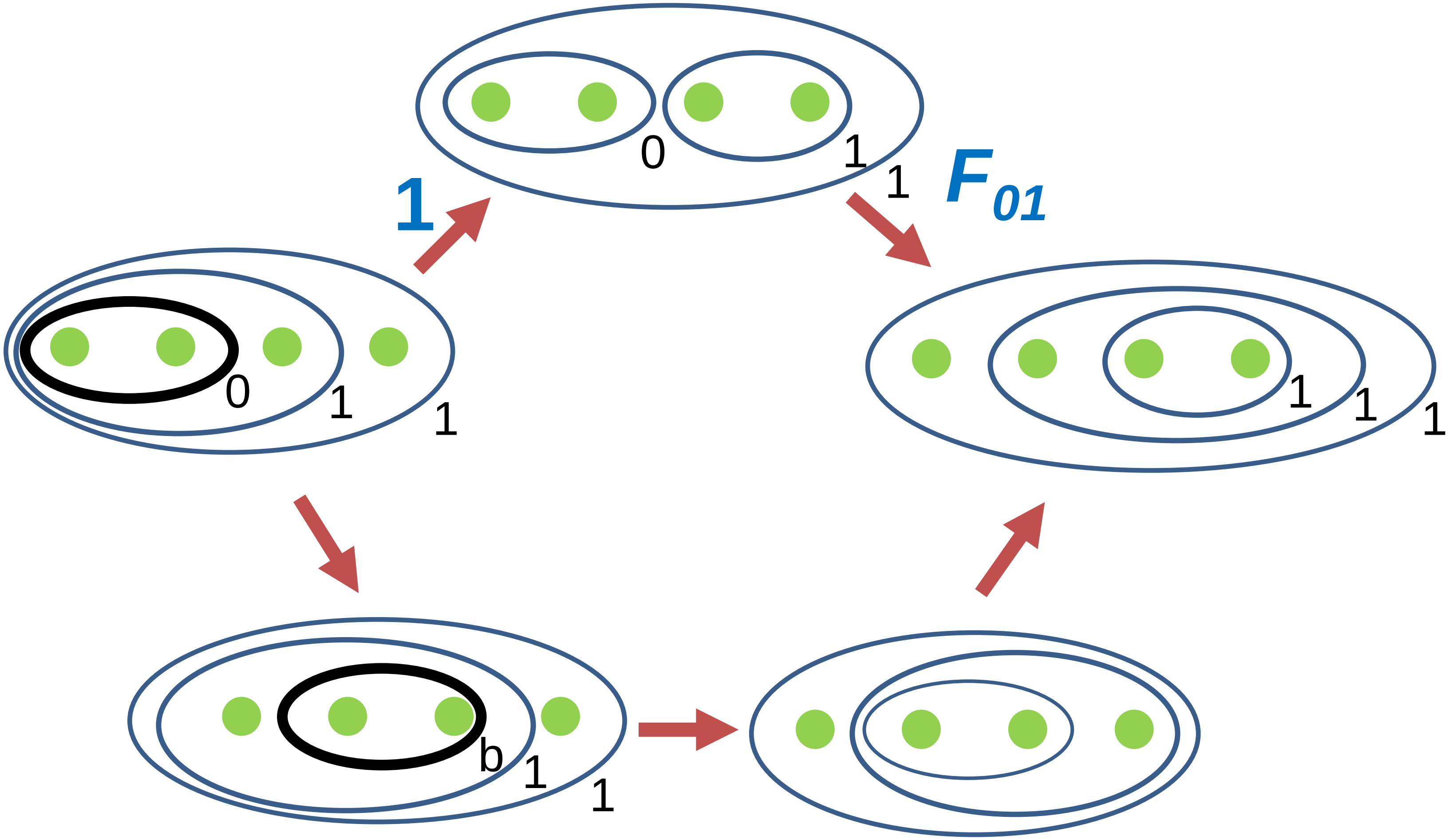
The Pentagon Equation



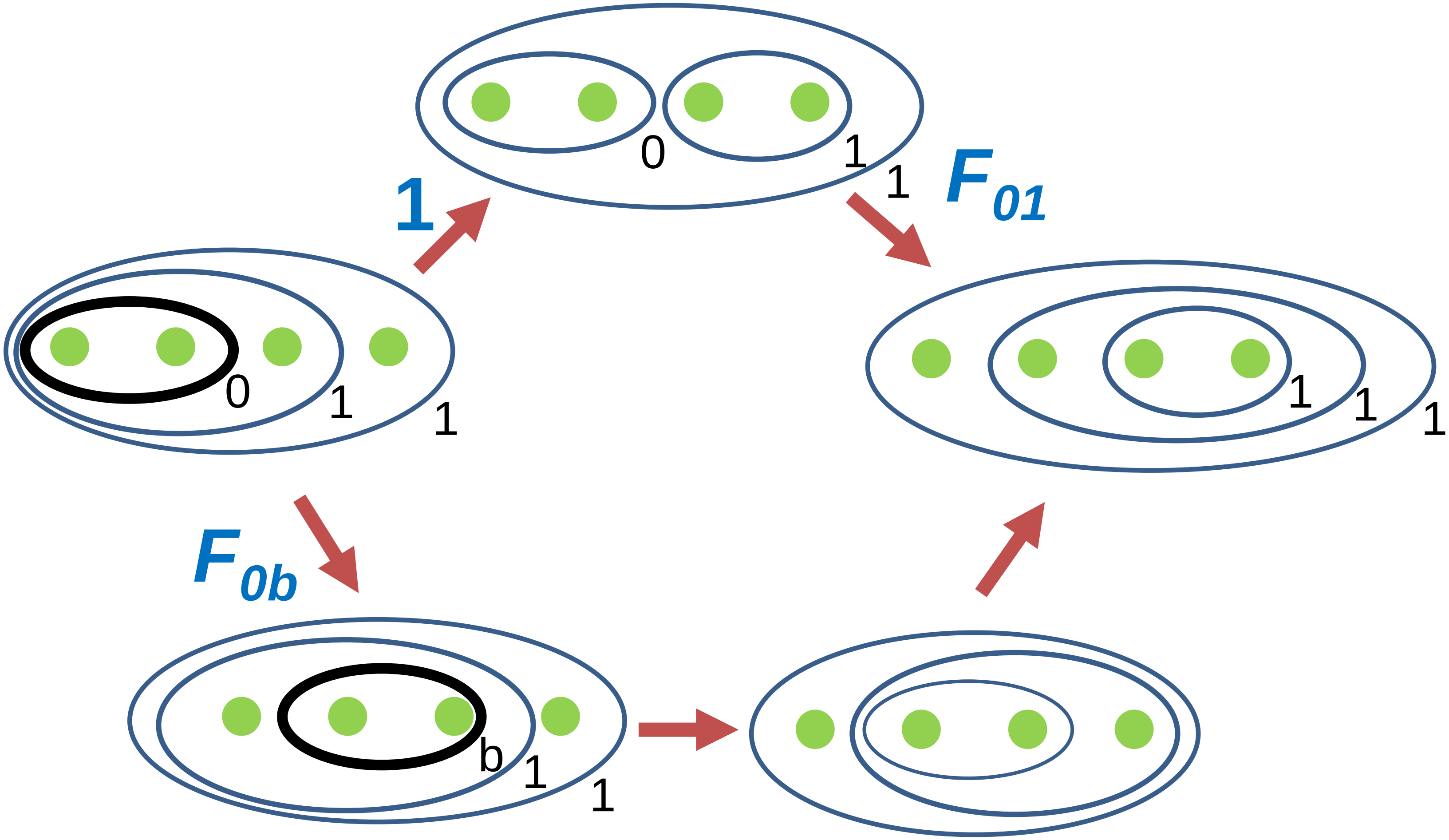
The Pentagon Equation



The Pentagon Equation



The Pentagon Equation



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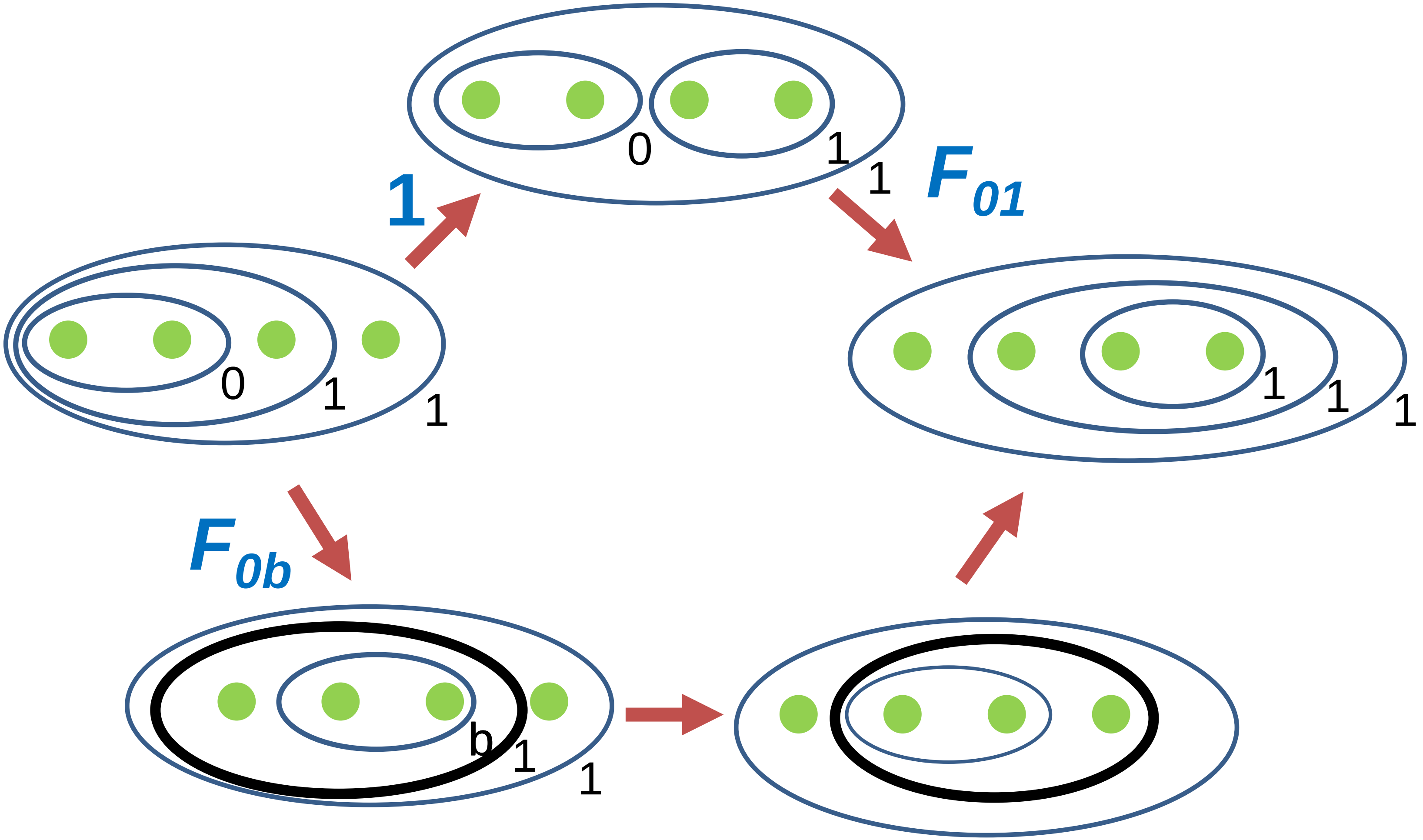

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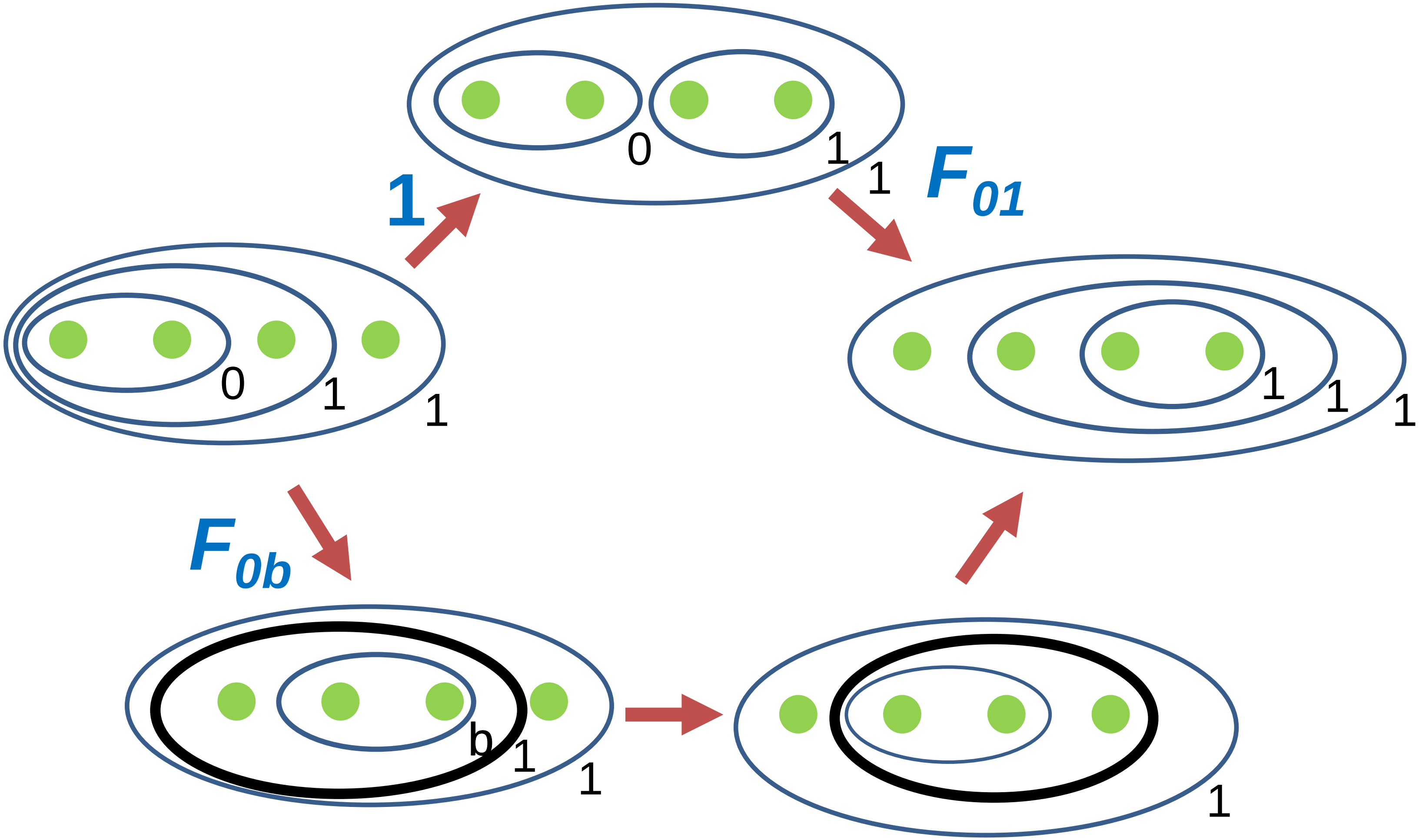
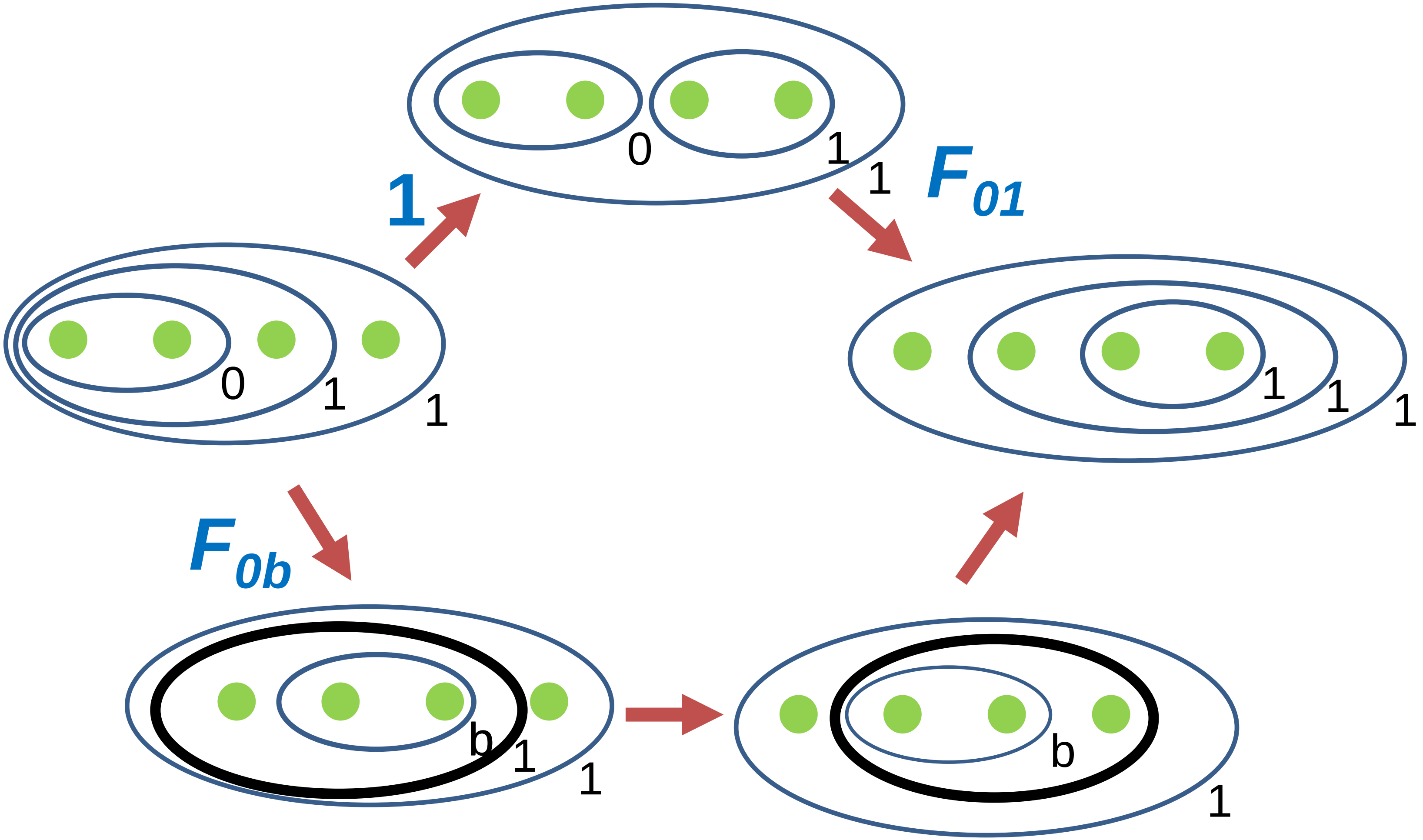
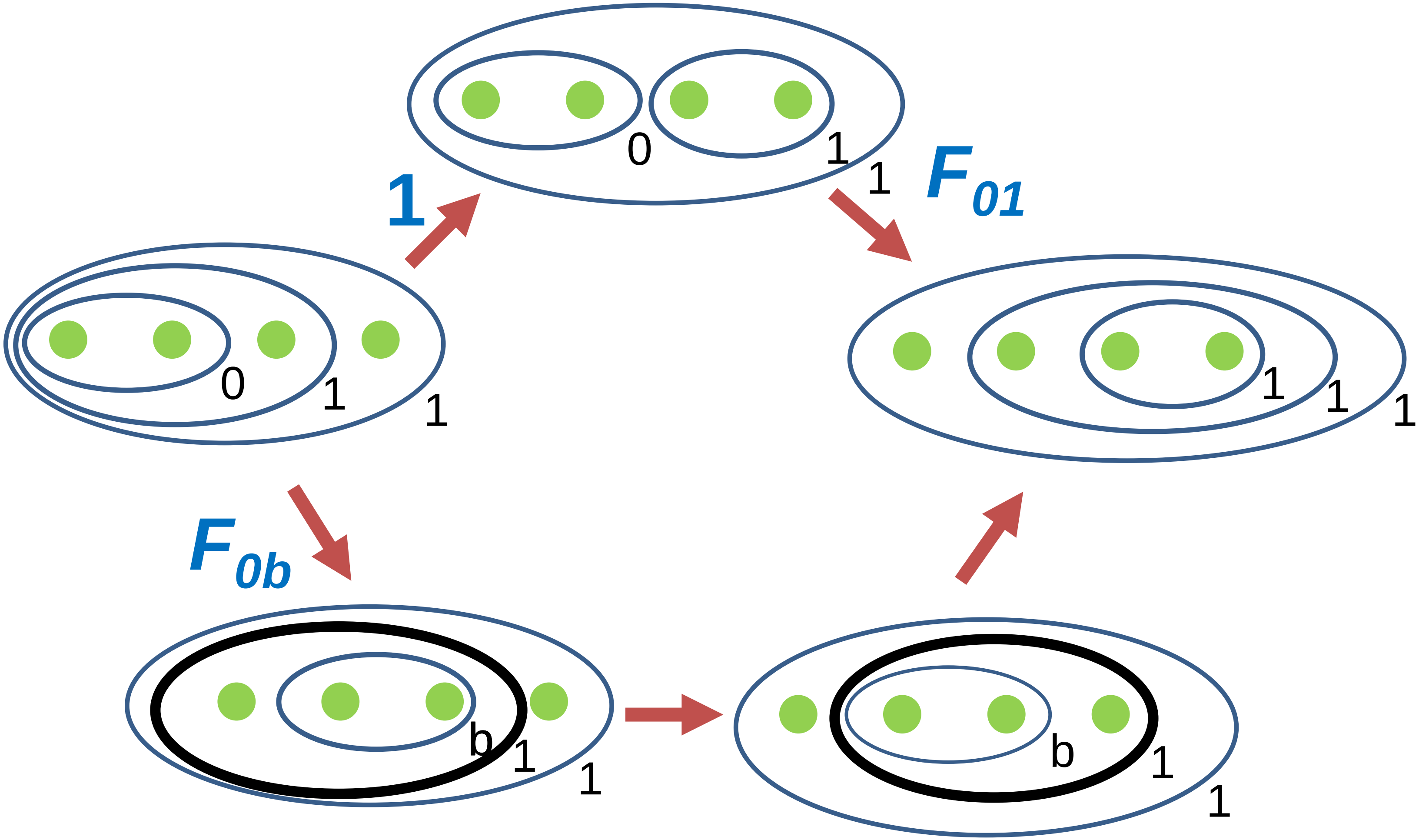


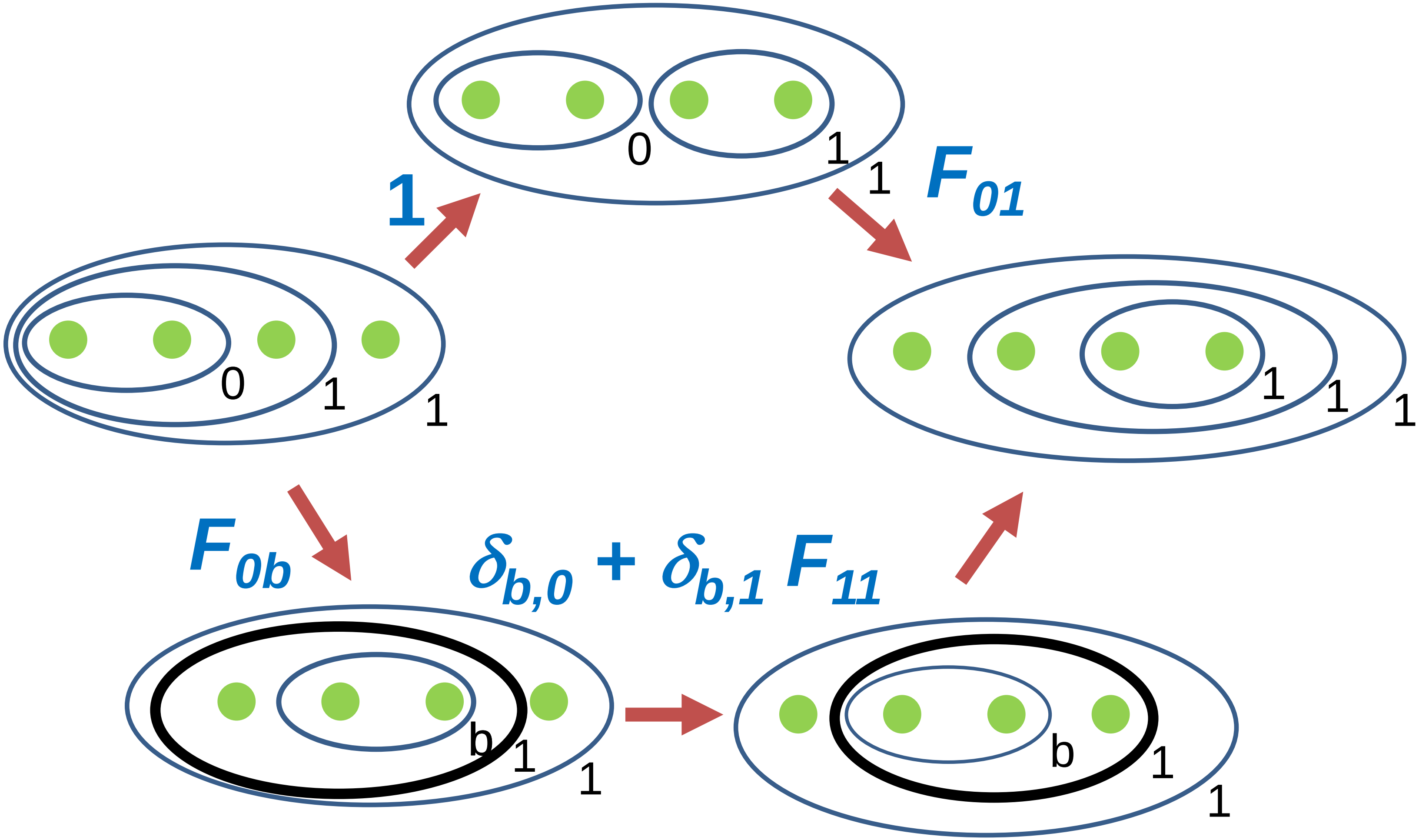
The Pentagon Equation



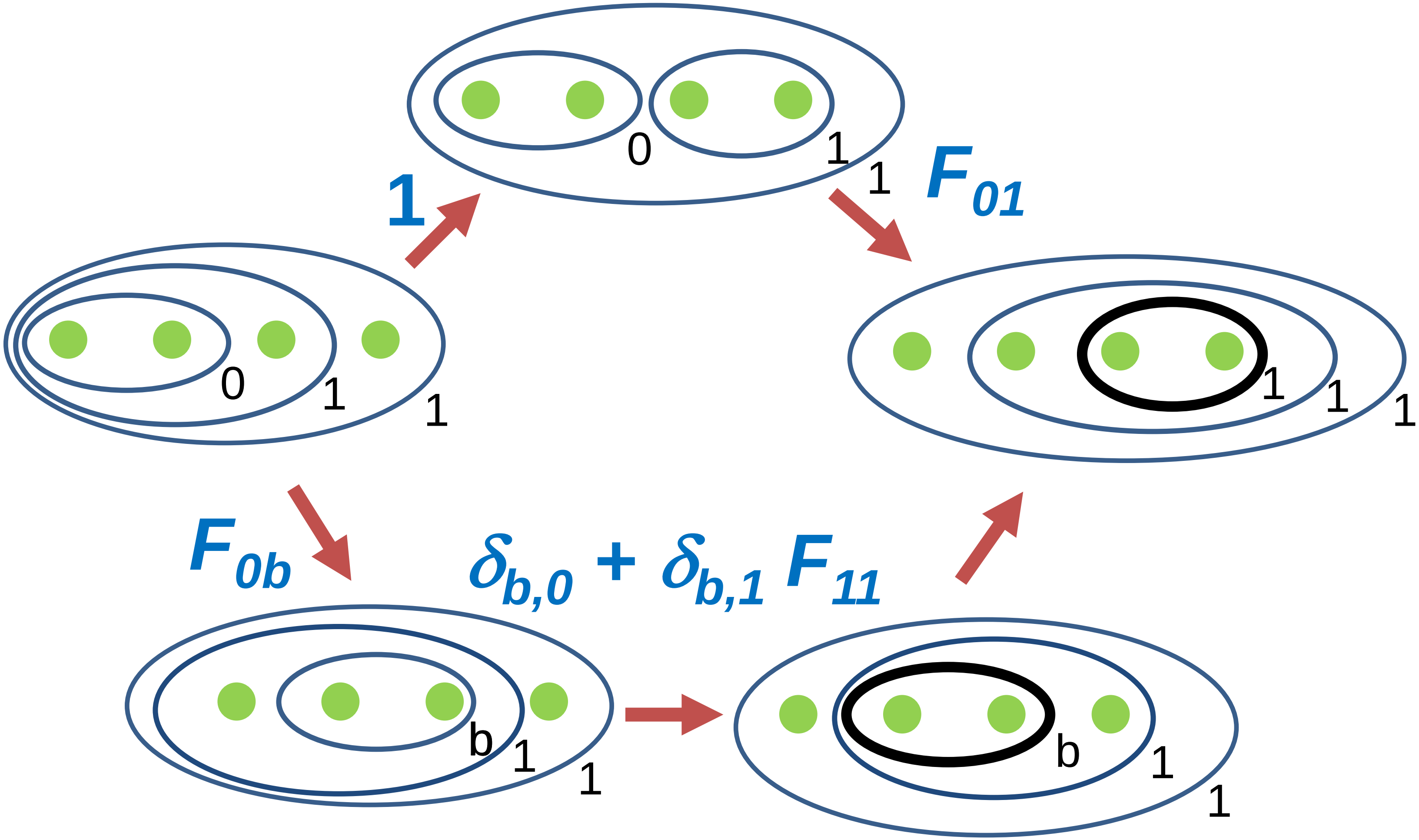
The Pentagon Equation



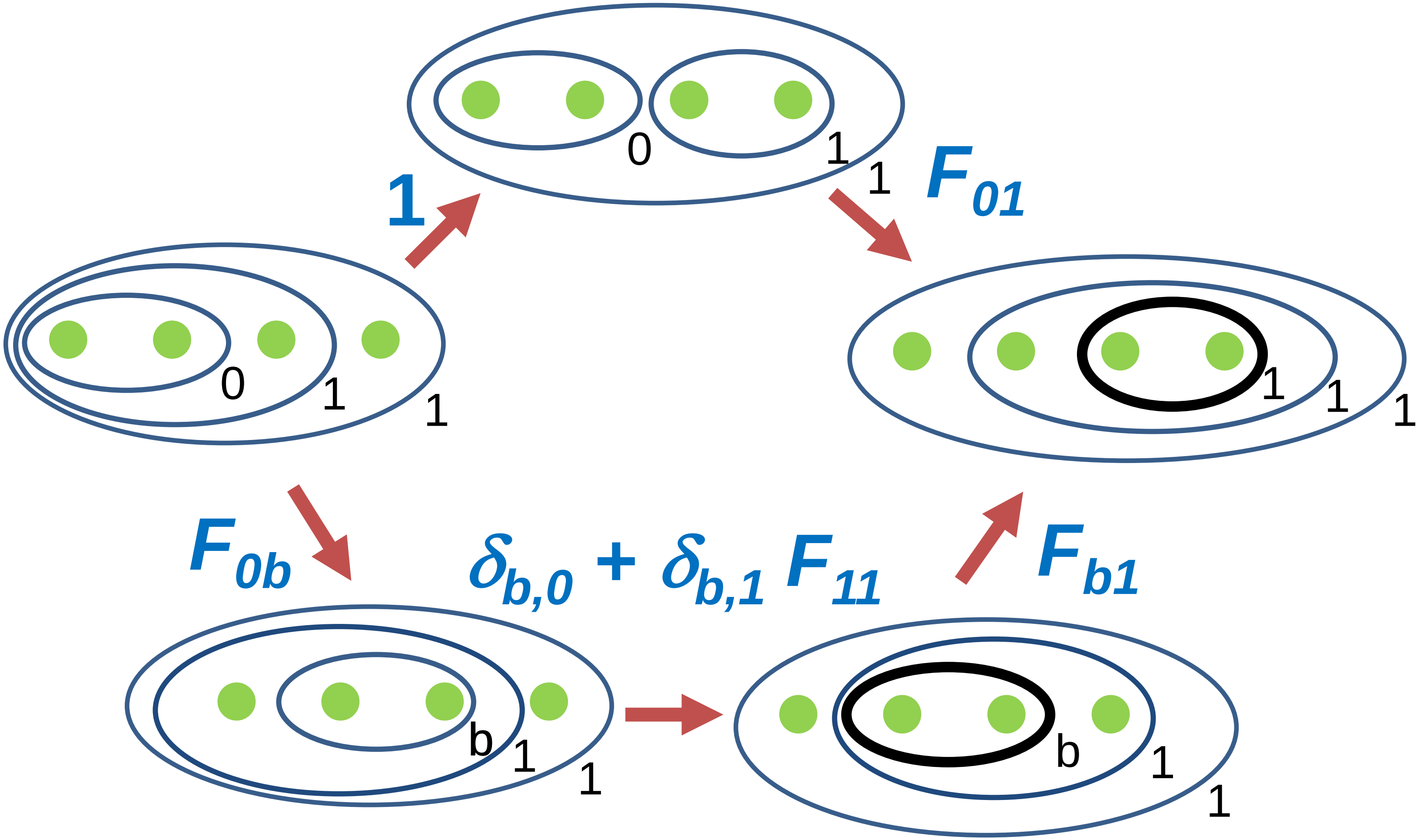
The Pentagon Equation



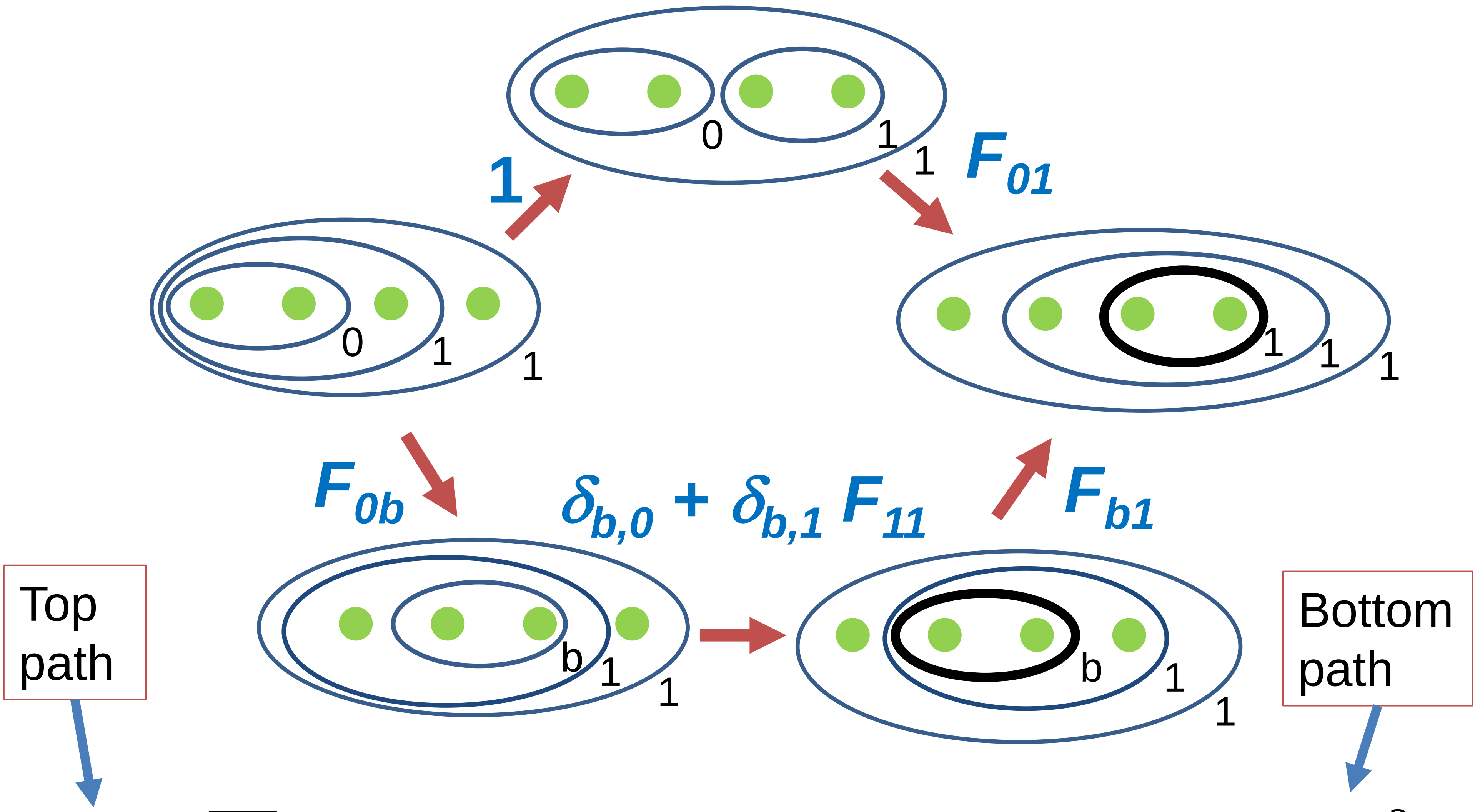
The Pentagon Equation



The Pentagon Equation

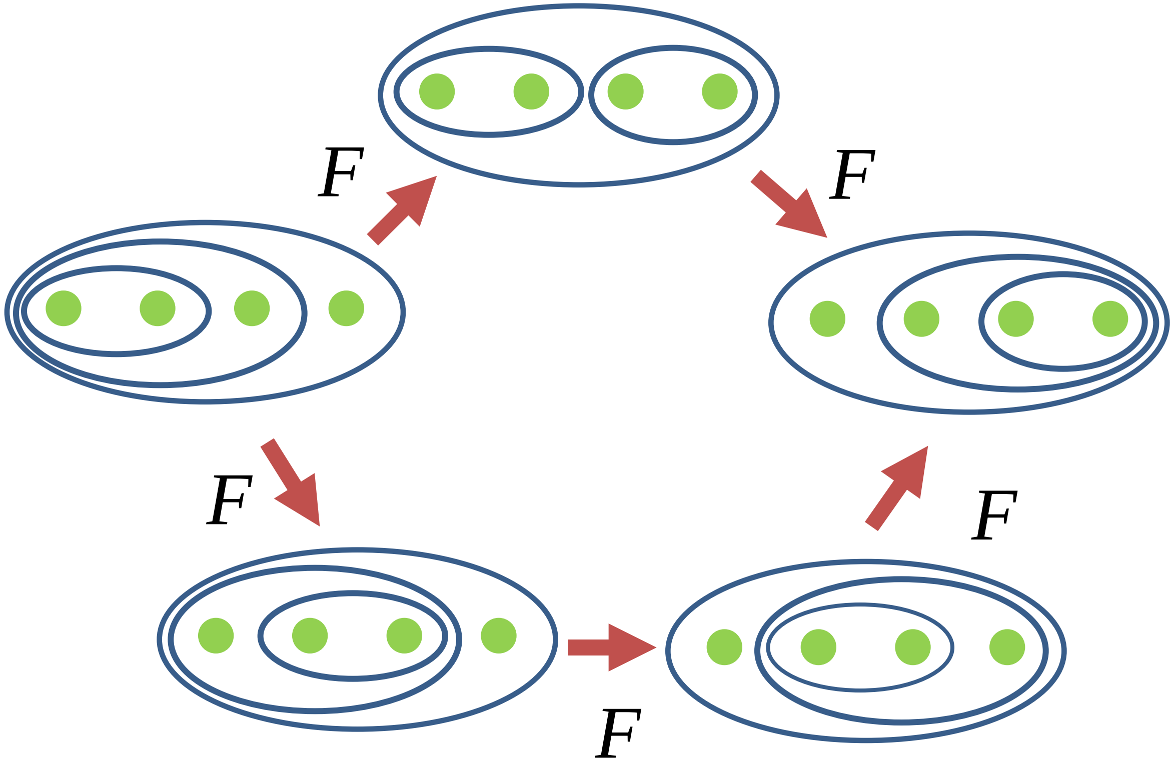


The Pentagon Equation



$$F_{01} = \sum_b F_{0b}\left(\delta_{b,0} + \delta_{b,1} F_{11}\right) F_{b1} = F_{00} F_{01} + F_{01} F_{11}{}^2$$

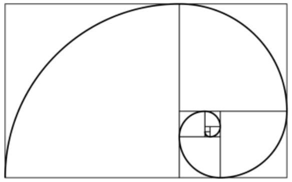
The Pentagon Equation



Unique unitary solution (up to irrelevant phase factors):

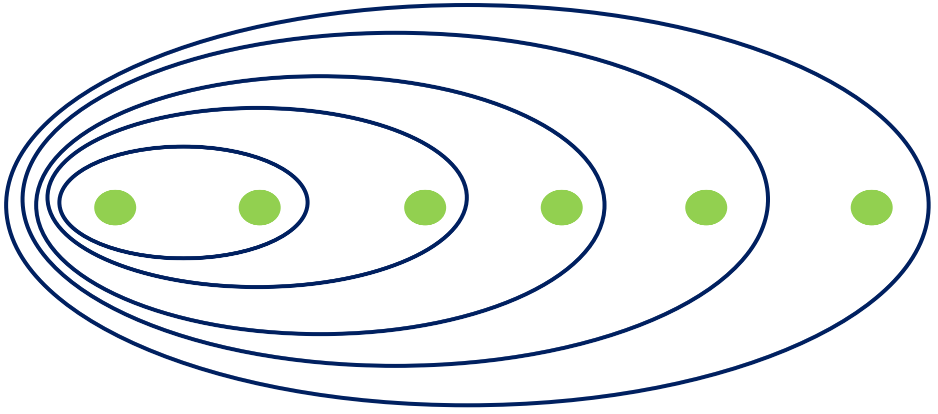
$$F = \begin{pmatrix} \varphi^{-1} & \varphi^{-1/2} \\ \varphi^{-1/2} & -\varphi^{-1} \end{pmatrix}$$

$$\varphi = \frac{\sqrt{5}+1}{2} \approx 1.618\cdots$$

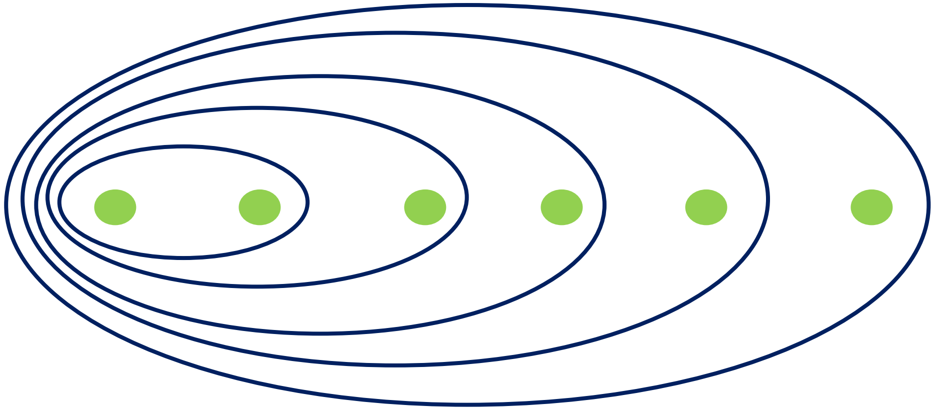


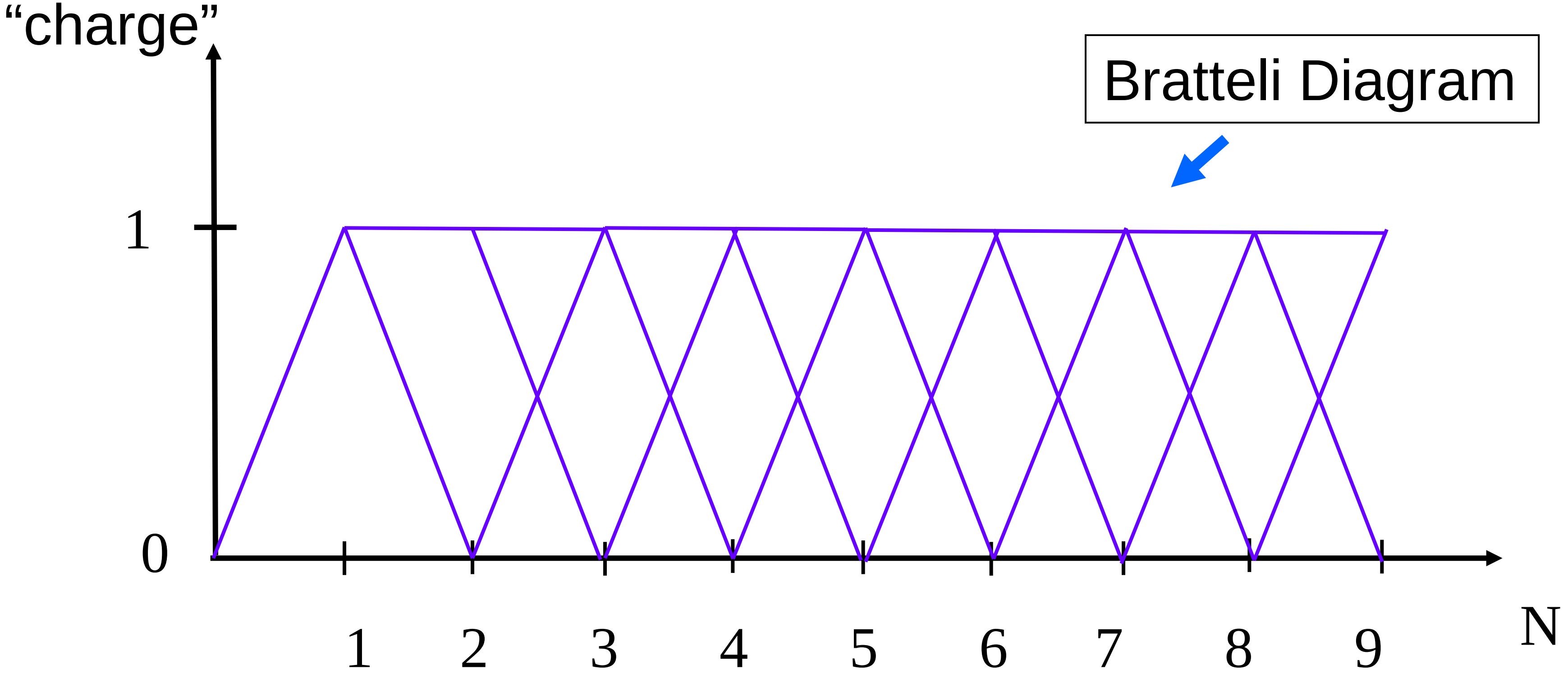
Golden Mean

Hilbert Space Dimensionality

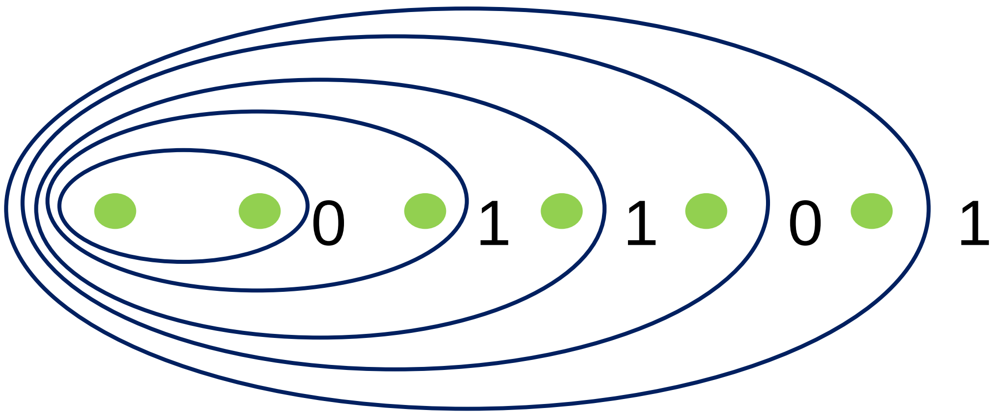


Hilbert Space Dimensionality

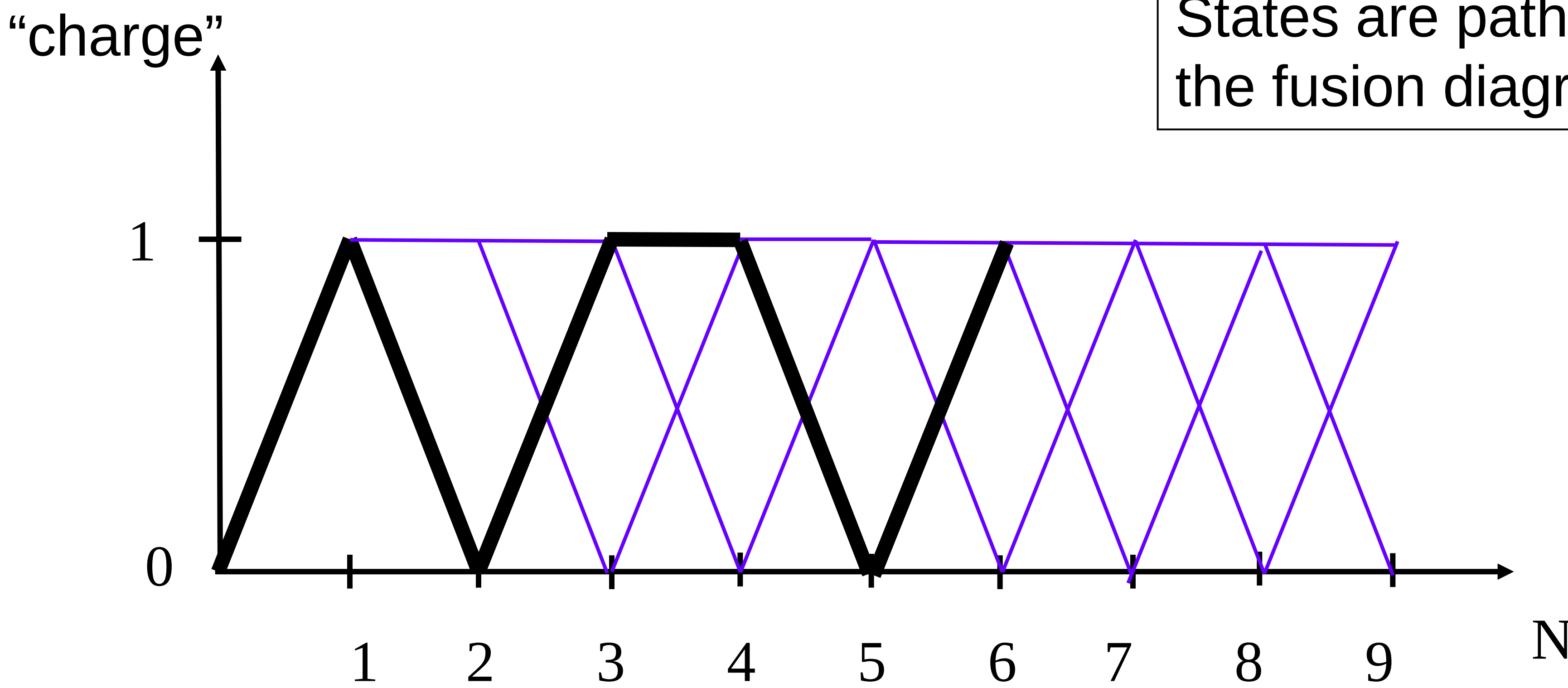




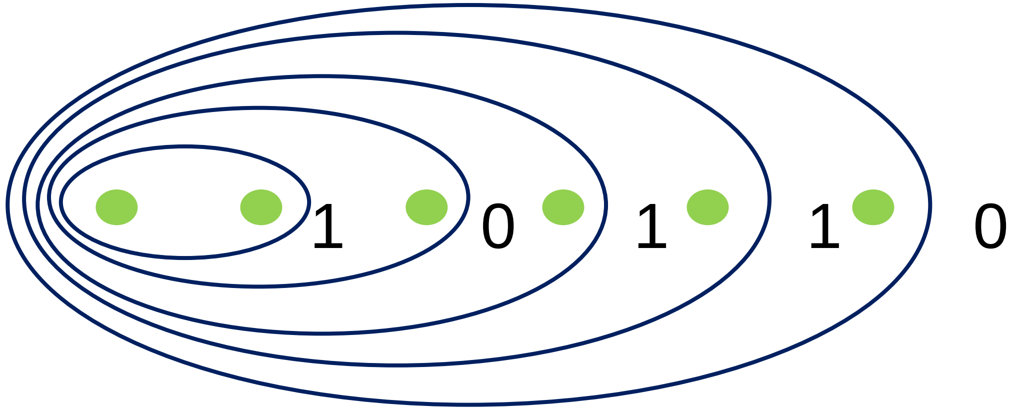
Hilbert Space Dimensionality



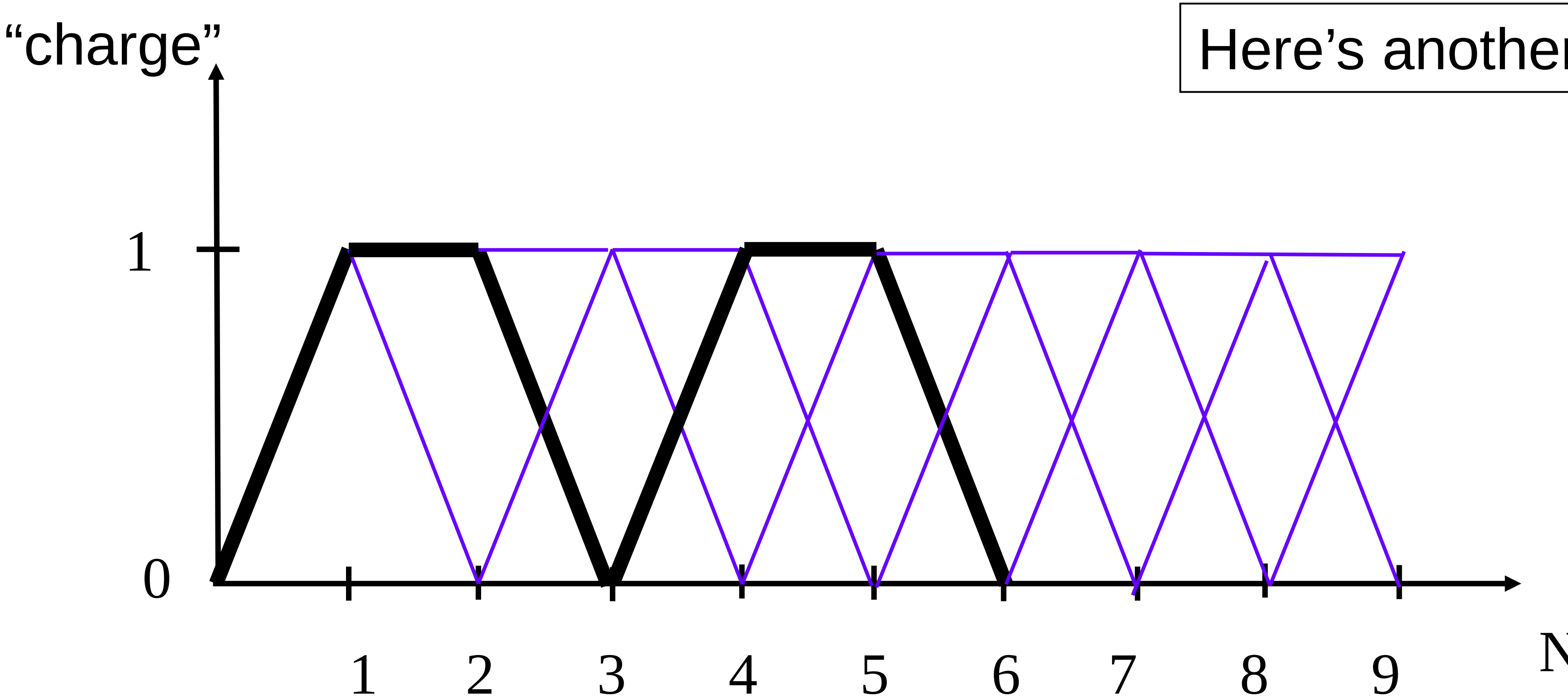
States are paths in the fusion diagram



Hilbert Space Dimensionality



Here's another one

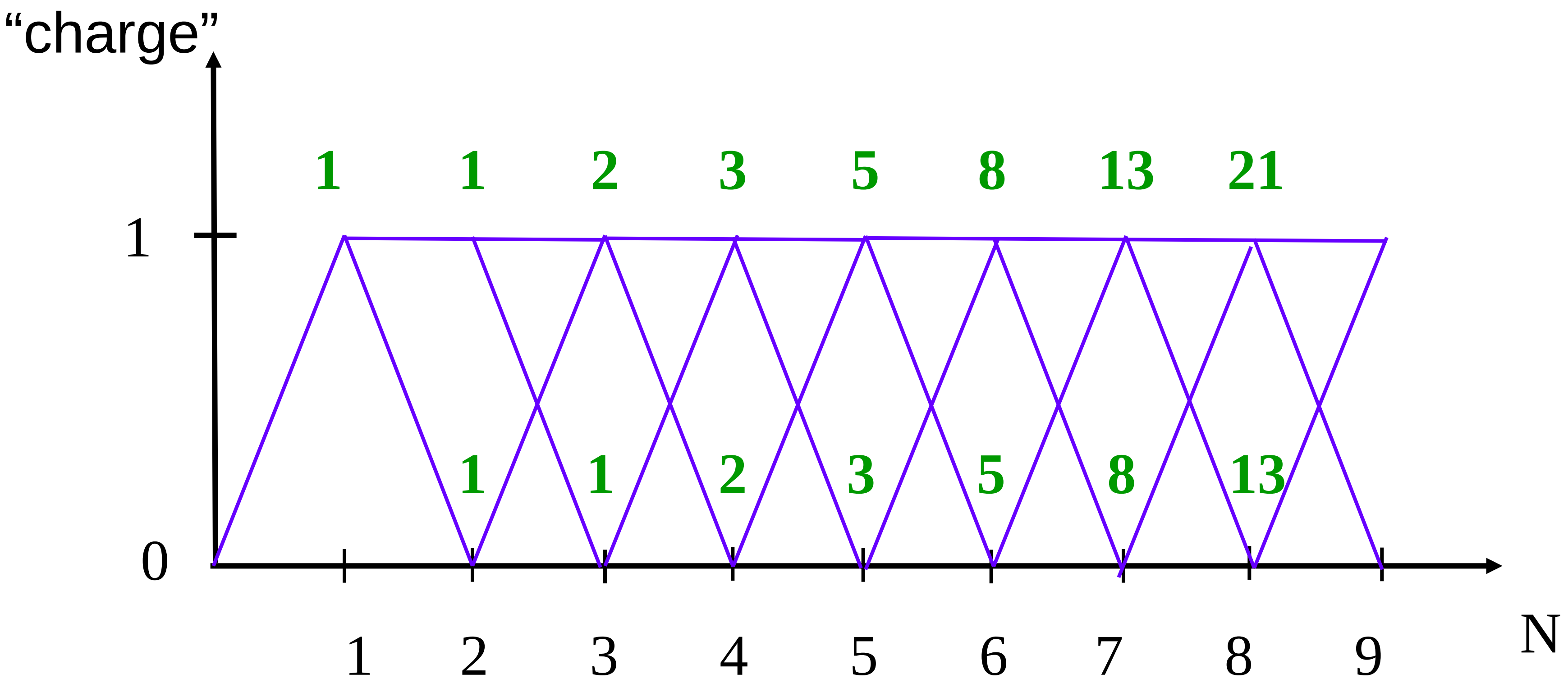


Hilbert Space Dimensionality

- Hilbert space dimensionality grows as the **Fibonacci sequence**!

→ Fibonacci Anyons

- Exponentially large in the number of quasiparticles **(deg ~ ϕ^N),** so big enough for quantum computing.

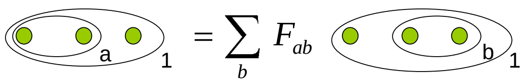


Problem 1. Pentagon Equation for Fibonacci Anyons.

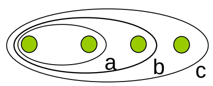
For Fibonacci anyons the 2×2 F matrix,

$$F = \begin{pmatrix} F_{00} & F_{01} \\ F_{10} & F_{11} \end{pmatrix},$$

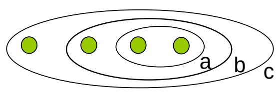
describes the following basis change,



The **pentagon equation** then equates the results of two distinct ways of using the F matrix to express a four anyon state from the basis,



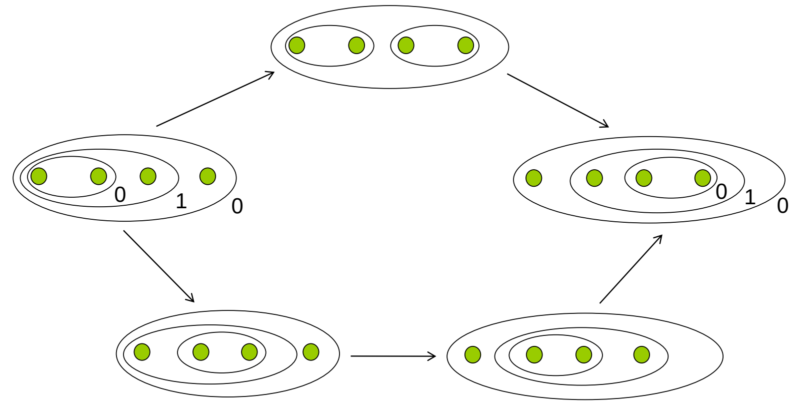
as a superposition of states from the basis,

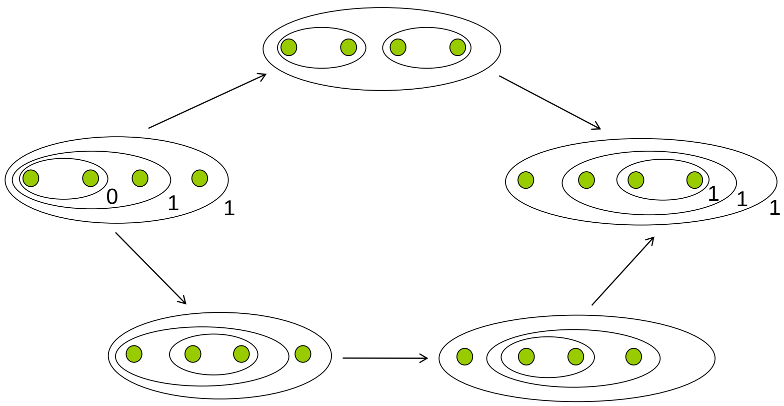


For each of the seven pentagon diagrams that follow, use the fact that the two paths (top and bottom) should yield the same amplitude for the contribution of the rightmost state in the expansion of the leftmost state to derive seven polynomial equations for F_{00}, F_{11} and the product $F_{01}F_{10}$.

Now solve these equations. You should find two solutions, only one of which yields a unitary F matrix if you take $F_{01} = F_{10} = \sqrt{F_{01}F_{10}}$. Find this 2×2 unitary F matrix.

You may find it convenient to express your answer in terms of $\tau = (\sqrt{5} - 1)/2 \simeq 0.62$, where τ is the inverse of the golden mean $\phi = (\sqrt{5} + 1)/2 \simeq 1.62$.

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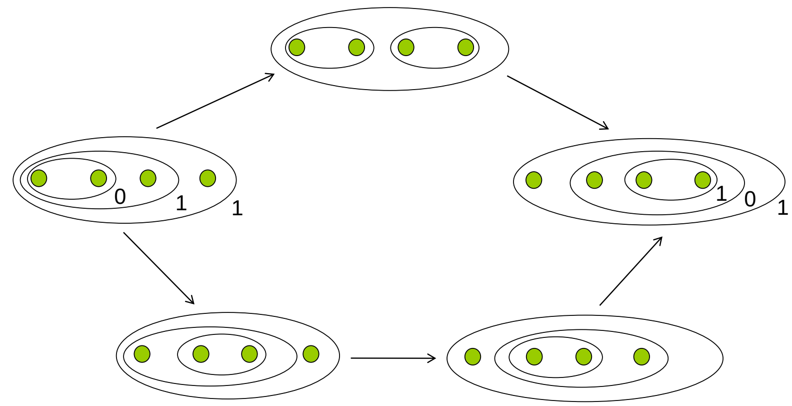
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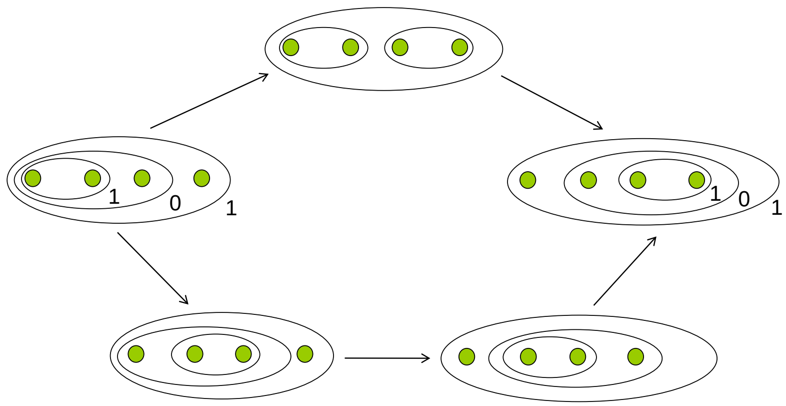
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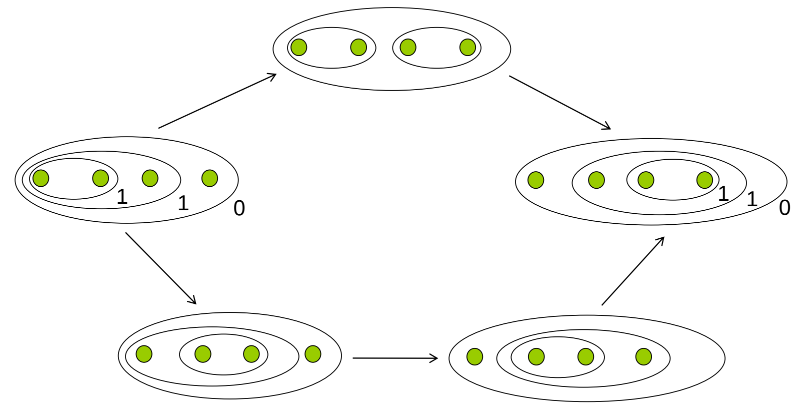
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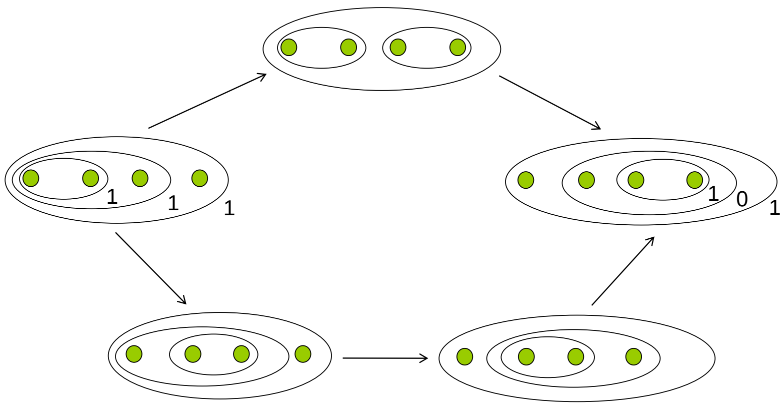
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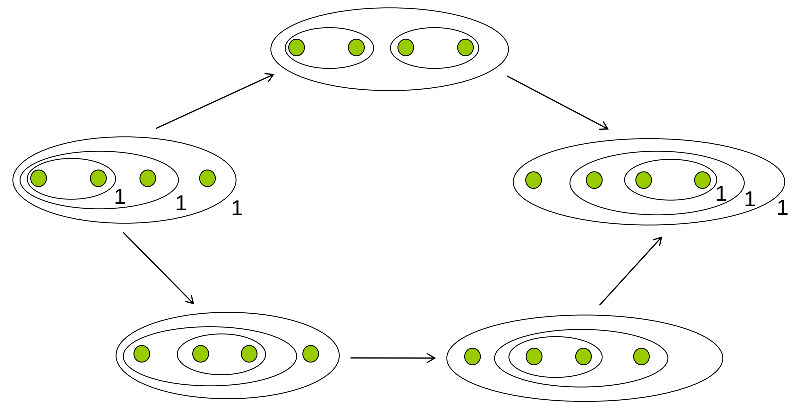
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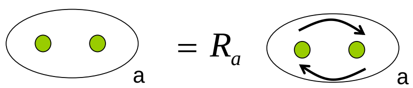
Problem 2. Hexagon Equation for Fibonacci Anyons.

For Fibonacci anyons the 2×2 R matrix,

$$R = \begin{pmatrix} R_0 & 0 \\ 0 & R_1 \end{pmatrix},$$

describes the phase factor acquired when anyons with a given total topological charge are exchanged in a clockwise manner,



.

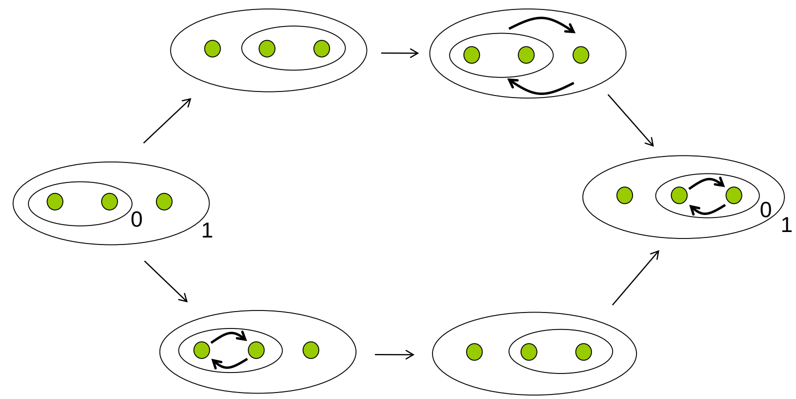
The **hexagon equation** then describes two different ways to use the F and R matrices to compute the effect of moving two anyons around a third.

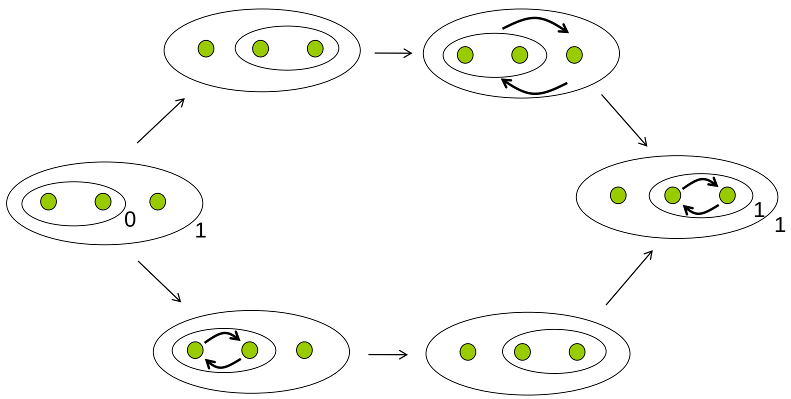
For each of the four hexagon diagrams that follow, use the fact that the two paths (top and bottom) should yield the same amplitude for the contribution of the rightmost state to the expansion of the state obtained by carrying out the anyon exchanges on the leftmost state to derive four polynomial equations for R_0 and R_1. (In this calculation you should use the unitary F matrix obtained in Problem 1).

Again you should find two solutions, this time corresponding to the two possible choices for the 'handedness' of the anyons.

The following identities might be useful in simplifying your results for R_0 and R_1,

$$\sin\left(\frac{3\pi}{5}\right) = \frac{\sqrt{10 - 2\sqrt{5}}}{4}, \quad \cos\left(\frac{3\pi}{5}\right) = \frac{1 - \sqrt{5}}{4}.$$

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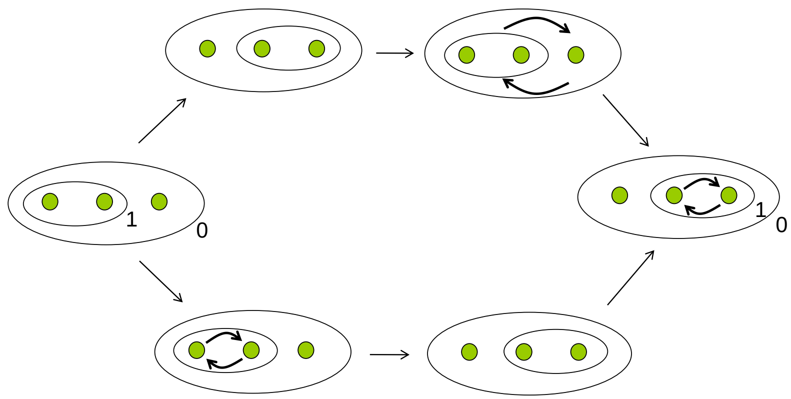
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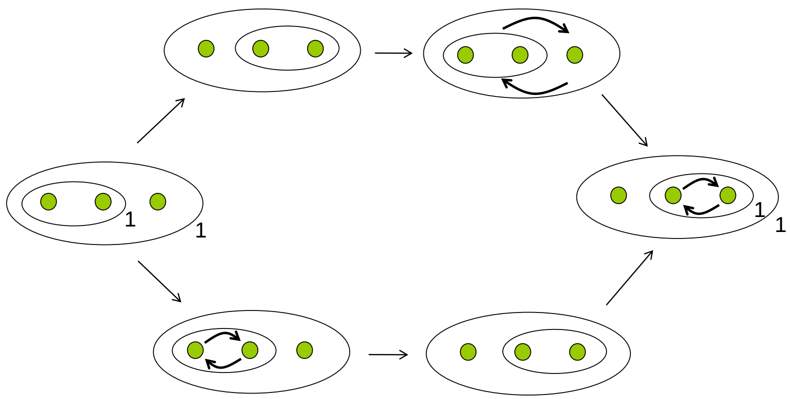
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