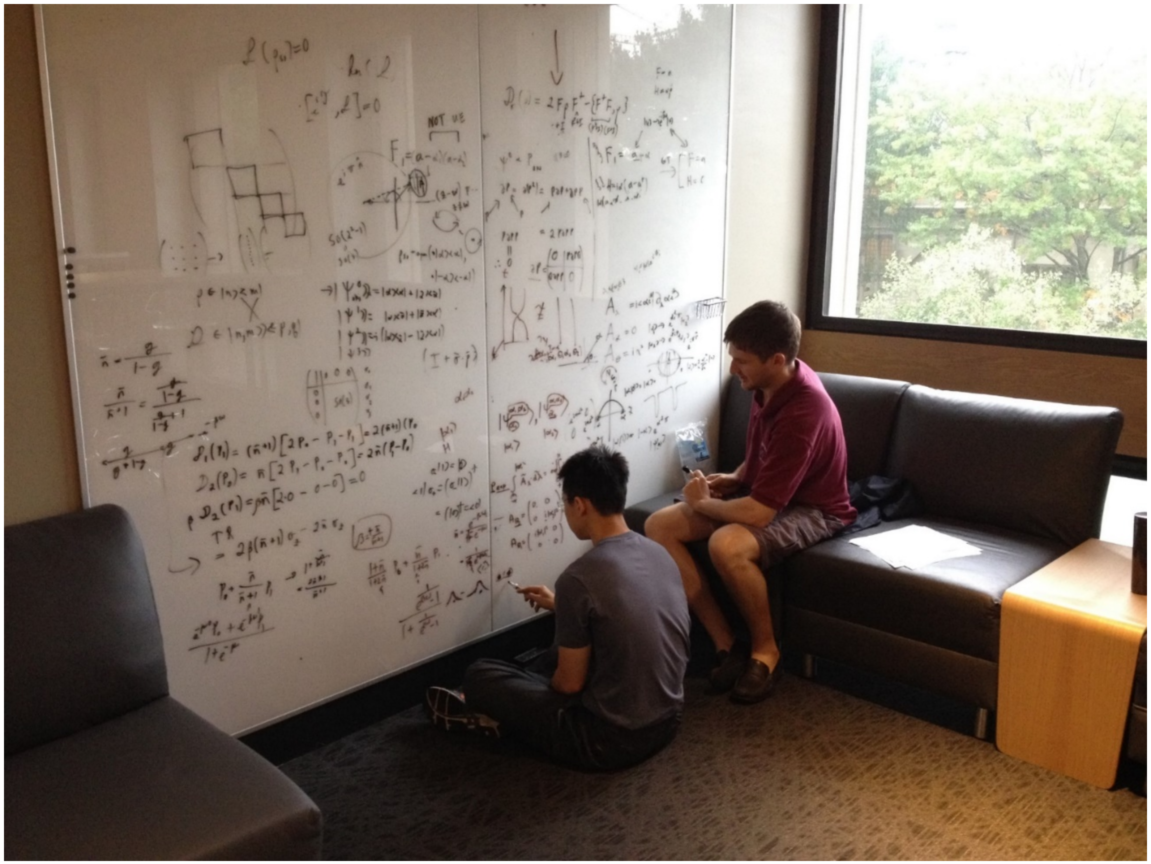
Entanglement Entropy, Maxwell's Demon and Quantum Error Correction



Experiment

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Quick Review of the Basics of Quantum Information

Quantum bits (‘qubits’)

Quantum information is stored in the physical states of a quantum system:

- atoms, molecules, ions, superconducting circuits, photons, mechanical oscillators, …

Quantum Information is Paradoxical

Is quantum information carried by waves or by particles?

YES!

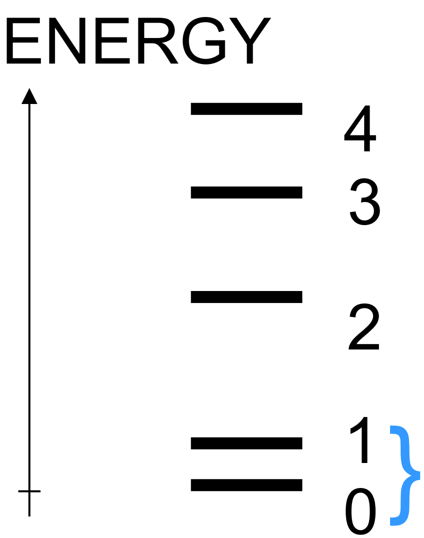
Is quantum information analog or digital?

YES!

Quantum information is digital:

Energy levels of a quantum system are discrete.

We use only the lowest two.



Measurement of the state of a qubit yields 1 classical bit of information.

Stern-Gerlach surprise.

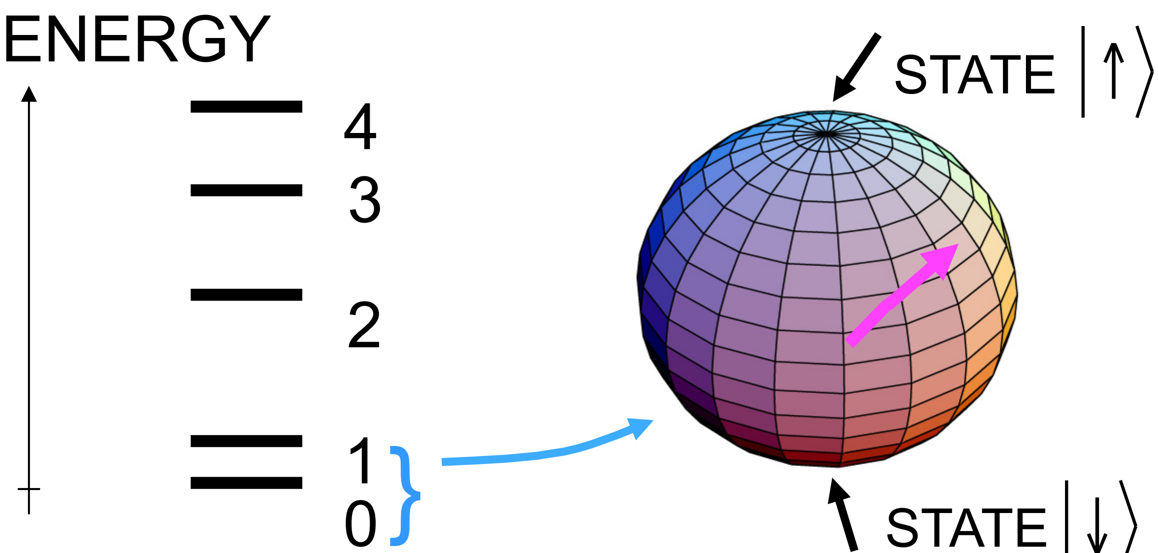
$$\text{excited state } 1 = |e\rangle = |\uparrow\rangle$$

$$\text{ground state } 0 = |g\rangle = |\downarrow\rangle$$

Quantum information is analog:

A quantum system with two distinct states can exist in an Infinite number of physical states ('superpositions') *intermediate* between $|\downarrow\rangle$ and $|\uparrow\rangle$.
(Requires infinite number of classical bits to specify)

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|\downarrow\rangle + e^{i\varphi}\sin\left(\frac{\theta}{2}\right)|\uparrow\rangle$$



$\theta =$ latitude

$\varphi =$ longitude

State defined by 'spin polarization vector' on Bloch sphere

Quantum information is analog/digital:

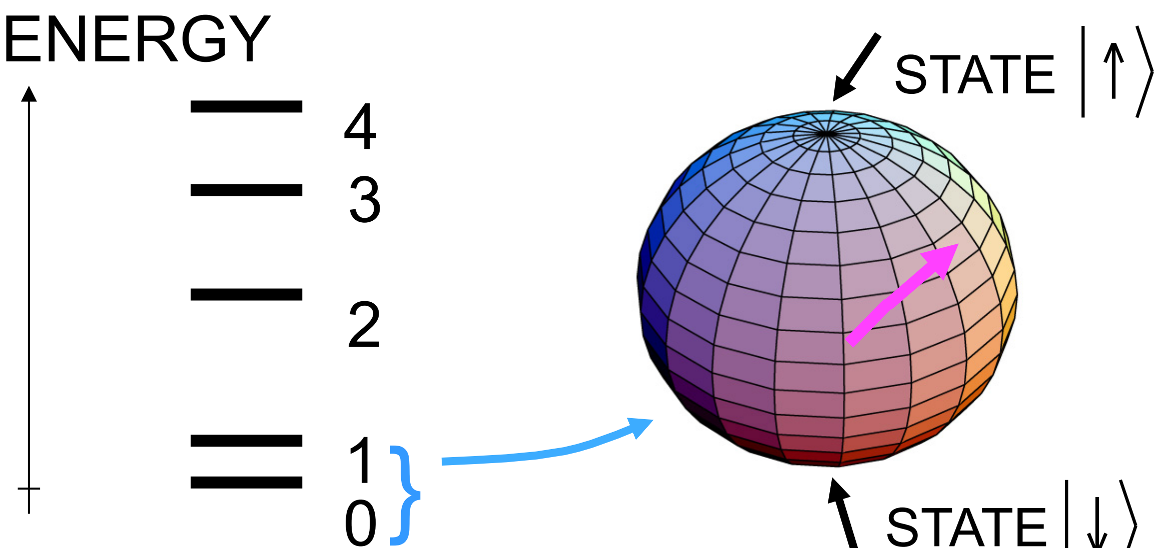
$$\vec{S} = \frac{\hbar}{2}(X, Y, Z)$$

Lie Algebra:

$$[X, Y] = 2iZ$$

State defined by ‘spin polarization vector’ on Bloch sphere.

Every two-level system is equivalent to a spin ½.



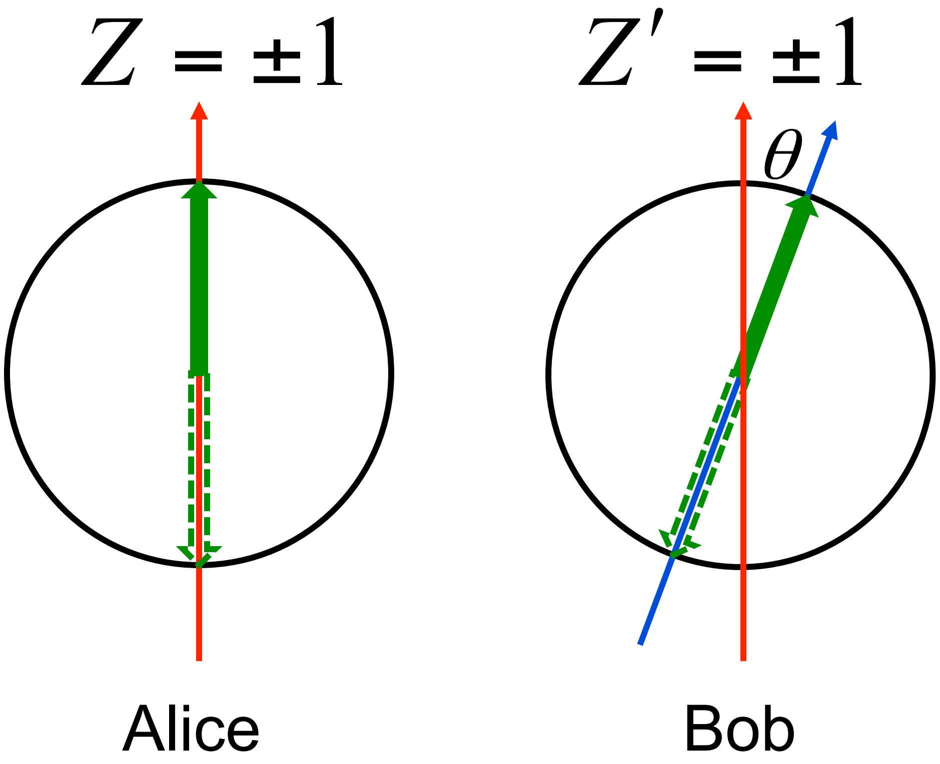
$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$Y = \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix}$$

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Quantum information is analog/digital:

Equivalently: a quantum bit is like a classical bit except there are an infinite number of encodings (aka 'quantization axes').



If Alice gives Bob a $Z = +1$,

Bob measures:

$$Z' = +1 \text{ with probability } P_+ = \cos^2\frac{\theta}{2}$$

$$Z' = -1 \text{ with probability } P_- = \sin^2\frac{\theta}{2}$$

'Back action' of Bob's measurement changes the state, but this is invisible to Bob.

The huge information content of quantum superpositions comes with a price:

Great sensitivity to noise perturbations and dissipation.

The quantum phase of superposition states is well-defined only for a finite 'coherence time' T_2

Example: qubit transition frequency noise

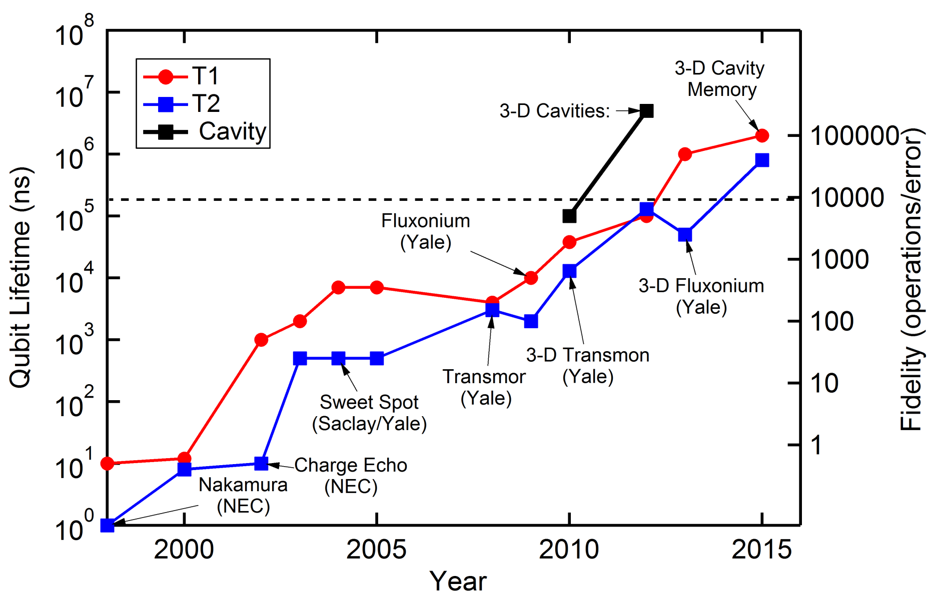
$$H(t) = \frac{\omega_0 + \delta\omega(t)}{2}\sigma_z; \quad |\psi(t)\rangle = e^{+i[\omega_0 t + \varphi(t)]/2}\alpha|\downarrow\rangle + e^{-i[\omega_0 t + \varphi(t)]/2}\beta|\uparrow\rangle$$

$$\varphi(t) = \underbrace{\int_0^t d\tau\,\delta\omega(\tau)}_{\text{random walk}}; \quad \left\langle e^{i\varphi(t)}\right\rangle = e^{-\frac{1}{2}\langle\varphi^2(t)\rangle} = e^{-\frac{t}{T_\varphi}}; \quad \frac{1}{T_\varphi} = \frac{1}{2}\underbrace{S_{\delta\omega\delta\omega}(0)}_{\text{noise spectral density}}$$

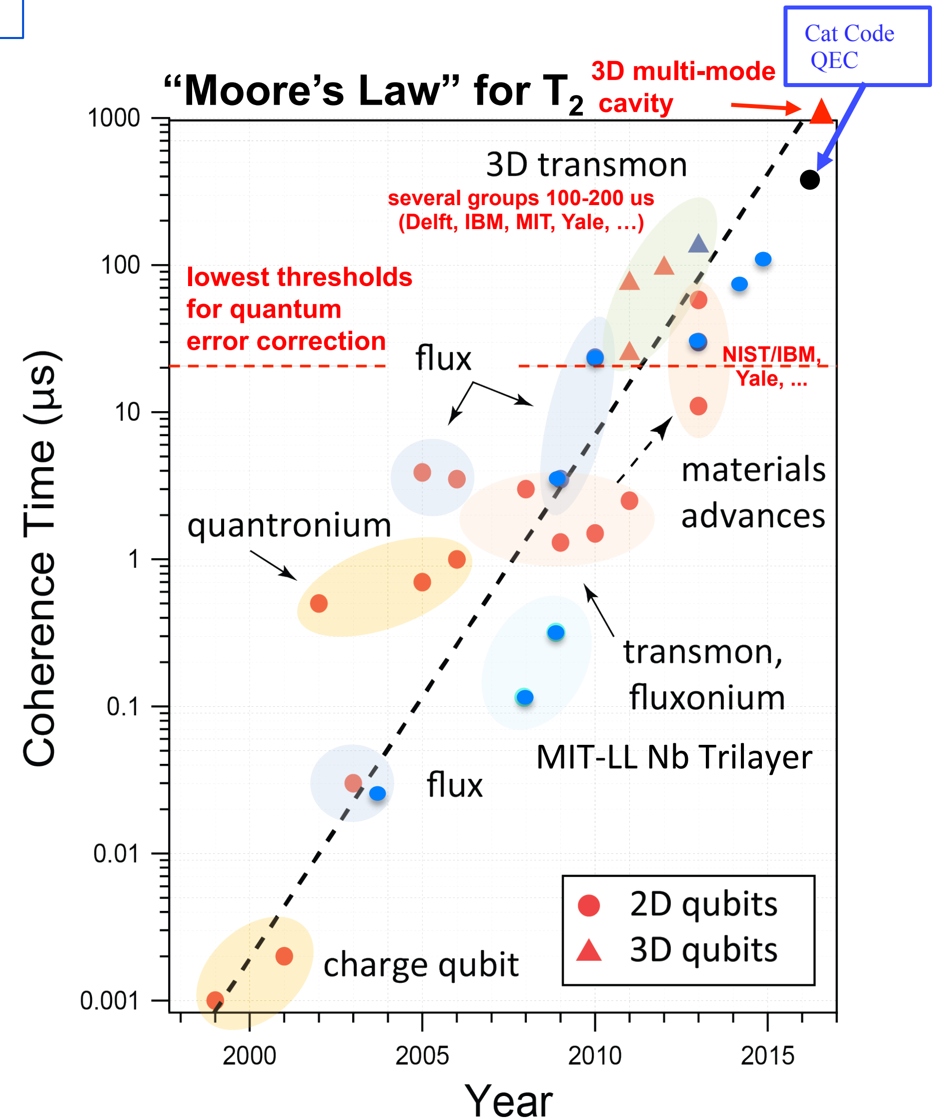
$$\boxed{\frac{1}{T_2} = \frac{1}{2T_1} + \frac{1}{T_\varphi}}$$

Defeating noise through clever engineering and qubit design.

Exponential Growth in SC Qubit Coherence



R. Schoelkopf and M. Devoret



Oliver & Welander, MRS Bulletin (2013)

Girvin’s Law:

There is no such thing as too much coherence.

We need quantum error correction!

The Quantum Error Correction Problem

I am going to give you an unknown quantum state.

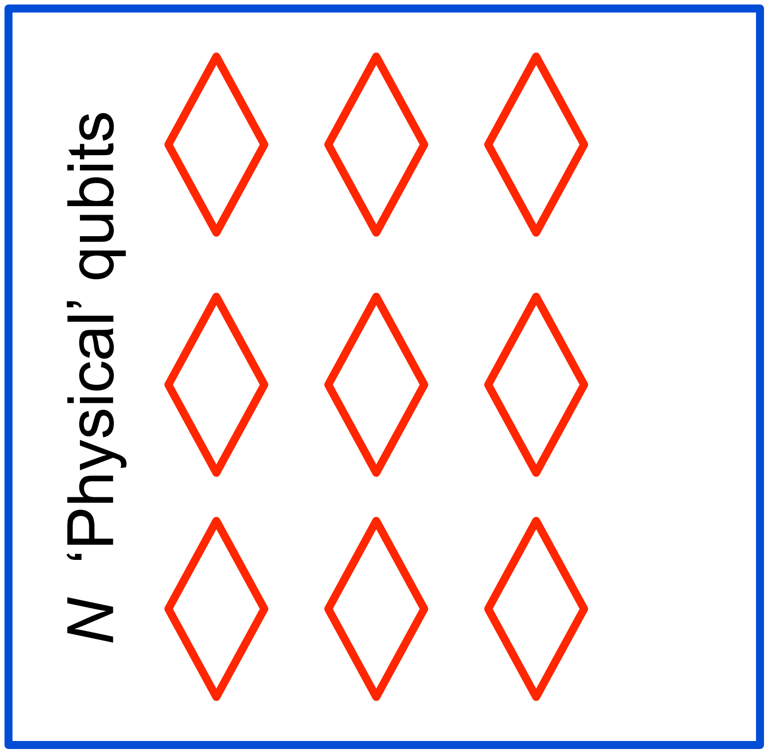
If you measure it, it will change randomly due to state collapse (‘back action’).

If it develops an error, please fix it.

***Mirable dictu*: It can be done!**

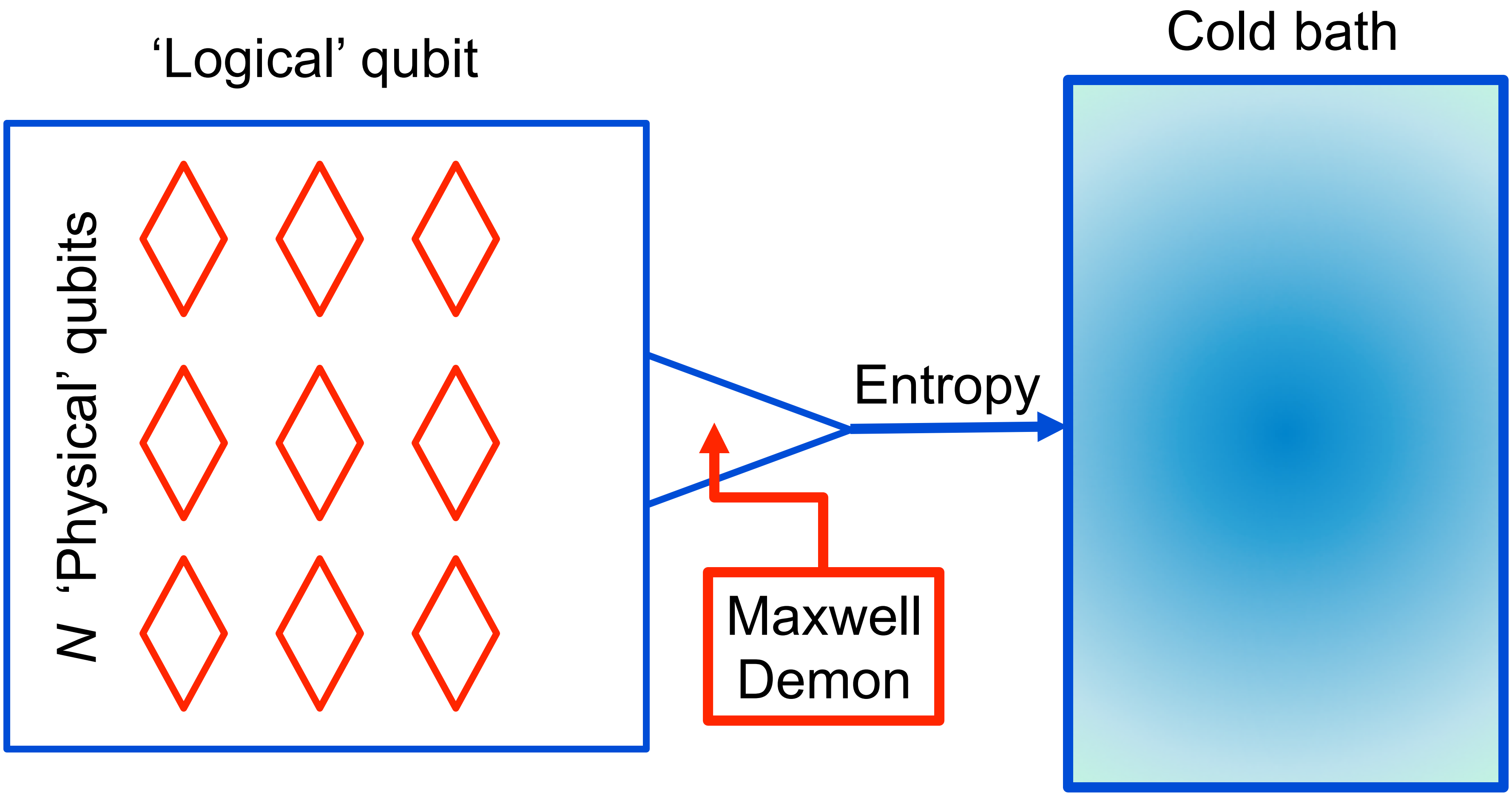
Quantum Error Correction for an unknown state requires storing the quantum information non-locally in (non-classical) *correlations* (entanglement) over multiple physical qubits.

‘Logical’ qubit



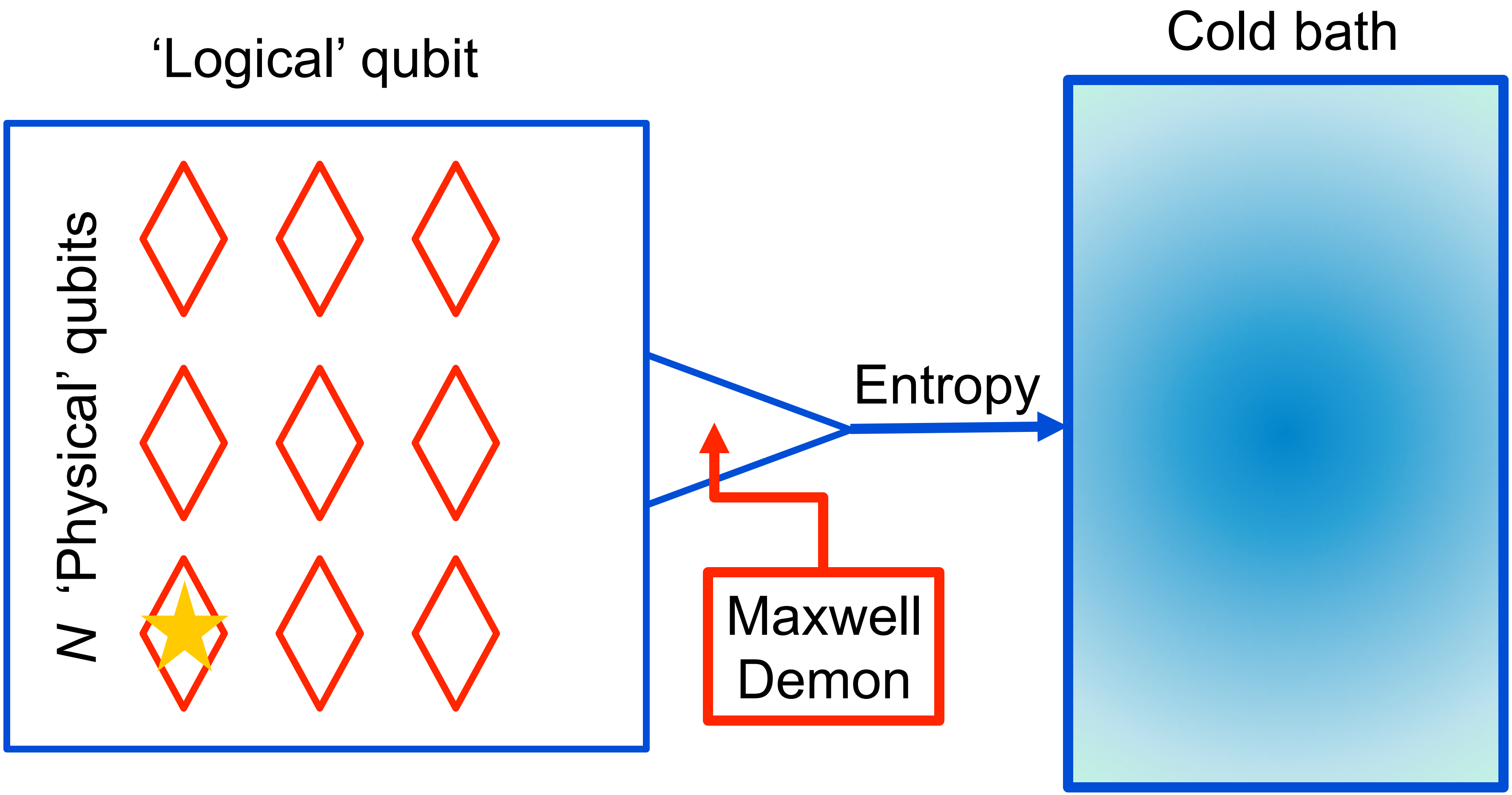
Non-locality: No single physical qubit can “know” the state of the logical qubit.

Quantum Error Correction



N qubits have errors *N* times faster. Maxwell demon must overcome this factor of *N* – *and not introduce errors of its own! (or at least not uncorrectable errors)*

Quantum Error Correction



QEC is an <u>emergent collective</u> phenomenon:
adding N-1 worse qubits to the 1 best qubit gives an improvement!

Let's start with classical error heralding

Classical duplication code: $0 \rightarrow 00 \quad 1 \rightarrow 11$

Herald error if bits do not match.

In	Out	# of Errors	Probability	Herald?
00	00	0	$(1-p)^2$	Yes
00	01	1	$(1-p)p$	Yes
00	10	1	$(1-p)p$	Yes
00	11	2	p^2	Fail

And similarly for 11 input.

Using duplicate bits:

-lowers channel bandwidth by factor of 2 (bad)

-lowers the fidelity from $(1-p)$ to $(1-p)^2$ (bad)

-improves unheralded error rate from p to p^2 (good)

In	Out	# of Errors	Probability	Herald?
00	00	0	$(1-p)^2$	Yes
00	01	1	$(1-p)p$	Yes
00	10	1	$(1-p)p$	Yes
00	11	2	p^2	Fail

And similarly for 11 input.

Quantum Duplication Code

No cloning prevents duplication

$$U\left(\alpha\left|\downarrow\right\rangle+\beta\left|\uparrow\right\rangle\right)\otimes\left|\downarrow\right\rangle=\left(\alpha\left|\downarrow\right\rangle+\beta\left|\uparrow\right\rangle\right)\otimes\left(\alpha\left|\downarrow\right\rangle+\beta\left|\uparrow\right\rangle\right)$$

Unknown quantum state

Ancilla initialized to ground state

Proof of no-cloning theorem:

α and β are unknown; Hence U cannot depend on them.

No such unitary can exist if QM is linear.

Q.E.D.

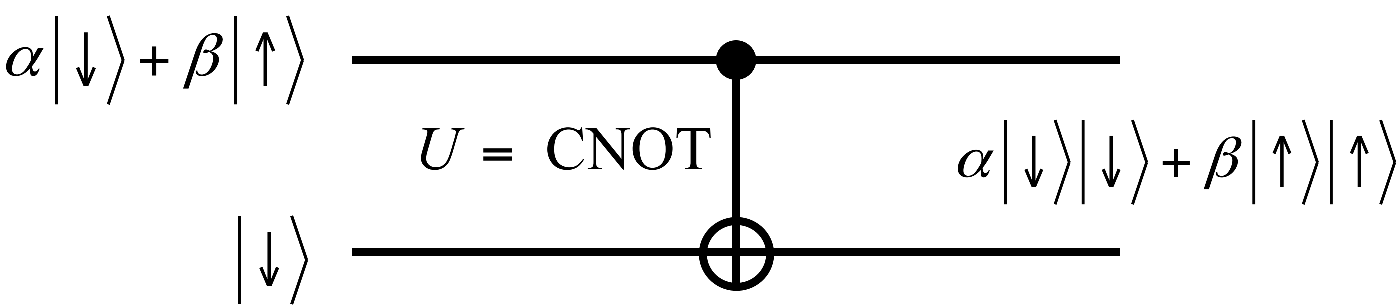
Don’t clone – entangle!

$$U\Big(\underbrace{\alpha|\downarrow\rangle+\beta|\uparrow\rangle}\Big)\otimes\underbrace{|\downarrow\rangle} = \alpha|\downarrow\rangle|\downarrow\rangle+\beta|\uparrow\rangle|\uparrow\rangle$$

Unknown quantum state

Ancilla initialized to ground state

Quantum circuit notation:



Heralding Quantum Errors

$$Z_1, Z_2 = \pm 1$$

Measure the Joint Parity operator: $\Pi_{12} = Z_1 Z_2$

$$\Pi_{12}\left|\uparrow\right\rangle\left|\uparrow\right\rangle = +\left|\uparrow\right\rangle\left|\uparrow\right\rangle$$

$$\Pi_{12}\left|\downarrow\right\rangle\left|\downarrow\right\rangle = +\left|\downarrow\right\rangle\left|\downarrow\right\rangle$$

$$\Pi_{12}\left|\uparrow\right\rangle\left|\downarrow\right\rangle = -\left|\uparrow\right\rangle\left|\downarrow\right\rangle$$

$$\Pi_{12}\left|\downarrow\right\rangle\left|\uparrow\right\rangle = -\left|\downarrow\right\rangle\left|\uparrow\right\rangle$$

$$\Pi_{12}\left(\alpha\left|\downarrow\right\rangle\left|\downarrow\right\rangle + \beta\left|\uparrow\right\rangle\left|\uparrow\right\rangle\right) = +\left(\alpha\left|\downarrow\right\rangle\left|\downarrow\right\rangle + \beta\left|\uparrow\right\rangle\left|\uparrow\right\rangle\right)$$

$\Pi_{12} = -1$ heralds single bit flip errors

Heralding Quantum Errors

$$\Pi_{12} = Z_1 Z_2$$

Not easy to measure a joint operator while not accidentally measuring individual operators!

(Typical 'natural' coupling is $M_Z = Z_1 + Z_2$)

$$\left|\uparrow\right\rangle\left|\uparrow\right\rangle \text{ and } \left|\downarrow\right\rangle\left|\downarrow\right\rangle \text{ are very different,}$$

$$\text{yet we must make that difference invisible}$$

But it can be done if you know the right experimentalists...

Heralding Quantum Errors

Example of error heralding:

$$|\Psi\rangle = \alpha|\downarrow\rangle|\downarrow\rangle + \beta|\uparrow\rangle|\uparrow\rangle$$

Introduce single qubit rotation error on 1 (say)

$$e^{i\frac{\theta}{2}X_1}|\Psi\rangle = \cos\frac{\theta}{2}|\Psi\rangle + i\sin\frac{\theta}{2}X_1|\Psi\rangle$$

Coherent superposition of no error and bit-flip error)

Relative weight of α, β is untouched.

Probability of error: $\sin^2\frac{\theta}{2}$

If no error is heralded, state collapses to $|\Psi\rangle$

and there is no error!

Heralding Quantum Errors

Example of error heralding:

$$|\Psi\rangle = \alpha|\downarrow\rangle|\downarrow\rangle + \beta|\uparrow\rangle|\uparrow\rangle$$

Introduce single qubit rotation error on 1 (say)

$$e^{i\frac{\theta}{2}X_1}|\Psi\rangle = \cos\frac{\theta}{2}|\Psi\rangle + i\sin\frac{\theta}{2}X_1|\Psi\rangle$$

Coherent superposition of no error and bit-flip error)

Relative weight of α, β is untouched.

Probability of error: $\sin^2\frac{\theta}{2}$

If error is heralded, state collapses to $X_1|\Psi\rangle$ and there is a full bitflip error. We cannot correct it because we don't know which qubit flipped.

Heralding Quantum Errors

Quantum errors are continuous (analog!).

But the detector result is discrete.

The measurement back action renders the error discrete (digital!)

– either no error or full bit flip.

Correcting Quantum Errors

Extension to 3-qubit code allows full correction of bit flip errors (only)

$$\left|\Psi\right\rangle = \alpha\left|\downarrow\right\rangle\left|\downarrow\right\rangle\left|\downarrow\right\rangle + \beta\left|\uparrow\right\rangle\left|\uparrow\right\rangle\left|\uparrow\right\rangle$$

$$\Pi_{12} = Z_1 Z_2 \text{ and } \Pi_{32} = Z_3 Z_2$$

Provide two classical bits of information to diagnose and correct all 4 possible bitflip errors:

$$I, X_1, X_2, X_3$$

Correcting Quantum Errors

Extension to 5,7,or 9-qubit code allows full correction of ALL single qubit errors

I (no error)

$\mathrm{X}_1,\ldots,X_N$ (single bit flip)

$Z_1,\ldots,Z_N$ (single phase flip; no classical analog)

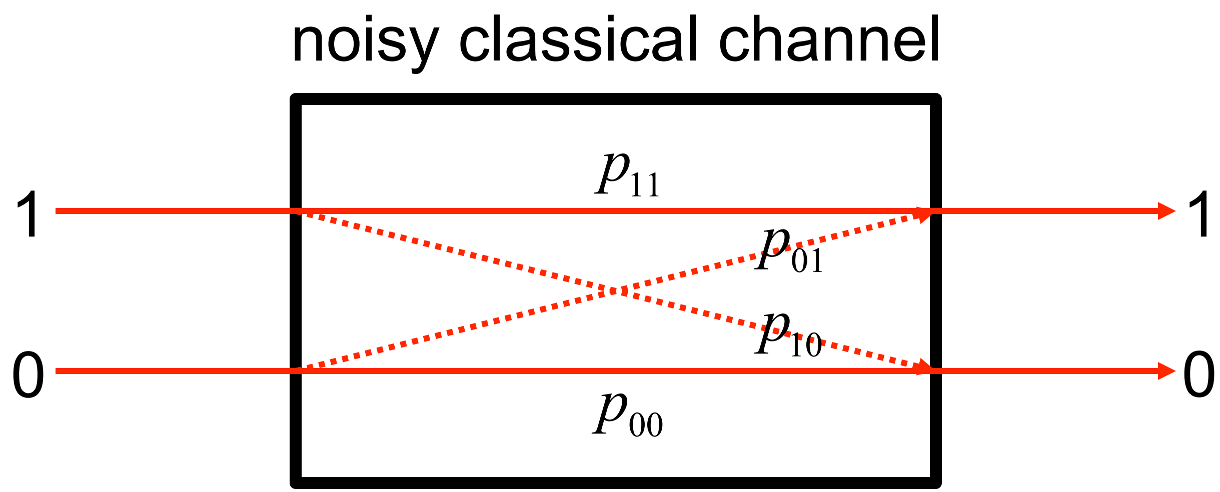
$Y_1,\ldots,Y_N$ (single bit AND phase flip; no classical analog)

For *N=5,* there are 16 errors and 32 states

32= 16 x 2

Just enough room to encode which error occurred and still have one qubit left to hold the quantum information.

Now for the Mathematical Details…



There are only two possible errors for a classical channel:

$1 \rightarrow 0$ with probability p_{10}

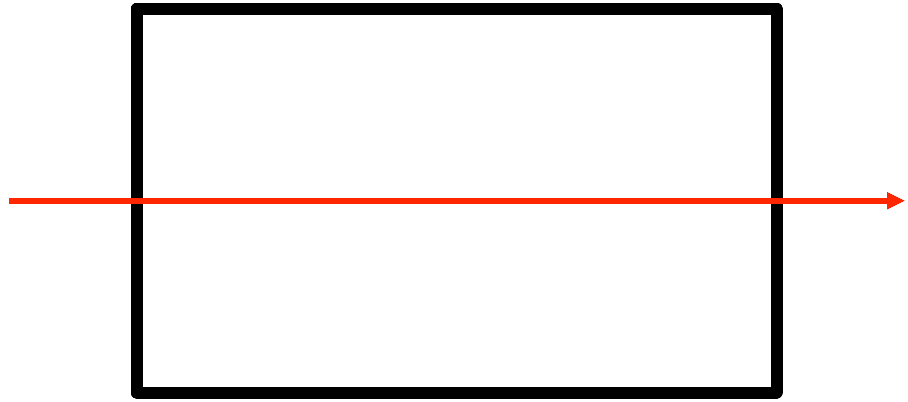
$0 \rightarrow 1$ with probability p_{01}

$$p_{11} + p_{10} = 1$$

$$p_{00} + p_{01} = 1$$

noiseless unitary quantum channel

$|\Psi_{\text{in}}\rangle$



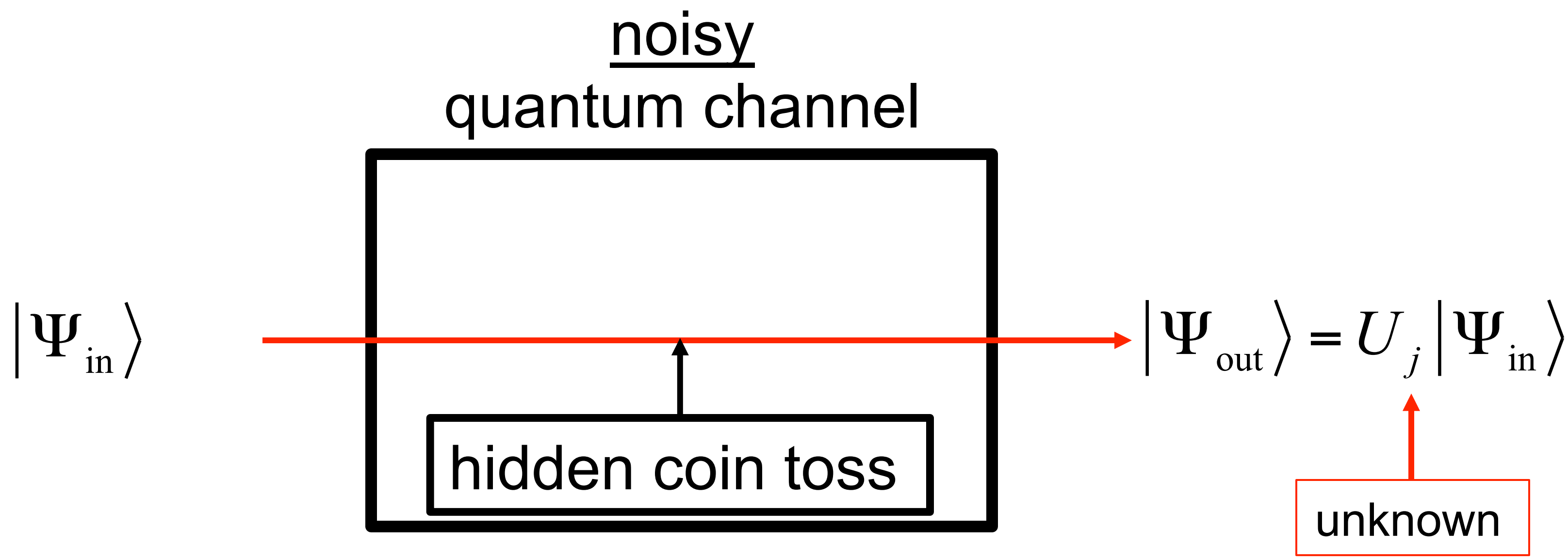
$|\Psi_{\text{out}}\rangle = U|\Psi_{\text{in}}\rangle$

Most general unitary for a single qubit:

$$U = \exp\left\{ i\frac{\theta}{2}\hat{n}\cdot\vec{\sigma} \right\} = \cos\frac{\theta}{2}\hat{I} + i\sin\frac{\theta}{2}\left[n_x\sigma_x + n_y\sigma_y + n_z\sigma_z \right]$$

Coherent superposition of 4 possible errors:

- identity
- bit flip
- bit-phase flip
- phase flip



Random unitaries:

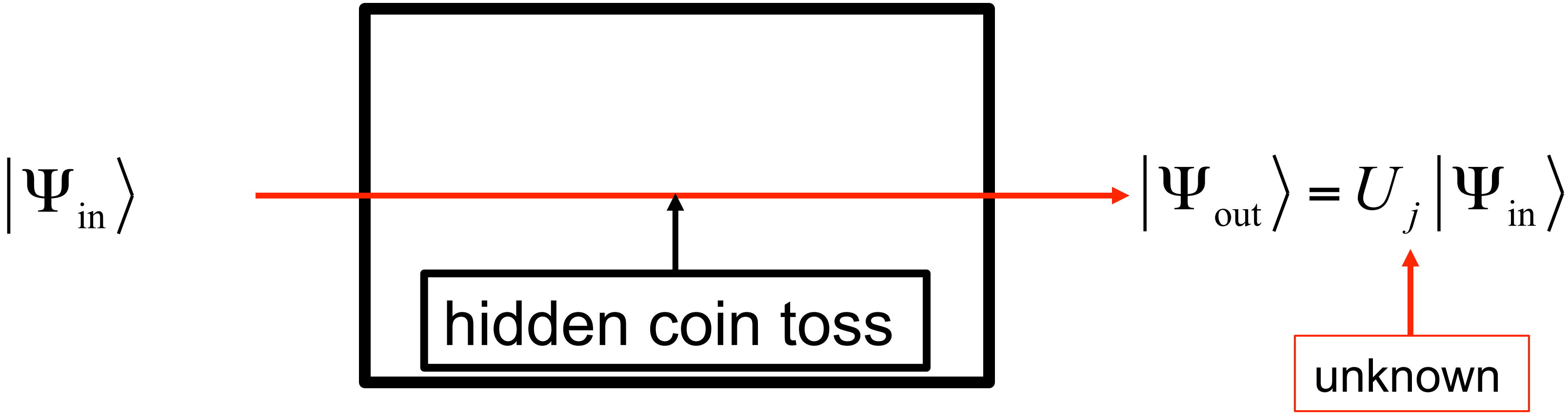
$$\rho_{\text{out}} = \sum_{j=1}^{N} p_j U_j |\Psi_{\text{in}}\rangle\langle\Psi_{\text{in}}| U_j^{\dagger}$$

$$\text{Tr}\rho_{\text{out}} = \sum_{j=1}^{N} p_j = 1$$

More generally:

$$\rho_{\text{out}} = \sum_{j=1}^{N} p_j U_j \rho_{\text{in}} U_j^{\dagger}$$

noisy
quantum channel



Random unitaries: $\rho_{\text{out}} = \sum_{j=1}^{N} p_j U_j \rho_{\text{in}} U_j^{\dagger}$

$$\text{Tr}\{\rho_{\text{out}} \ln \rho_{\text{out}}\} \geq \text{Tr}\{\rho_{\text{in}} \ln \rho_{\text{in}}\}$$

Example: depolarizing channel

$$U_1 = I, \quad p_1 = 1 - 3Ú/4,$$
$$U_2 = \sigma_x, \quad p_2 = Ú/4$$
$$U_3 = \sigma_y, \quad p_3 = Ú/4$$
$$U_4 = \sigma_z, \quad p_4 = Ú/4$$

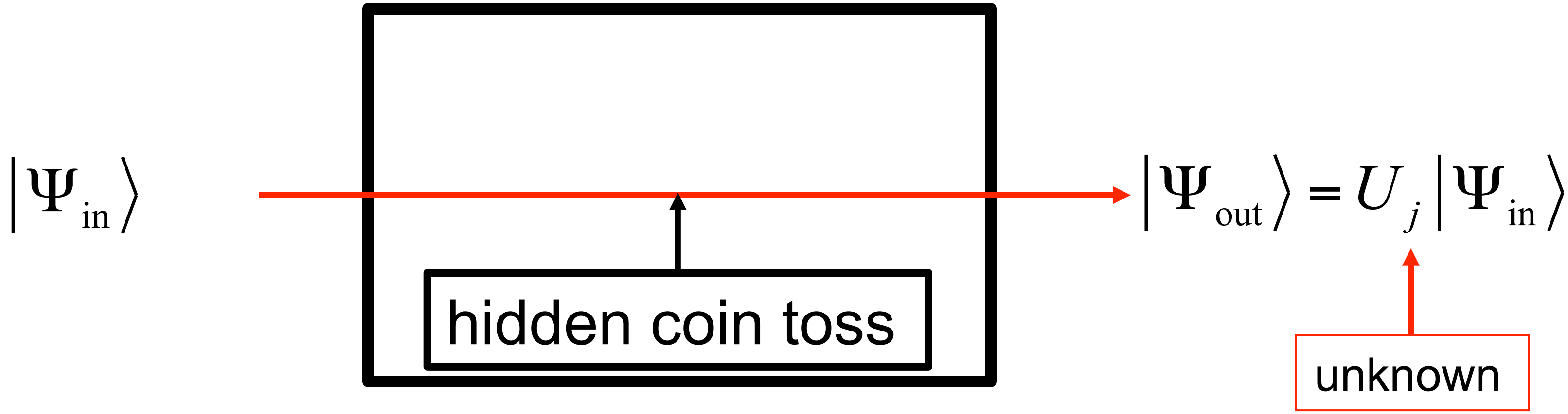
Homework exercise:

$$\rho_{\text{out}} = Ú\left(\frac{I}{2}\right) + (1 - Ú)\rho_{\text{in}}$$

Fully mixed state

Untouched state

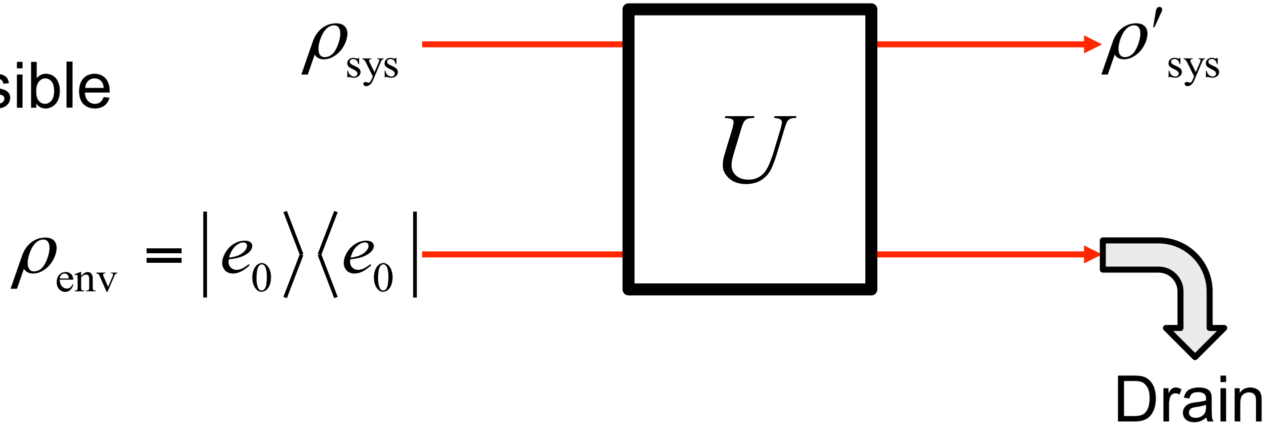
<u>noisy</u> quantum channel



Random unitaries: $\rho_{\text{out}} = \sum_{j=1}^{N} p_j U_j \rho_{\text{in}} U_j^{\dagger}$ $\qquad \text{Tr}\{\rho_{\text{out}} \ln \rho_{\text{out}}\} \geq \text{Tr}\{\rho_{\text{in}} \ln \rho_{\text{in}}\}$

N.B. Random unitaries are <u>not</u> the most general possible quantum channel. (They are always unital, mapping *I* to *I*.)

Most general possible quantum channel:



$$\rho'_{sys} = \mathrm{Tr}_{env}\left\{U\left[\rho_{sys} \otimes \rho_{env}\right]U^\dagger\right\} = \sum_{k=1}^{d^2} E_k \rho_{sys} E_k^\dagger$$

$$\sum_{k=1}^{d^2} E_k^\dagger E_k = I$$

$d =$ dim sys Hilbert space

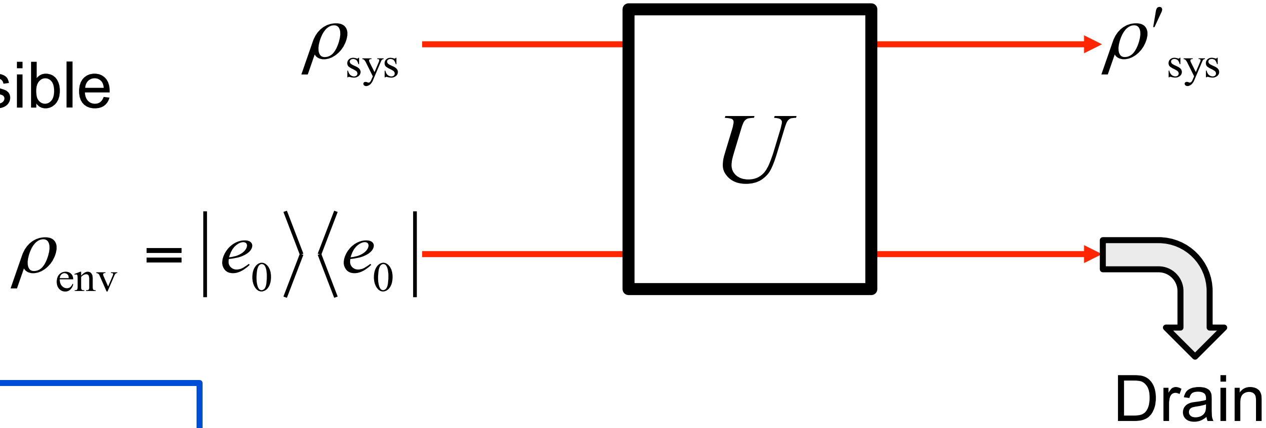
Kraus operators E_k need not be unitary

$\mathrm{E}_k = \langle e_k | U | e_0 \rangle$ is an operator on the system space

dim env Hilbert space need only be* d^2

*See however: *Phys. Rev. B* **95**, 134501 (2017)
where we prove that repeated unitaries and measurements of a single *d*=2 ancilla can synthesize *any quantum channel*

Most general possible quantum channel:



$$\rho'_{\text{sys}} = \sum_{k=1}^{d^2} E_k \rho_{\text{sys}} E_k^\dagger$$

$$\sum_{k=1}^{d^2} E_k^\dagger E_k = I$$

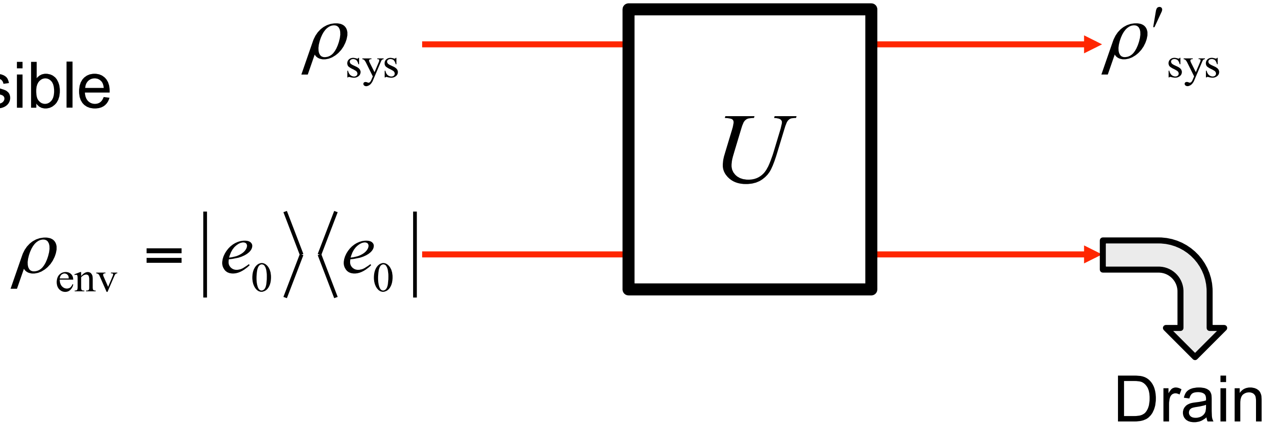
CPTP:
completely positive,
trace-preserving map

Kraus representation is <u>not</u> unique:

$$E_k \rightarrow K_k = S_{km} E_m$$

is equivalent for any unitary mapping S among the errors

Most general possible quantum channel:



$$\rho'_{\text{sys}} = \sum_{k=1}^{d^2} E_k \rho_{\text{sys}} E_k^\dagger$$

$$\sum_{k=1}^{d^2} E_k^\dagger E_k = I$$

Arbitrary channel can *__decrease__ the entropy!*

Example:
"Reset channel"

If $E_k = |1\rangle\langle k|$

then $\rho'_{\text{sys}} = |1\rangle\langle 1|$

An arbitrary quantum channel is the most general possible operation on a quantum system.

Therefore <u>if quantum error correction is possible</u>, it can be performed via a quantum channel

$$\rho'_{\text{sys}} = \sum_{k=1}^{d^2} E_k \rho_{\text{sys}} E_k^\dagger$$ ‘error map’

$$\rho_{\text{sys}} = \sum_{k=1}^{d^2} R_k \rho'_{\text{sys}} R_k^\dagger$$ ‘recovery map’

Let the ‘system’ be N physical qubits. A logical qubit encoded in sys consists of two orthogonal ‘words’ in the Hilbert of sys

$$\text{code} = \text{span}\left\{|W_0\rangle, |W_1\rangle\right\}$$

$$P_{\text{code}} = |W_0\rangle\langle W_0| + |W_1\rangle\langle W_1|$$

Knill-Laflamme condition

A recovery map for a set of errors $\left\{E_1, E_2, \ldots, E_N\right\}$ exists if

$$P_{\text{code}} E_i^{\dagger} E_j P_{\text{code}} = \alpha_{ij} P_{\text{code}}$$

where α is a Hermitian matrix.

Knill-Laflamme condition

A recovery map for a set of errors $\{E_1, E_2, \ldots, E_N\}$ exists if

$$P_{\text{code}} E_i^\dagger E_j P_{\text{code}} = \alpha_{ij} P_{\text{code}}$$

where α is a Hermitian matrix.

"Proof:" Let $S\alpha S^\dagger = d$ diagonalize α. Let $K = SE$.

$$P_{\text{code}} K_i^\dagger K_j P_{\text{code}} = d_{ij} P_{\text{code}}$$

Different error states are orthogonal and hence identifiable by measurement of the projector

$$\Pi_j = \frac{K_j P_{\text{code}} K_j^\dagger}{d_{jj}}, \quad (\Pi_j)^2 = \Pi_j$$

Given knowledge of which error occurred, there exists a unitary map from the error state back to the original state in the code space.

Errors can be non-unitary (increase entropy)
But Knill-Laflamme condition says we can correct them with unitaries if the choice of unitary is conditioned on measurement result.

$$\Pi_j = \frac{K_j P_{\text{code}} K_j^\dagger}{d_{jj}}, \quad (\Pi_j)^2 = \Pi_j$$

Next up: Quantum Error Correction Codes for Bosonic Modes (microwave photons)