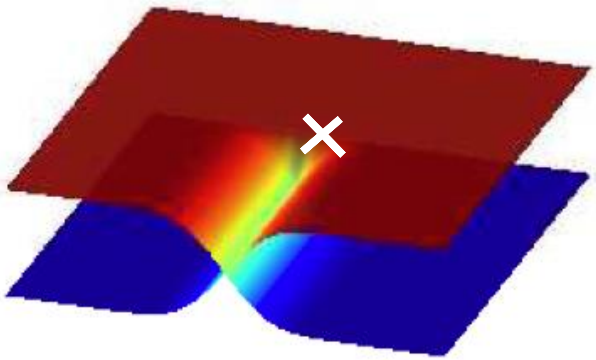
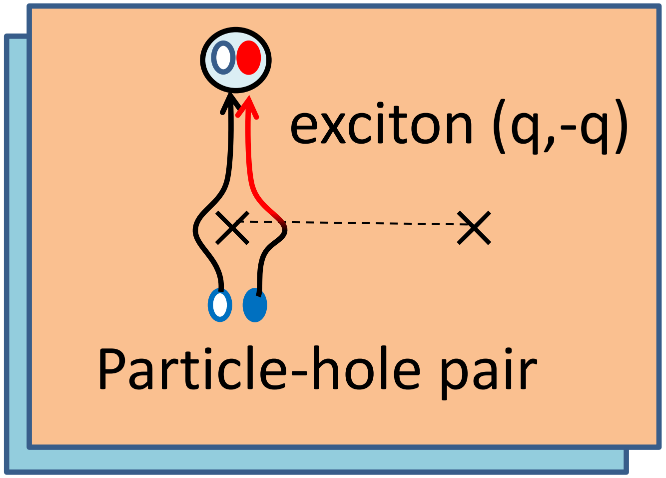
Lecture 2

Realizations of genons and twist defects

A sketch of the first lecture

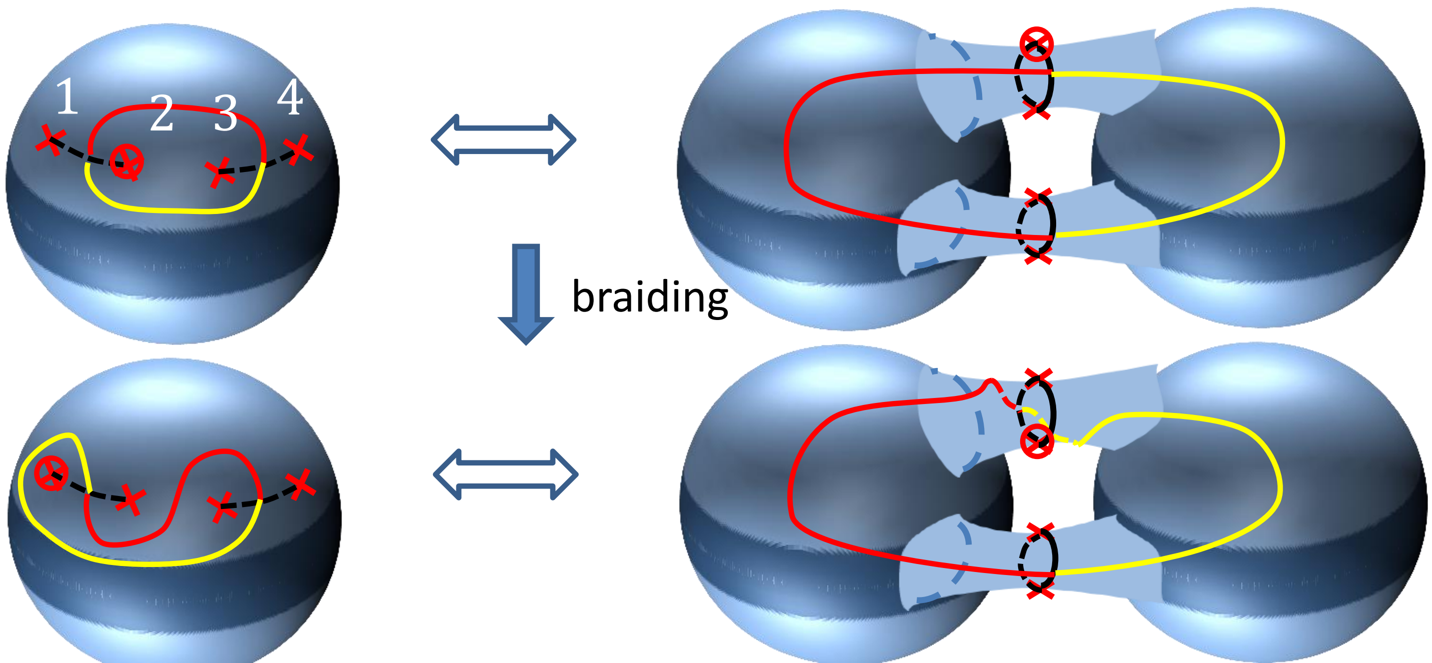


genons as the end of branchcut line



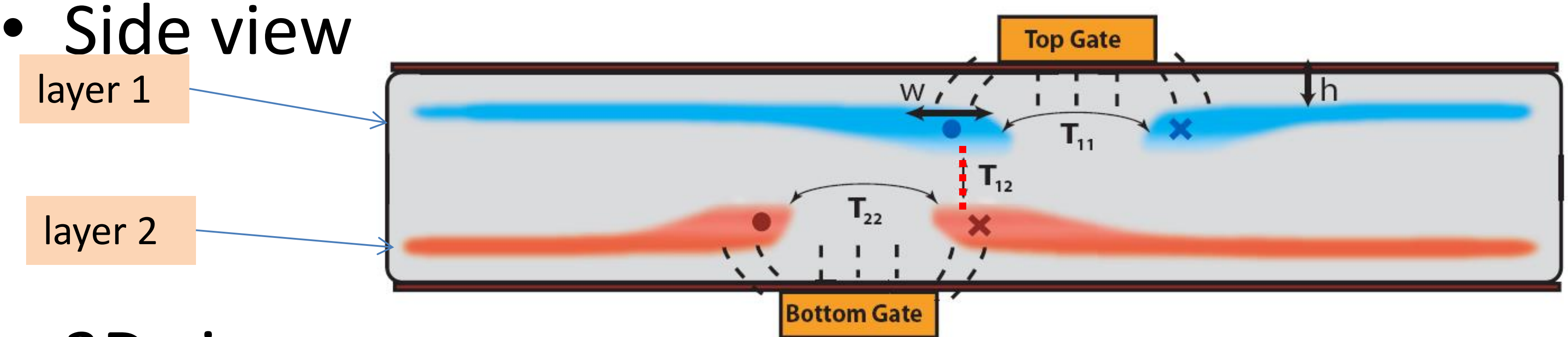
Parafermion zero modes and $\sqrt{m-l}$ quantum dimension

Non-Abelian statistics

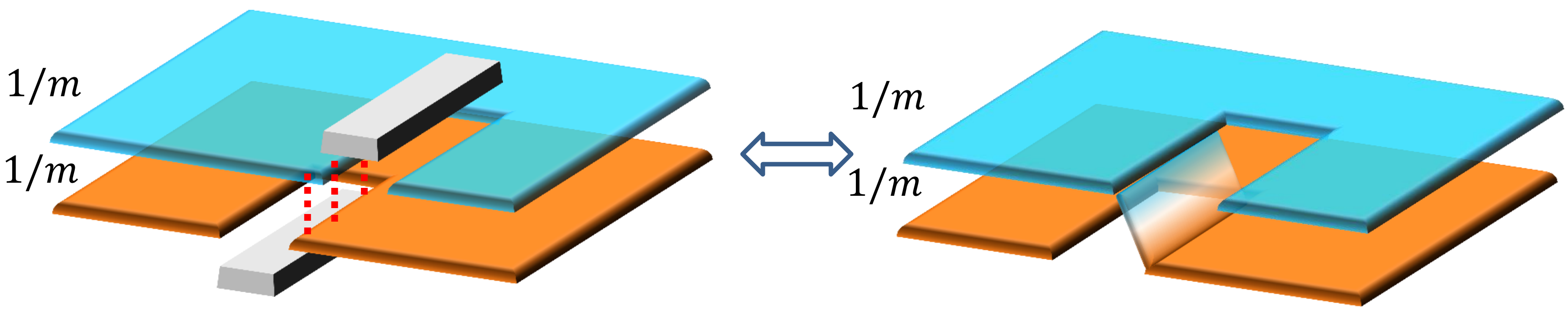


Realization 1: Bilayer FQH states

- Bilayer FQH states with top and bottom gates
- Inter-layer edge state tunneling
- Side view



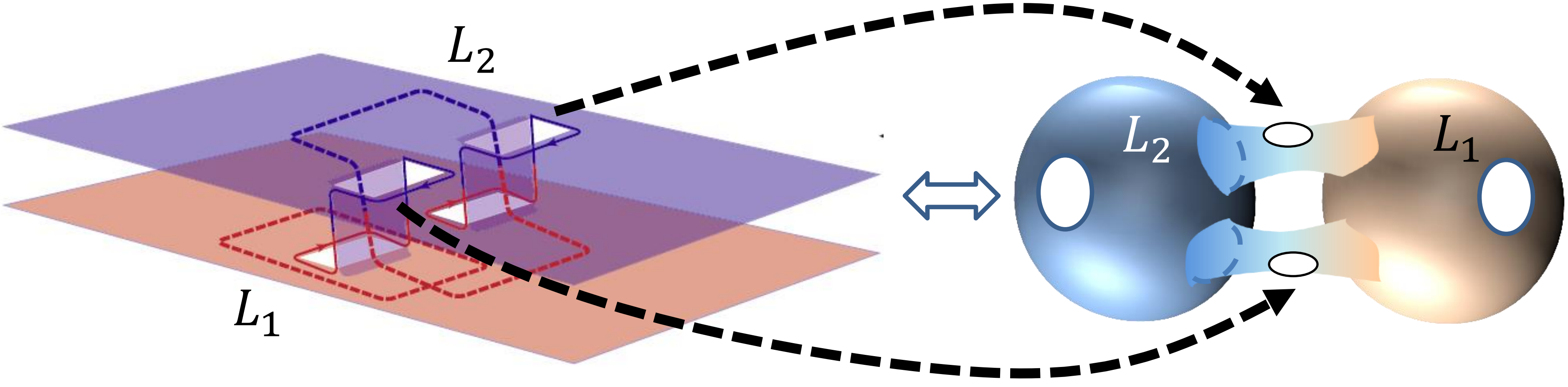
- 3D view



Barkeshli & XLQ, '13

Realization 1: Bilayer FQH states

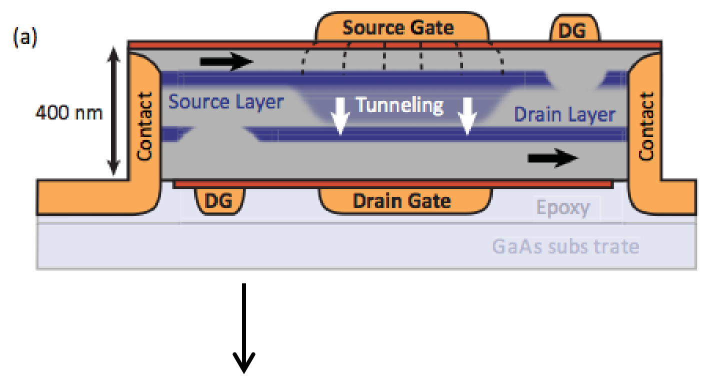
- With proper distance between two layers, it is possible to induce relevant inter-edge tunneling over the bridge, so that the two layers are connected by a "staircases".
- Staircases are different from branch-cut lines, but good enough for changing topology. Adding a staircase (one pair of gates) adds genus by 1.➔One staircase = 2 genons
- Uncoupled edges do not destroy topological protection.



Realization 1: Bilayer FQH states

- Conditions/Requirements:
  ⋆ Separate depletion of top and bottom layers
  ⋆ Strong inter-edge interaction making the inter-edge tunneling relevant

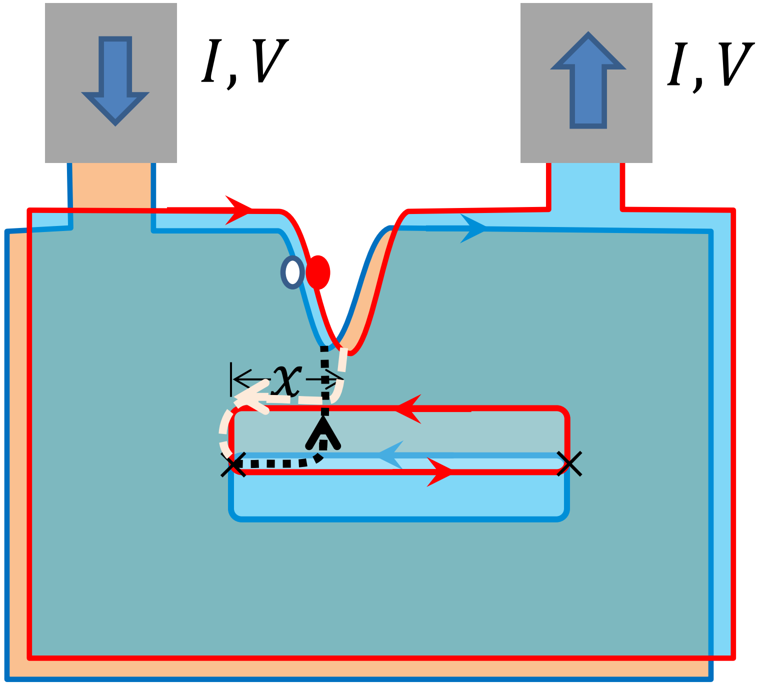
  Dimensionless interaction strength $1 > \lambda > \frac{m^2-4}{m^2+4}$
  ⋆ Inter-layer tunneling t large enough to gap the edge states, but much smaller than the bulk gap
- Separate depletion of two layers by top and bottom gates, and tunned tunneling in bilayer system has been demonstrated



(Goldhaber-Gordon group, ’10,’11)

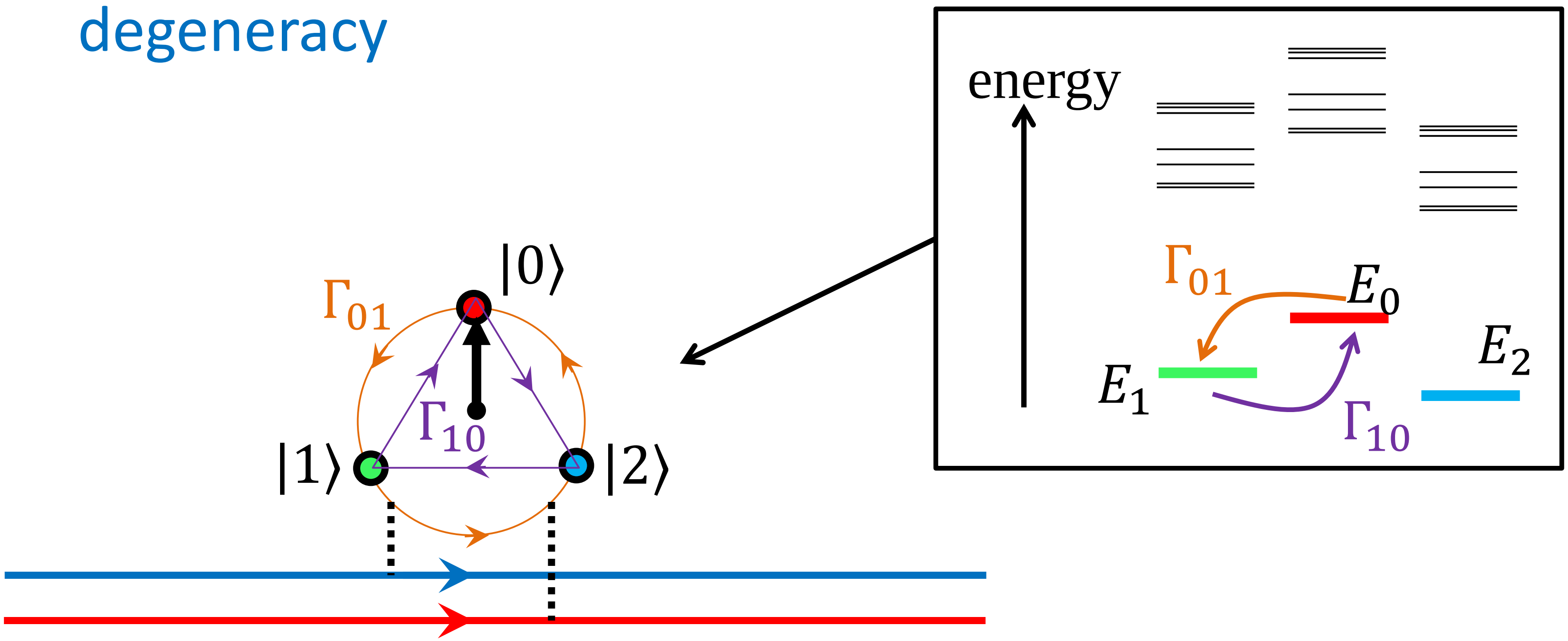
Experimental consequences (1): Zero bias peak

- Tunnelling current in the inter-layer channel
- Tunnelling of $(q, -q)$ type quasi-particles
- Edge plays the role of an "STM tip"
- Only when the tip position is at the end of the staircase, a zero bias peak appears.
- Each end of the staircase is a local "parafermion" zeromode.
- The zero mode is exponentially localized, even if there are gapless edges



Experimental consequences (1): Zero bias peak

- Finite size effect leads to an exponential splitting between the topological ground states
- Multiple peaks in the tunneling conductance
- An experimental probe of topological ground state degeneracy

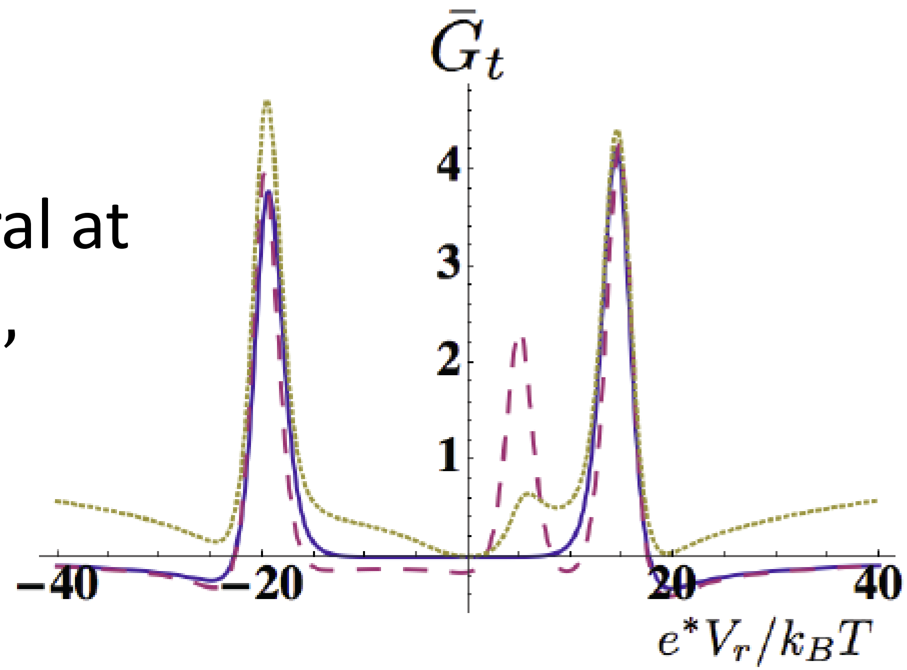


Experimental consequences (1): Zero bias peak

- Tunneling conductance can be calculated from a master equation approach for the three-state rotor model (for bilayer Laughlin 1/3 state)

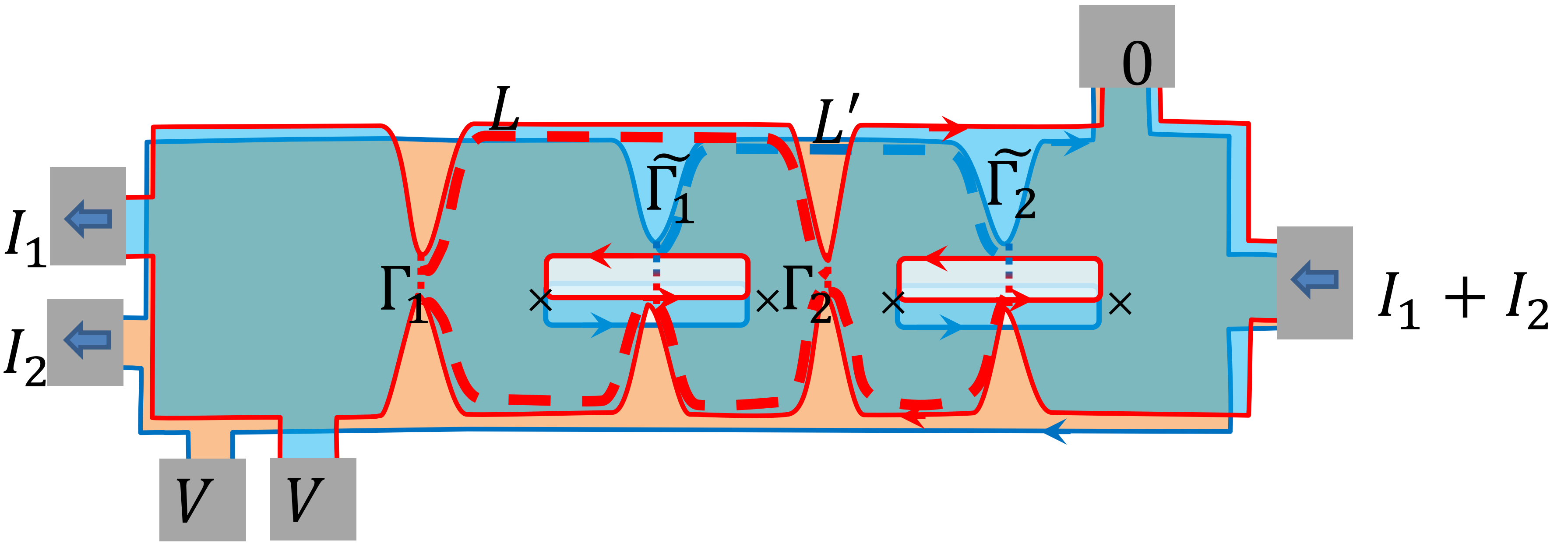
$$\frac{dp_n}{dt} = \sum_{l=1}^{m-1} [\Gamma_{n+l,n} p_{n+l} - \Gamma_{n,n+l} p_n] = 0,$$

- Three peaks exist in general at energies $E_2 - E_1, E_3 - E_2$, $E_1 - E_3$. (Barkeshli&Oreg&Qi in preparation)



Experimental consequences (2): Quantum interference

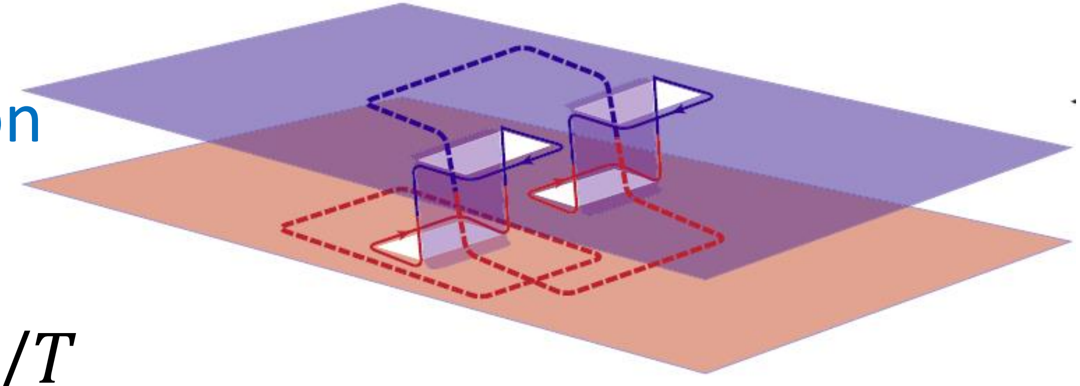
- Two staircases. Four QPC's. Two non-commuting interference loops L_1, L_2



- Quasi-particle tunneling at $\widetilde{\Gamma}_1$ changes the topological charge in $\Gamma_1\Gamma_2$ loop.

Experimental consequences (2): Quantum interference

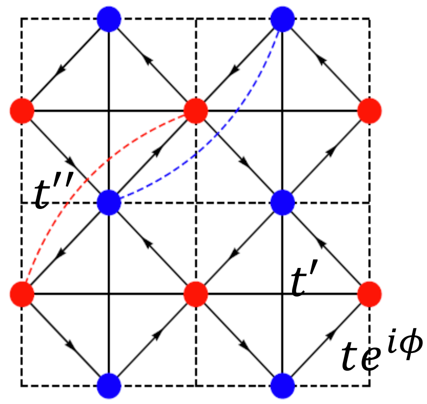
- Current noise cross correlation
  $S_{12}(t) = \frac{1}{2}\langle\{I_1(t), I_2(0)\}\rangle - \langle I_1(t)\rangle\langle I_2(0)\rangle$
- A quasi-particle tunneling in loop 1 changes the charge in loop 2 permanently

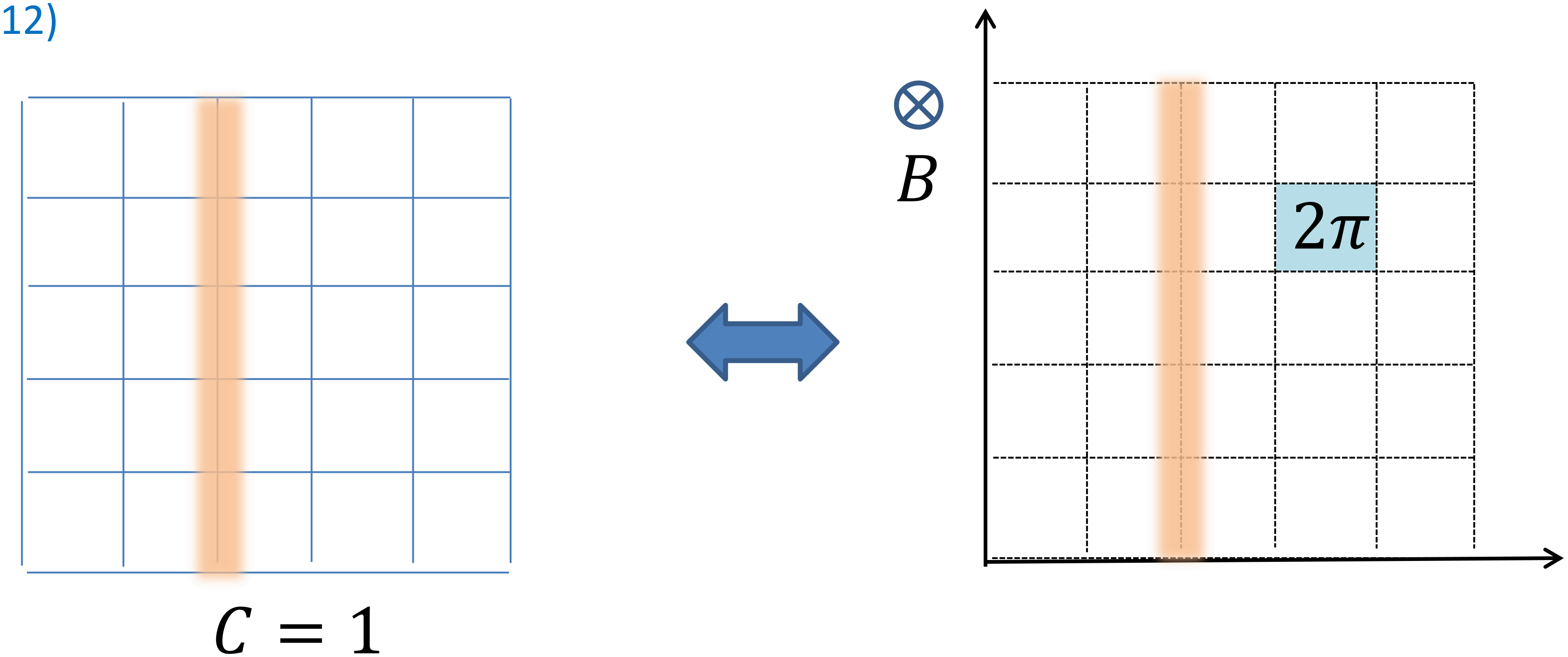


- ➔ Long time correlation even at finite temperature
  $S_{12}(t) \neq 0$ for $|t| \gg 1/T$
  (but $|t| \ll \tau$ the exponentially long life time of the topological states.) The nonlocal contribution is proportional to $\Gamma_1\Gamma_2\widetilde{\Gamma_1}\widetilde{\Gamma_2}$.

Realization 2: Fractional Chern Insulators

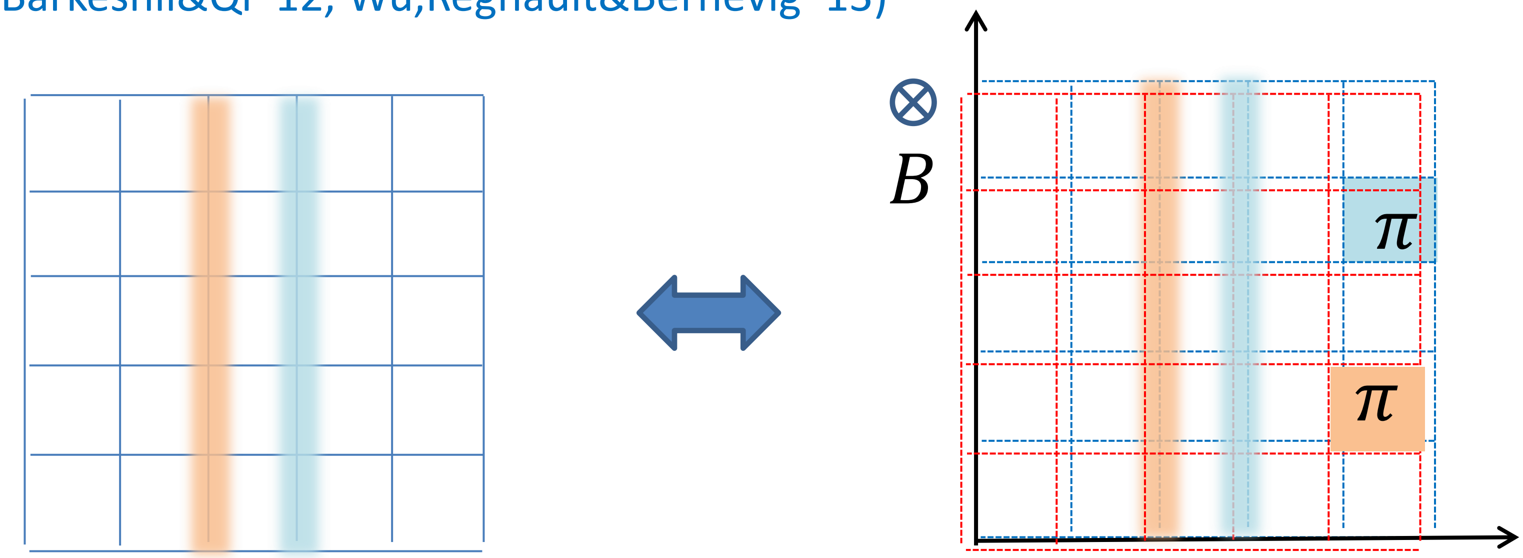
- Fractional Chern insulators (FCI) are lattice FQH states with no magnetic field (Sun et al, Neupert, et al, Tang et al, PRL 2011)
- Fractional Chern insulators can be mapped to fractional quantum Hall states (XLQ '11, see also Scaffidi&Moller, arxiv '12, Wu,Regnault&Bernevig, PRB '12, Liu&Bergholtz arxiv' 12)



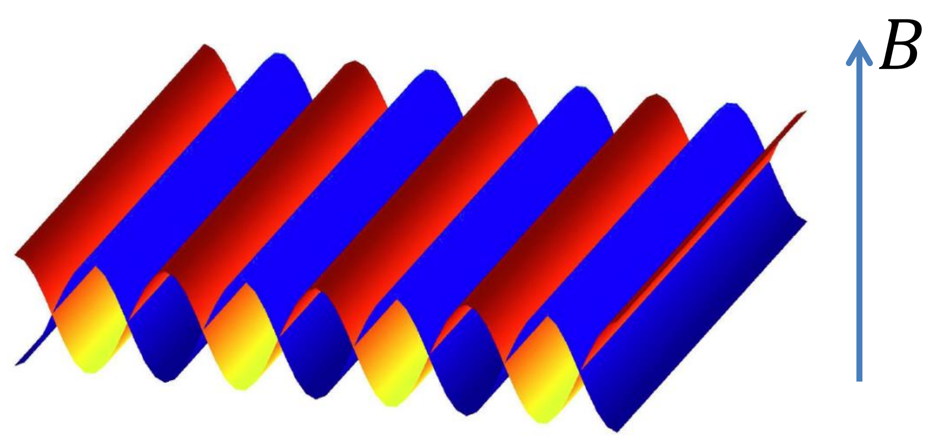


FCI with higher Chern number

- FCI with a Chern number 2 band are mapped to bilayer FQH states, with the two layers “nested” w/ each other
  (Barkeshli&Qi ‘12, Wu,Regnault&Bernevig ‘13)



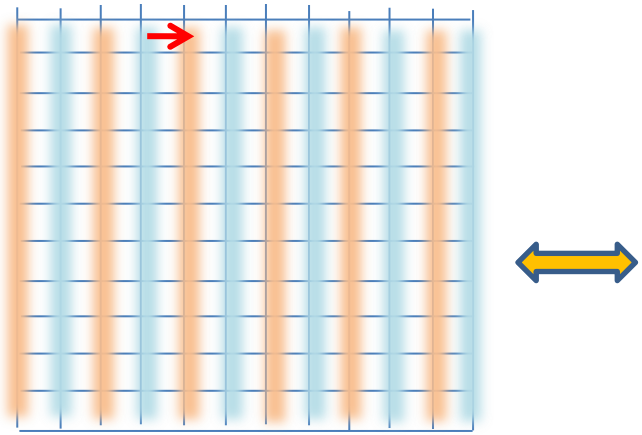
$C = 2$

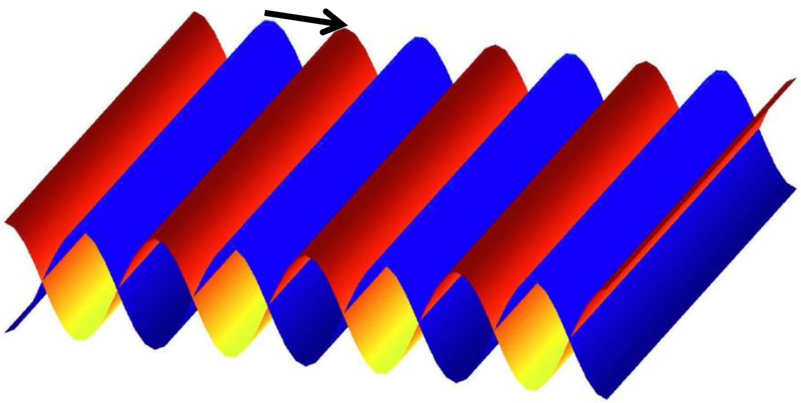


- Realizing bilayer FQH states with an enhanced translation symmetry.

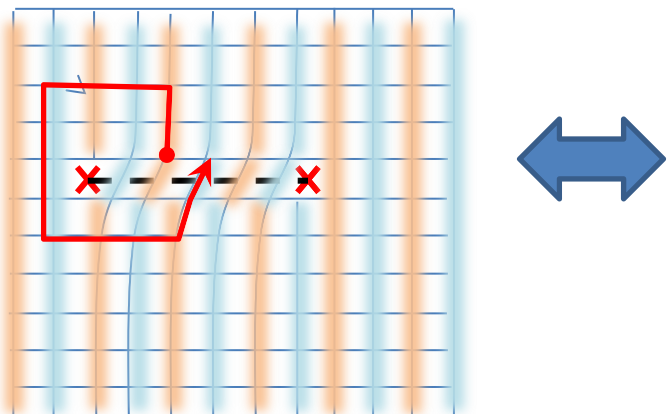
Topological nematic states

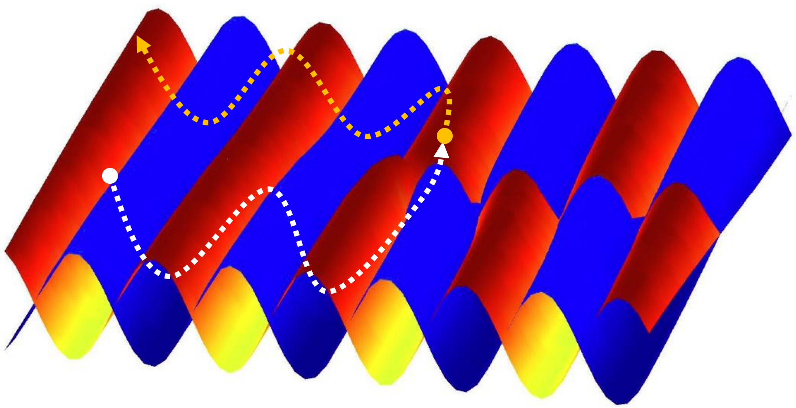
- Lattice translation exchanges the two layers





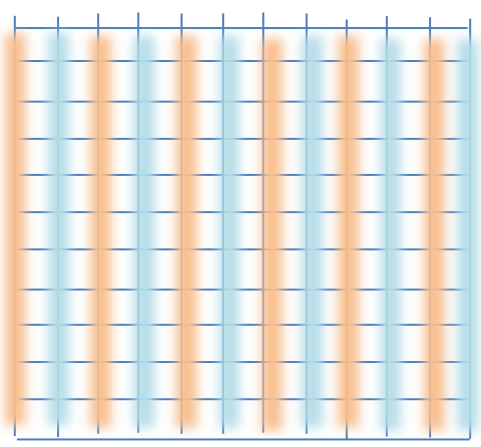
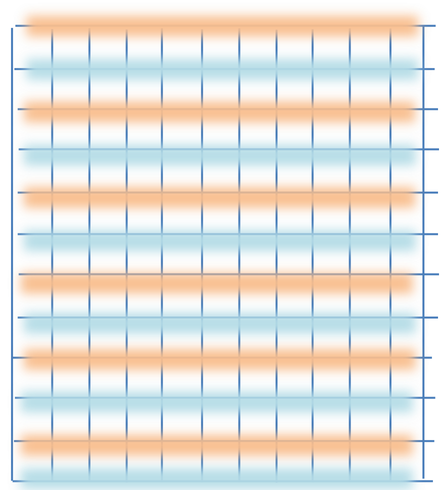
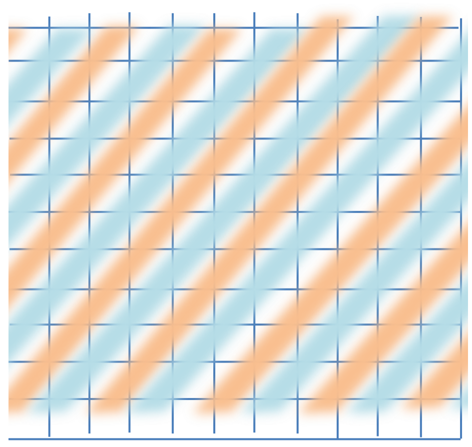
- ➔Branch cut between the two layers can be created by lattice dislocation! Rotation symmetry is broken "topologically". This state is called a topological nematic state





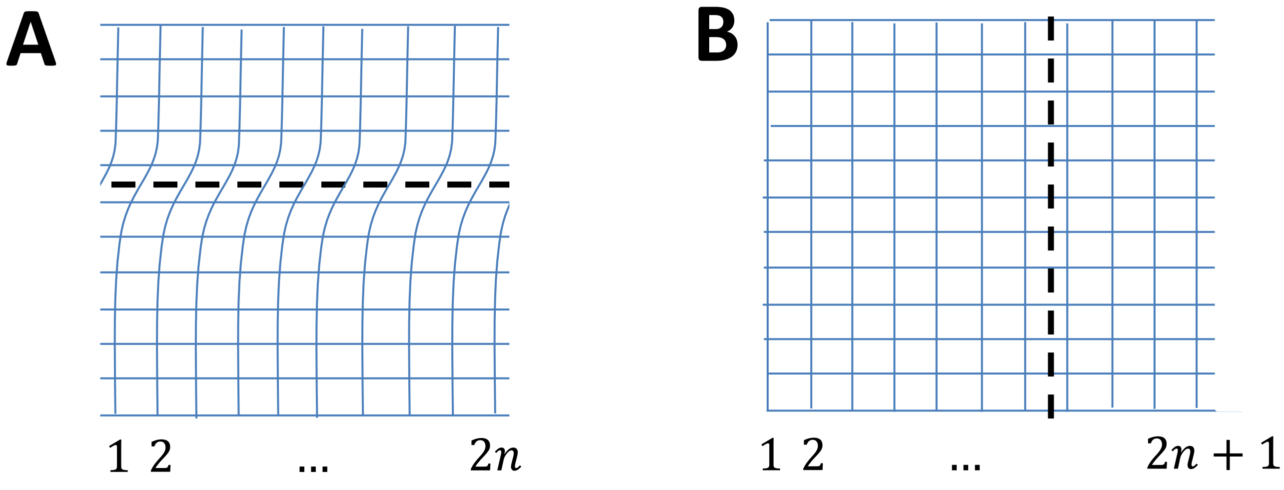
Topological nematic states

- Dislocations become genons.
- Advantages of this realization: genons are point defects with log interaction.
- Z_N generalization can be done for Chern number N.
- Three types of topological nematic states

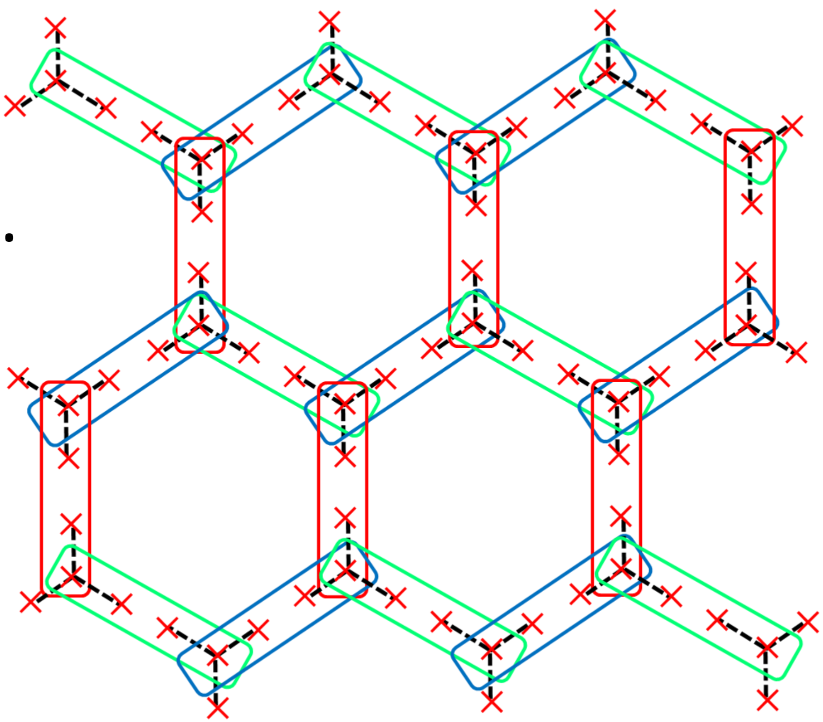
Numerical probe of topological nematic states

- Ground state degeneracy depending on system size and twisted boundary condition.
- For even by odd lattice (B), or lattice with a twist (A), the ground state degeneracy is reduced from $|m^2 - l^2|$ to $|m + l|$. (Barkeshli&Qi PRX '12)
- Verified recently in exact diagonalization (Wu&Jain&Sun 1309.1698)



An "application" of genons: generalized Kitaev model

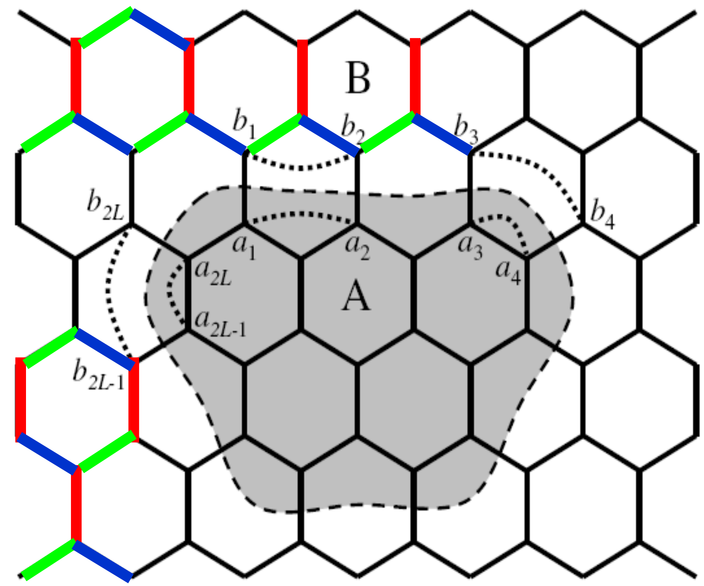
- Twist defects carry parafermion zero modes, which are generalizations of Majorana zero modes.
- Twist defects can be used as "slave particles" which are useful for solving certain spin models.
- Goal: obtain spin models with non-Abelian topologically ordered phases.



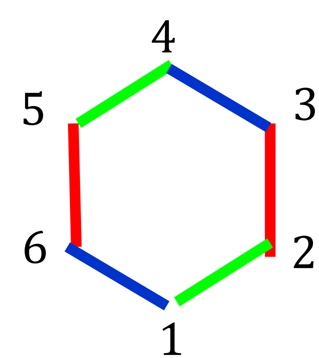
Hong-Chen Jiang, Maissam Barkeshli, Ronny Thomale&XLQ, in preparation.

Kitaev's honeycomb model

$$H = -\sum_{x-link} J_x \sigma_i^x \sigma_j^x - \sum_{y-link} J_y \sigma_i^y \sigma_j^y \text{ - } \sum_{z-link} J_z \sigma_i^z \sigma_j^z$$



- An exact solvable spin model. (Kitaev '06)
- Conserved quantities on each plaquette $O_I = \prod_{\langle ij \rangle \in I} H_{ij}$ $= \sigma_{1z}\sigma_{2y}\sigma_{3x}\sigma_{4z}\sigma_{56}\sigma_{6x}$
- $[H, O_I] = 0$
- For fixed value of O_I, 2 residual states per unit cell

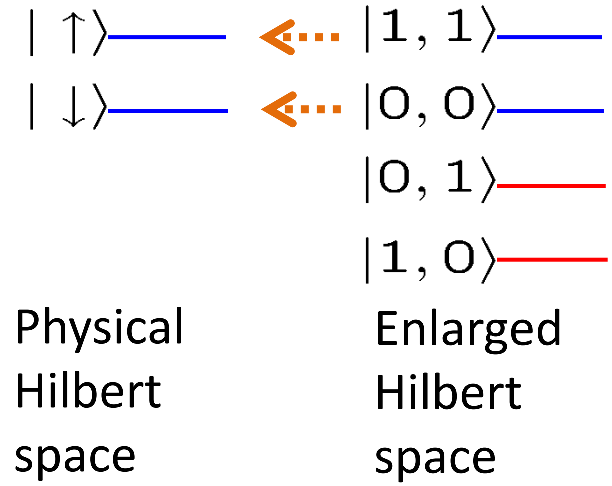


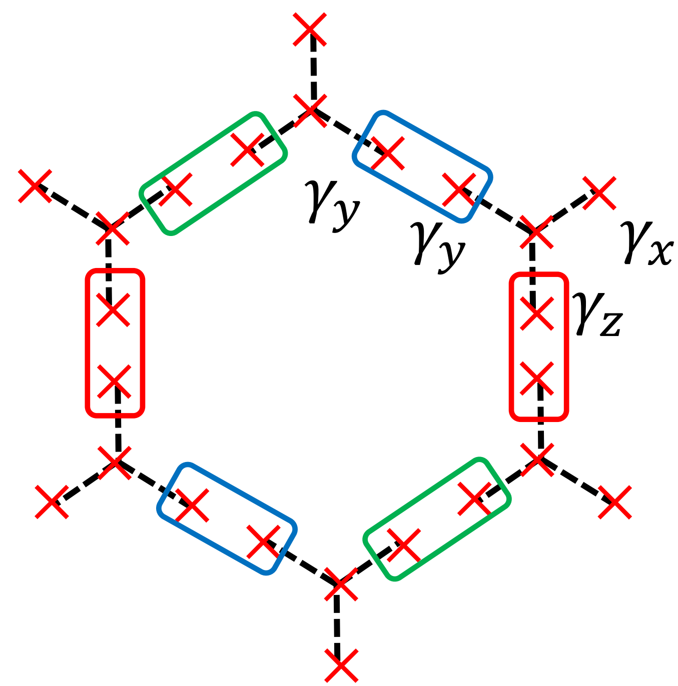
Kitaev's honeycomb model

- Majorana representation
- The spins can be written as bilinear form of the Majorana fermions:

$$\sigma_i^{x,y,z} = i\gamma_i^{x,y,z}\eta_i$$

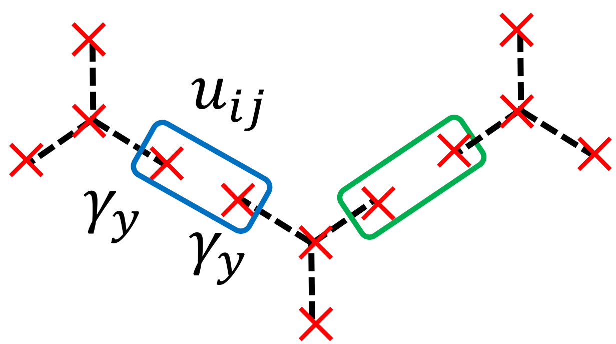
- A local constraint $\gamma_i^x \gamma_i^y \gamma_i^z \eta_i = 1$ projects the Majorana fermion Hilbert space back to the spin Hilbert space.
- In Majorana fermions, the conserved quantity is $O_I = \prod_{\langle ij \rangle \in I} i\gamma_i^{\alpha_{ij}} \gamma_j^{\alpha_{ij}}$

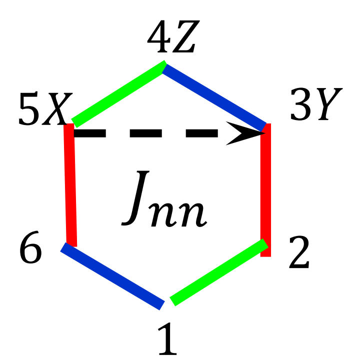




Kitaev's honeycomb model

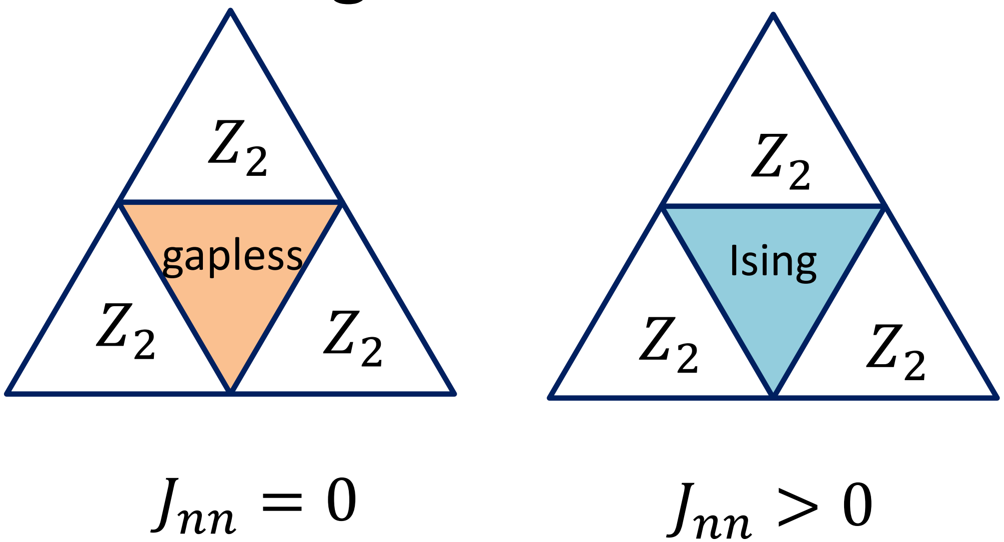
- In the enlarged Hilbert space, the Majorana fermions $i\gamma_i^{\alpha_{ij}}\gamma_j^{\alpha_{ij}} = u_{ij}$ is classical, and η_i fermions are free (with a quadratic Hamiltonian)
- $H = \sum_{ij} u_{ij} J_{ij} i\eta_i\eta_j$
- More generic models can be defined by adding terms of the form $H_{ij}H_{jk}$ or similar terms along longer chains, which translates to $\eta_i\eta_k u_{ij}u_{jk}$.

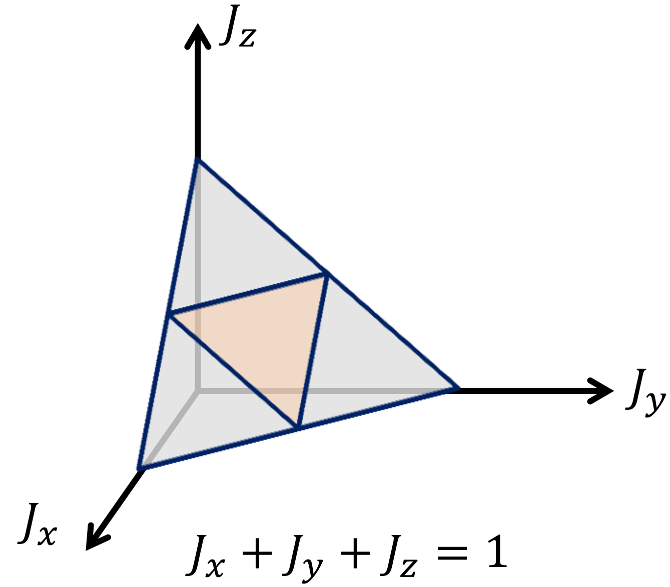




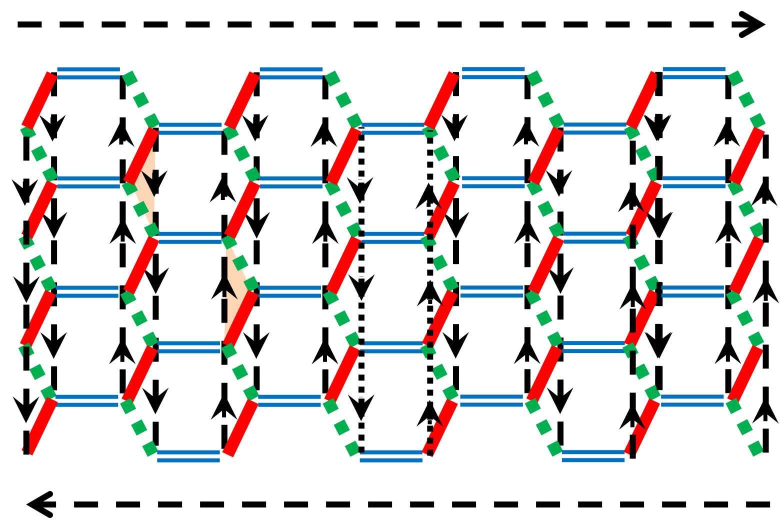
Kitaev's honeycomb model

- Phase diagram



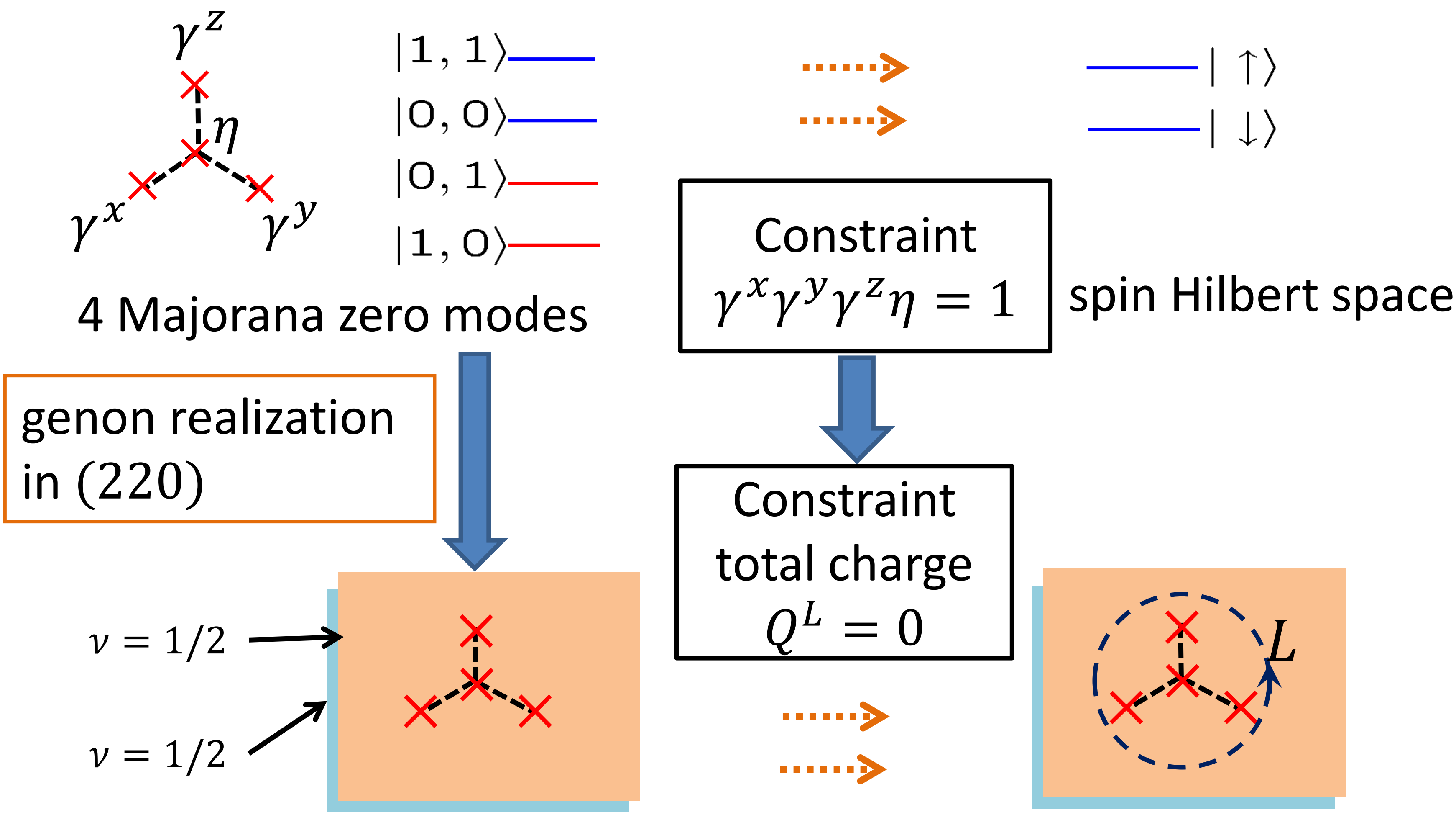


- Non-Abelian Ising anyon phase appears with finite J_{nn}
- Chiral Majorana edge states
- Plaquette operator O_I is the Z_2 π flux.



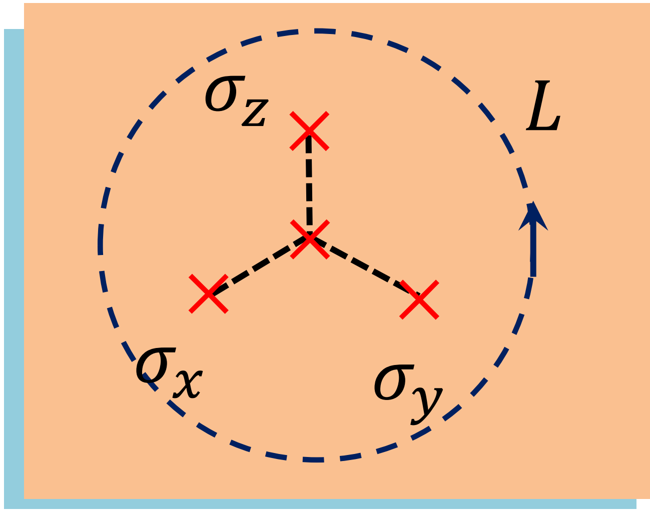
Genon realization of the Kitaev model

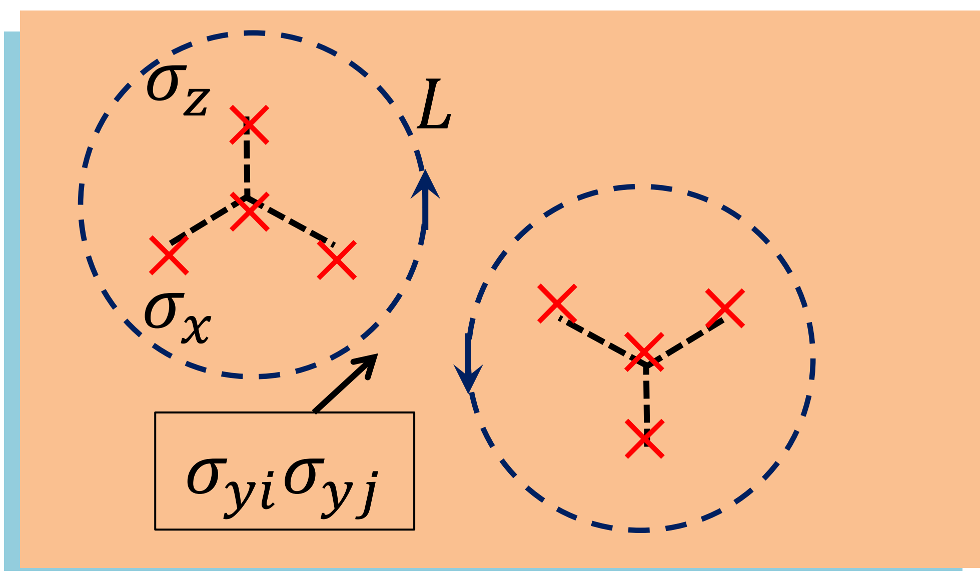
- Genons/twist defects can be used to realize the Majorana representation of the Z_2 Kitaev model.



Genon realization of the Kitaev model

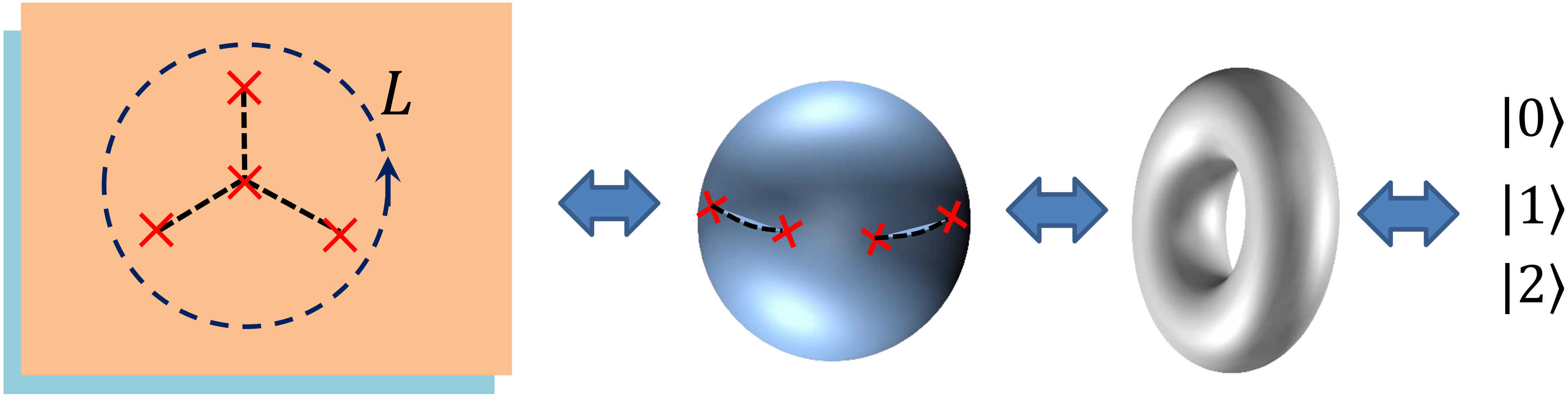
- Spin operators ➔ Wilson loop operators around 2 genons
- Interaction terms in the Hamiltonian ➔ Wilson loop operators around 4 genons
- All terms commute with the constraints





Build the Z_n Kitaev model with genons

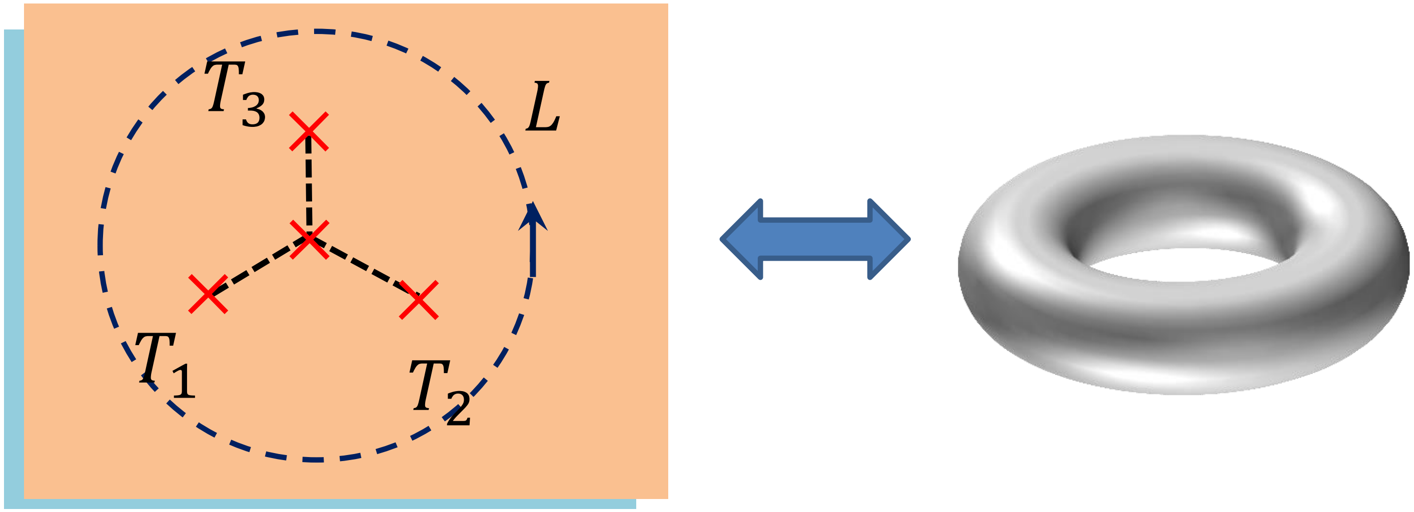
- Once the Majorana zero modes are represented by genons, the model can be easily generalized to parafermionic genons
- Example: 4 genons in (330) state (Laughlin $1/3$ in each layer).
- Degeneracy 3^2 ➔ Adding constraint that the particle type at loop L to be trivial➔ 3 states, equivalent to 1/3 Laughlin state on a torus



Build the Z_n Kitaev model with genons

- The spin operators $T_{1,2,3}$ can be realized by Wilson loops. Each pair of Wilson loops have one crossing, leading to the algebra

$$T_i T_j = \omega T_j T_i, i < j, \omega = e^{\frac{i2\pi}{n}}$$



Build the Z_n Kitaev model with genons

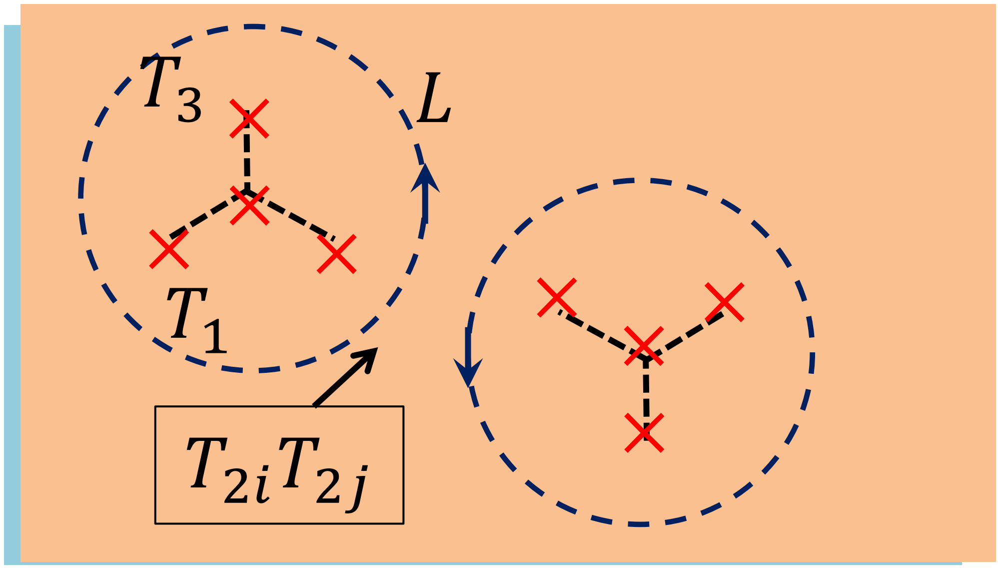
- Each site of the spin model is a Z_n rotor.
- For $n=3$, the explicit form is ($\omega = e^{i2\pi/3}$)

$$T_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}, T_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, T_3 = \begin{pmatrix} 0 & 0 & \omega^2 \\ 1 & 0 & 0 \\ 0 & \omega & 0 \end{pmatrix}$$

- Hamiltonian has the same form as Z_2 case

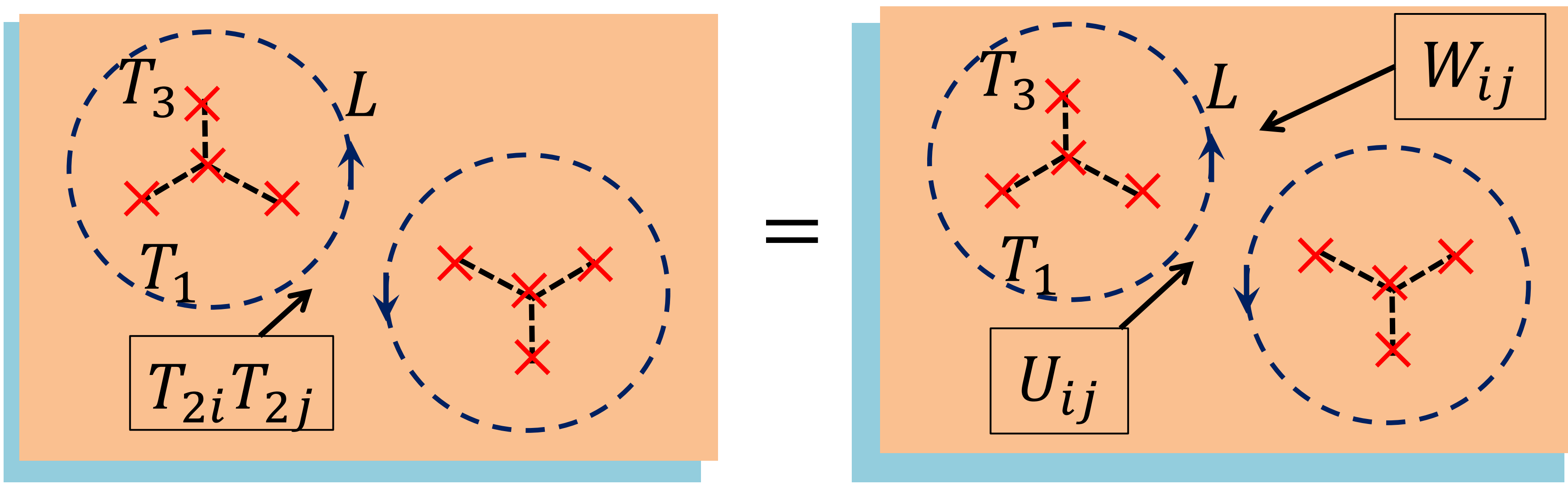
$$H = -\sum_{x-link} J_x T_i^1 T_j^1 - \sum_{y-link} J_y T_i^2 T_j^2 - \sum_{z-link} J_z T_i^3 T_j^3 + \mathrm{h.c.}$$

- Hamiltonian can be realized by Wilson loop operators
- Related to the anyon lattice models (Ludwig et al)



Emergent Z_n gauge field

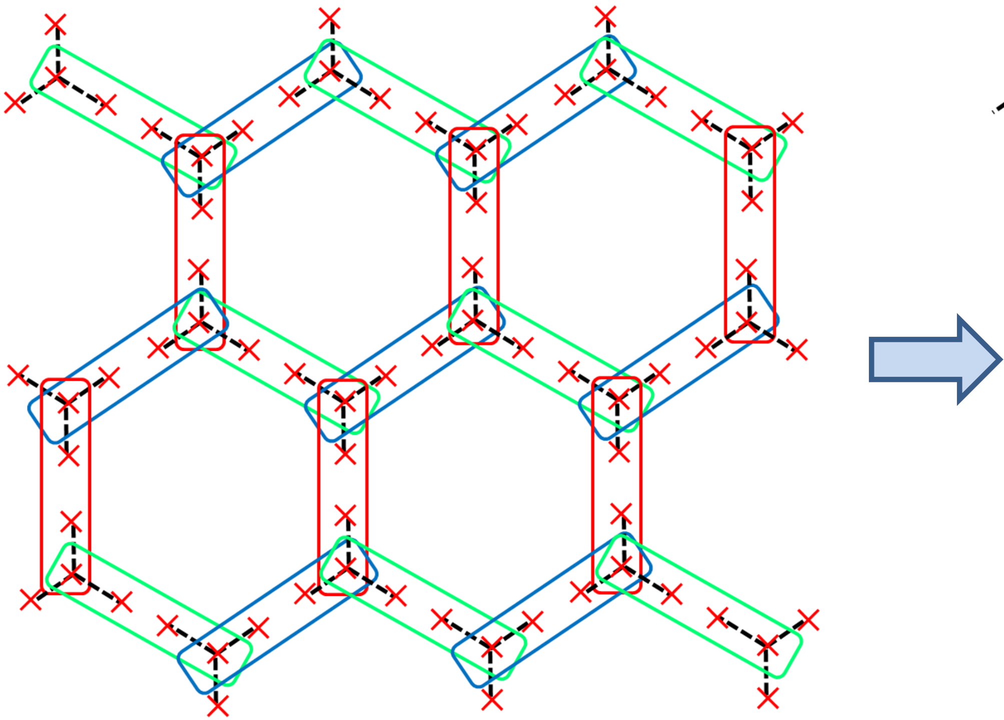
- If we temporarily release the constraint at each site, conserved quantities U_{ij} can be defined on the links, which are Z_n gauge fields

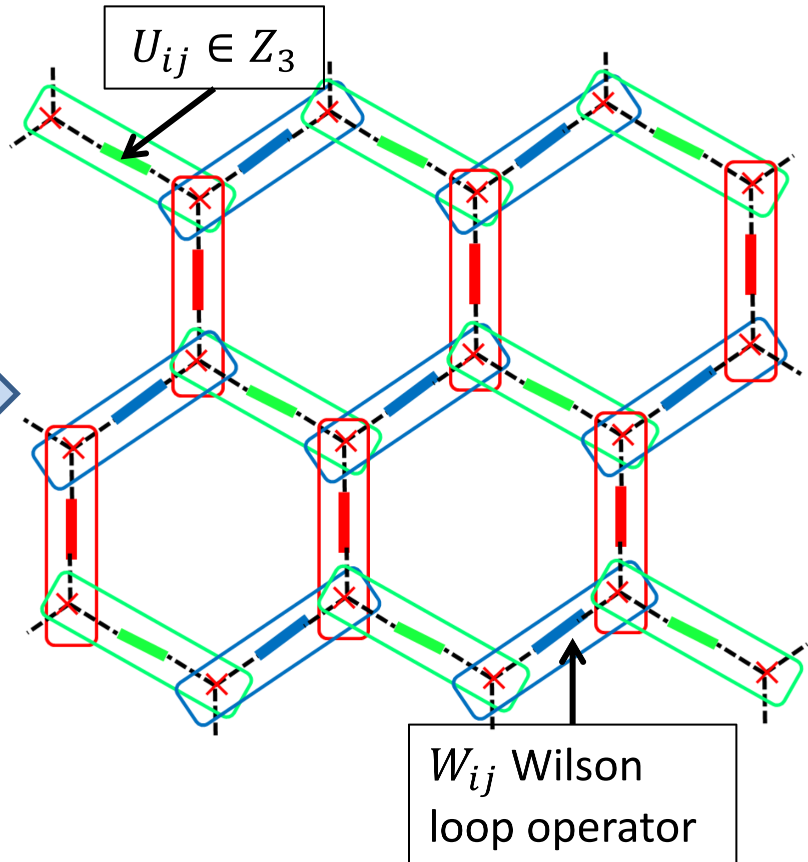


- $H = -\sum_{\langle ij \rangle} J_{ij} \left(U_{ij} W_{ij} + h.c. \right)$

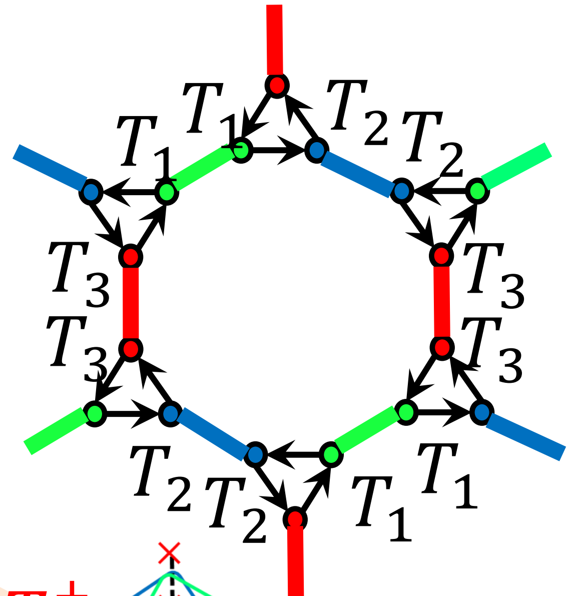
Emergent Z_n gauge field

- The Z_n Kitaev model describes parafermion hopping on a honeycomb lattice, coupled with the emergent Z_n gauge field.
- On-site constraint ➔ Projection to gauge invariant states

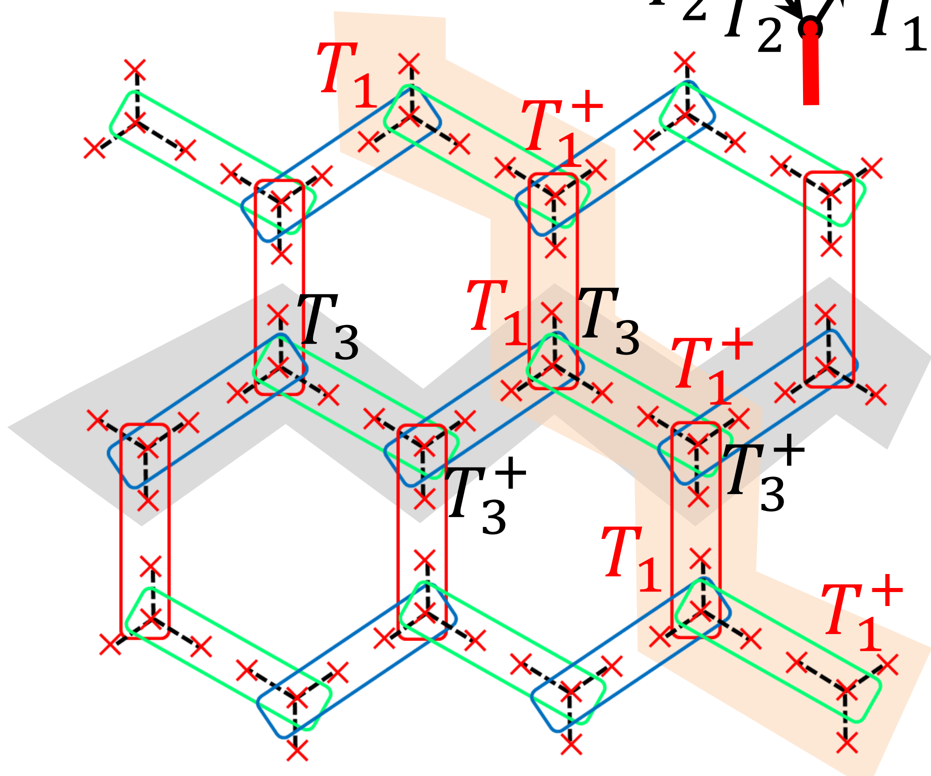




Properties of the Z_n Kitaev model

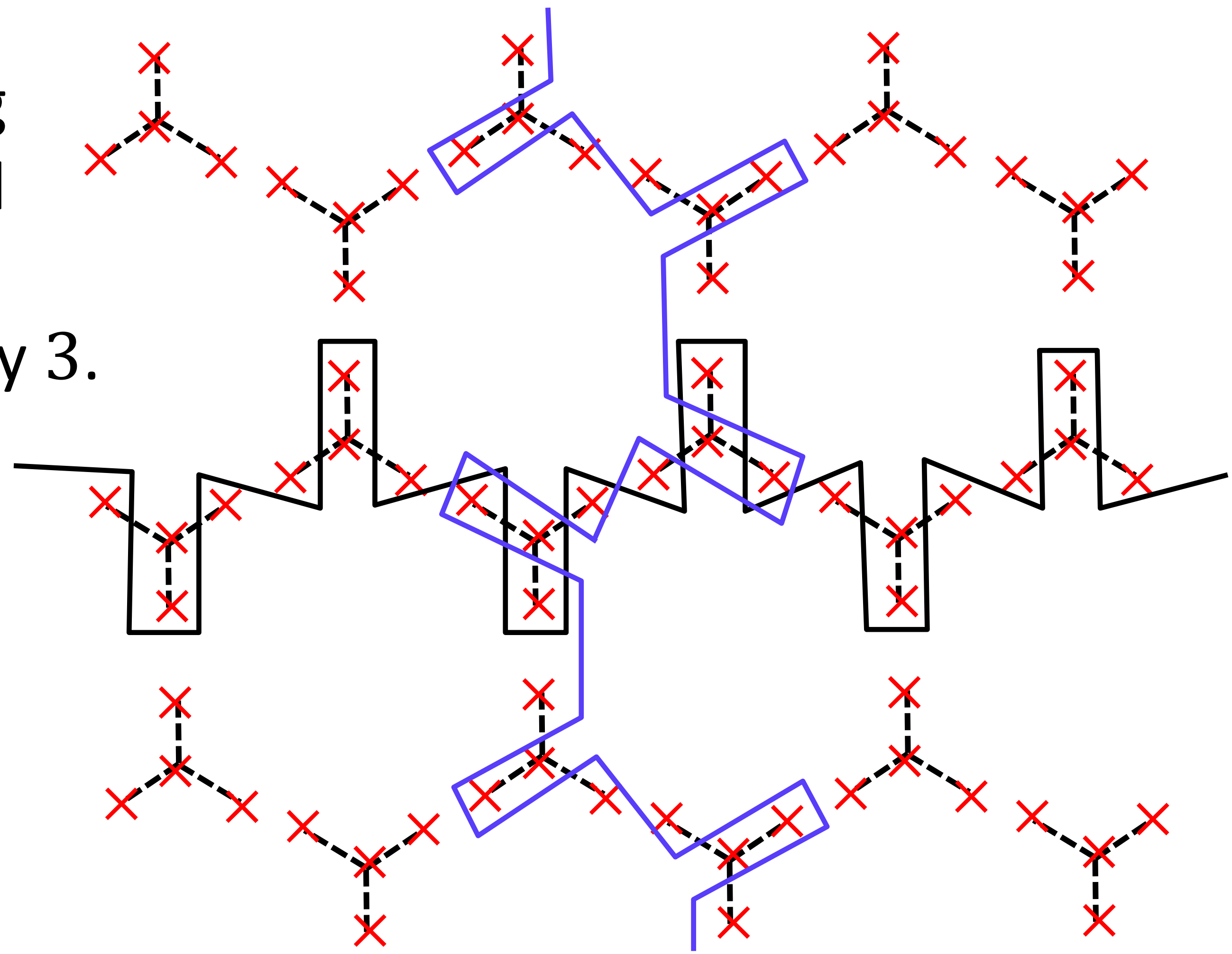


- **Plaquette conserved quantities**: Each plaquette has a conserved Z_3 flux $O_I = \prod_{<ij>\in I} U_{ij}$. Returning to the spin model, O_I is a product of Hamiltonian terms.
- **Large loop operators on a torus:**
- $L_3 = T_3 T_3^+ T_3 T_3^+ \ldots$
- $L_1 = T_1 T_1^+ T_1 T_1^+ \ldots$
- $[L_{1,3}, H] = 0$, $L_1 L_3 = L_3 L_1 \omega^2$,



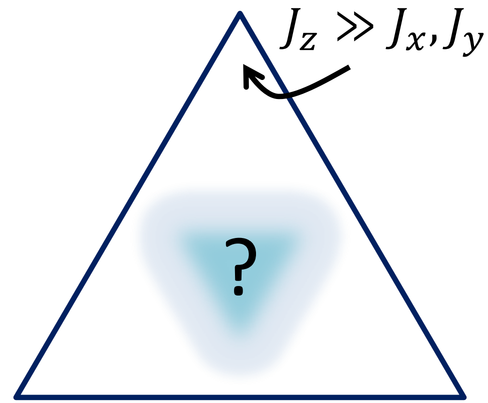
Large loop operators as Wilson loops

- The large loop operators can be understood as Wilson loops.
- Non-commuting large loops lead to exact ground state degeneracy 3.

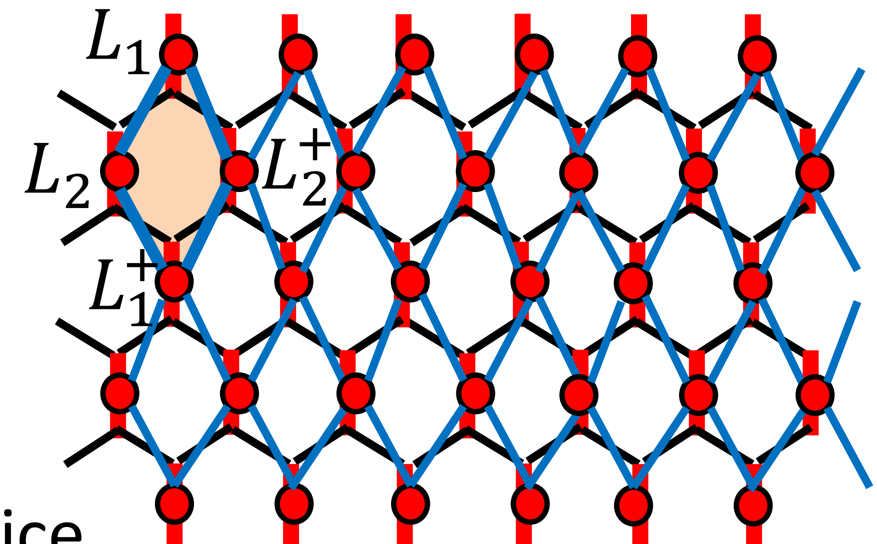


Abelian phase: Z_n toric code

- The Z_n model is not solvable.
- In the an-isotropic limit, one can obtain a Z_n toric code (lattice gauge theory) phase. (Kitaev '03)



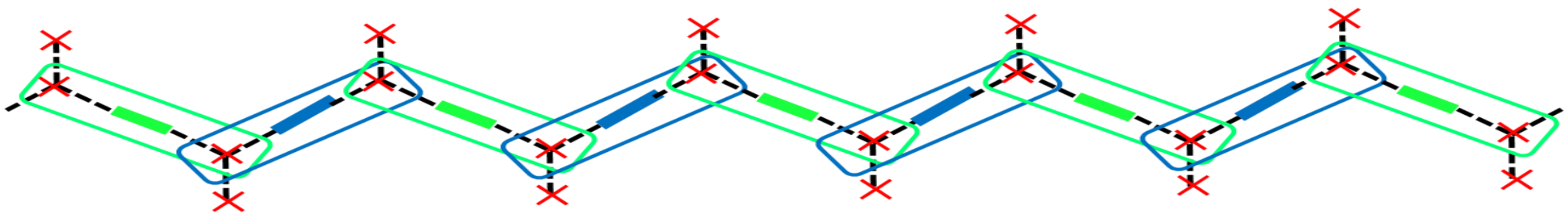
- $J_z \gg J_x, J_y$ limit
- Strong coupling along red bonds $J_z T_{3i} T_{3j}$
- The low energy state of each bond is a n-state rotor
- The rotors form a square lattice with the dynamics of Z_n gauge theory.



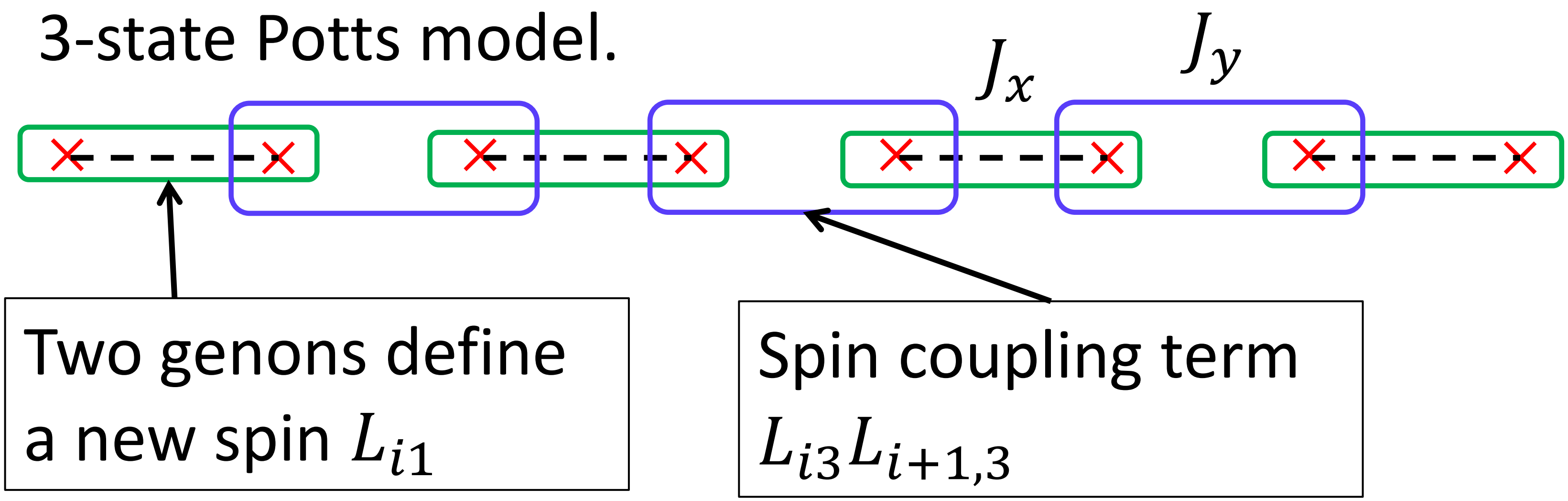
- $H_{\text{eff}} = -J_{\text{eff}} \sum_I L_1 L_2 L_1^+ L_2^+$ an exact solvable model.

Attacking the isotropic limit: starting from single chain

- A single chain of this Z_n Kitaev model is mapped to a parafermion chain (Fradkin-Kadanoff '80, P. Fendley '12)

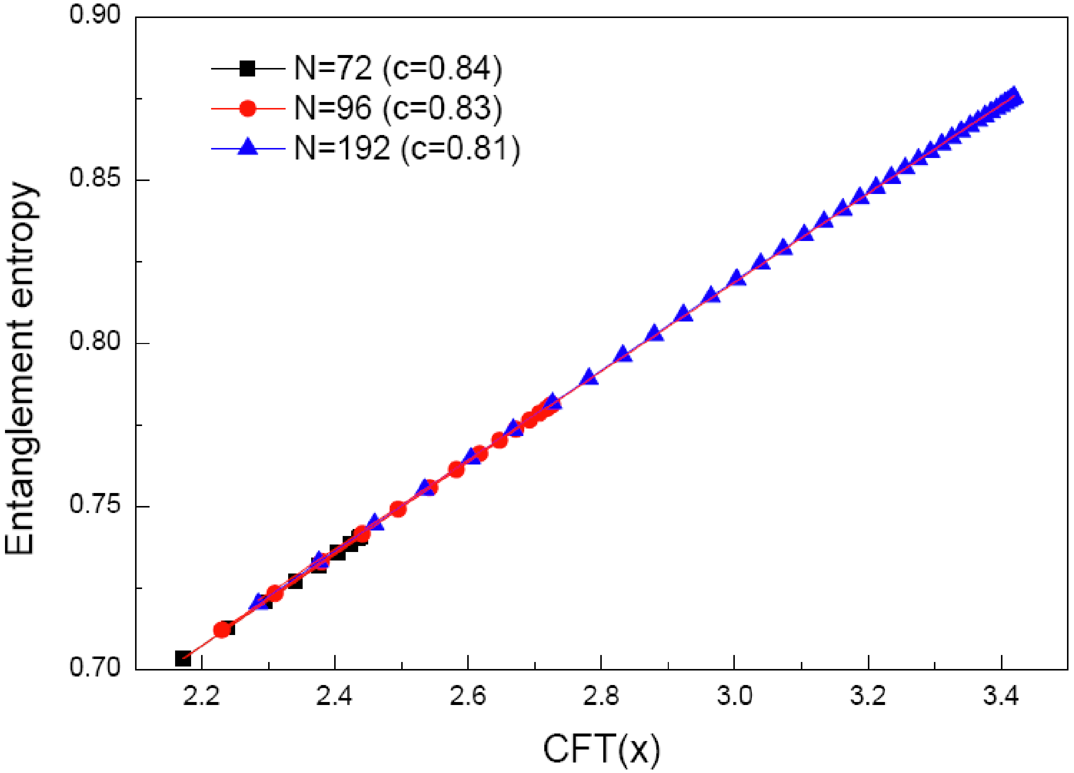


- Genons carrying a parafermion zero mode at each site.
- For $n = 3$ case, the coupled genons are equivalent to a 3-state Potts model.



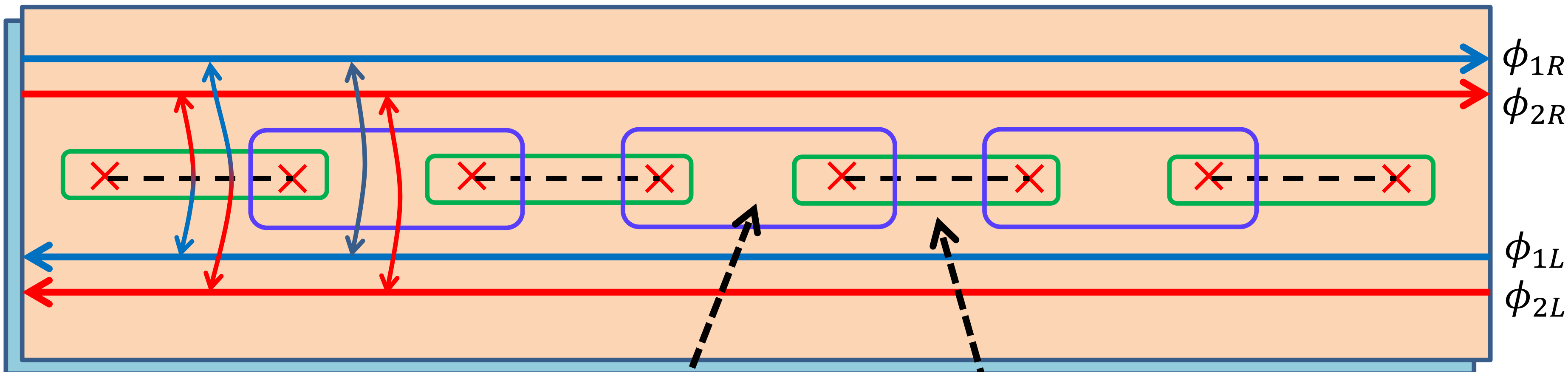
Single chain: Z_3 Potts model CFT

- For $J_x = J_y$, the model is critical (P. Fendley '12)
- 3-state Potts model CFT $c = \frac{4}{5}$
- Verified by entanglement entropy from DMRG



Single chain: FQH edge state picture

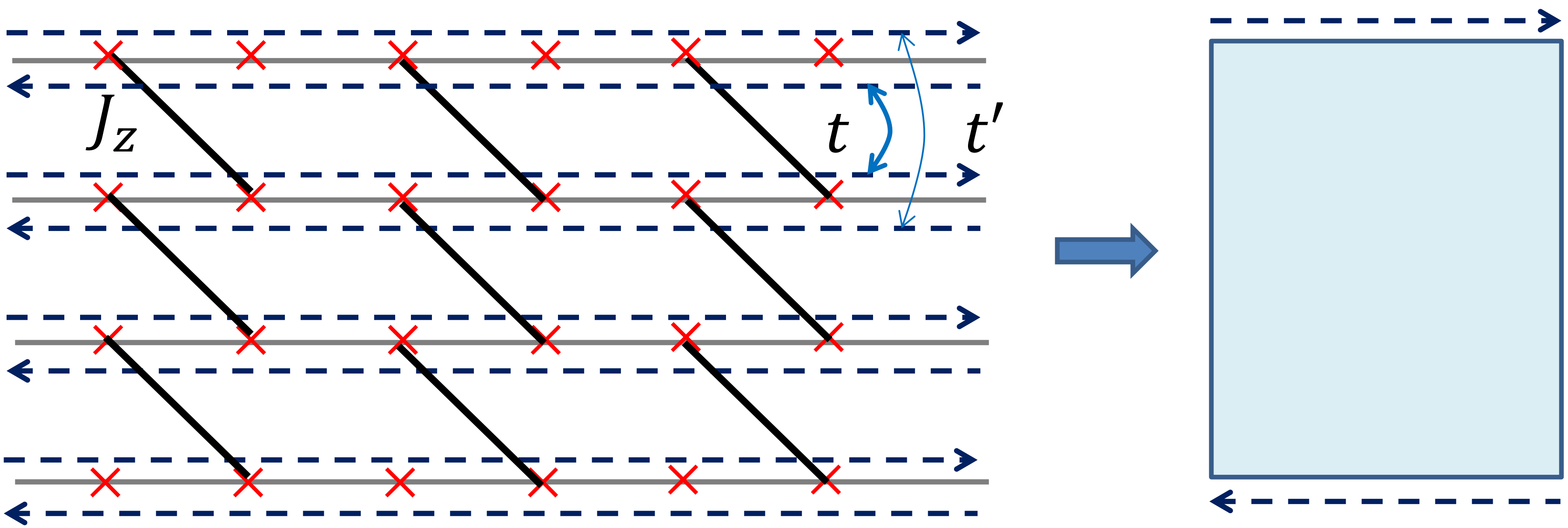
- Genon arrays can be understood as FQH edges with alternating mass terms.



- Chiral Luttinger liquid theory description (Wen)
  $\phi_- = \phi_1 - \phi_2$ sector:
  $$\mathcal{L} = \left(\partial_\mu \phi_-\right)^2 + V_1(x)\cos m\phi_- + V_2(x)\cos m\theta_-$$
- Critical point described by parafermion CFT when the two regions have the same length. (*c.f.* Lecheminant et al '02)

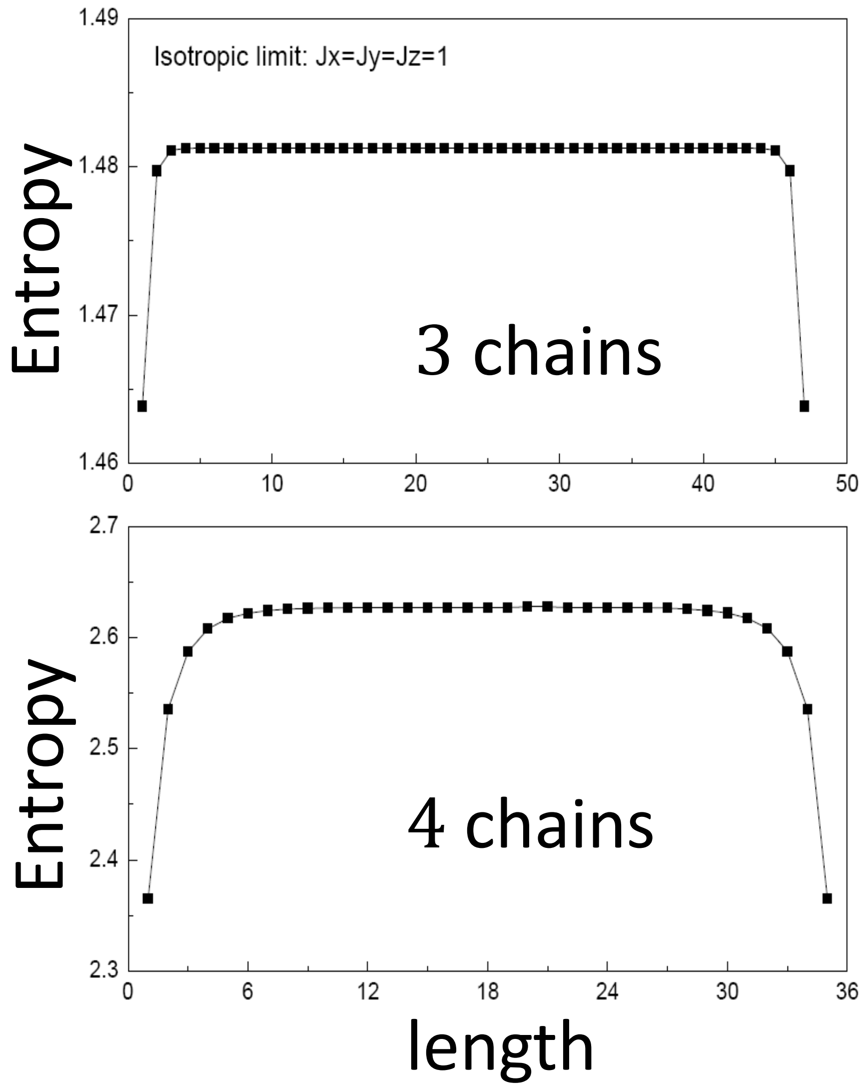
Coupled chains: towards a non-Abelian phase

- The 2D Hamiltonian breaks time-reversal. (Different from Z_2 case)
- If the chains are coupled chirally, we can obtain a 2D chiral topological phase. (R. Mong et al '13)



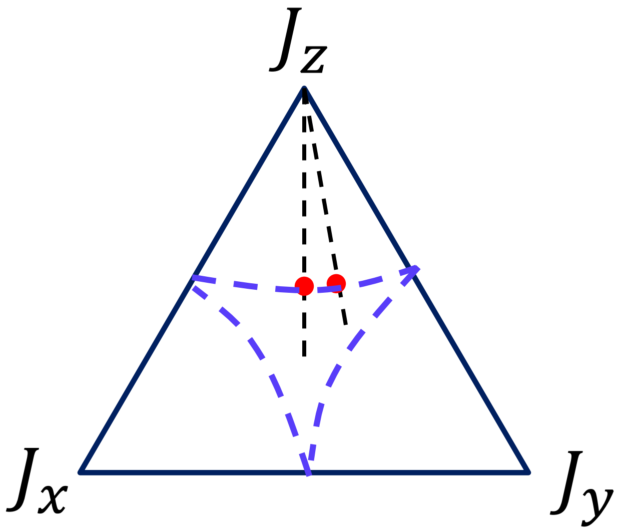
On-going numerical results: Gapped isotropic phase

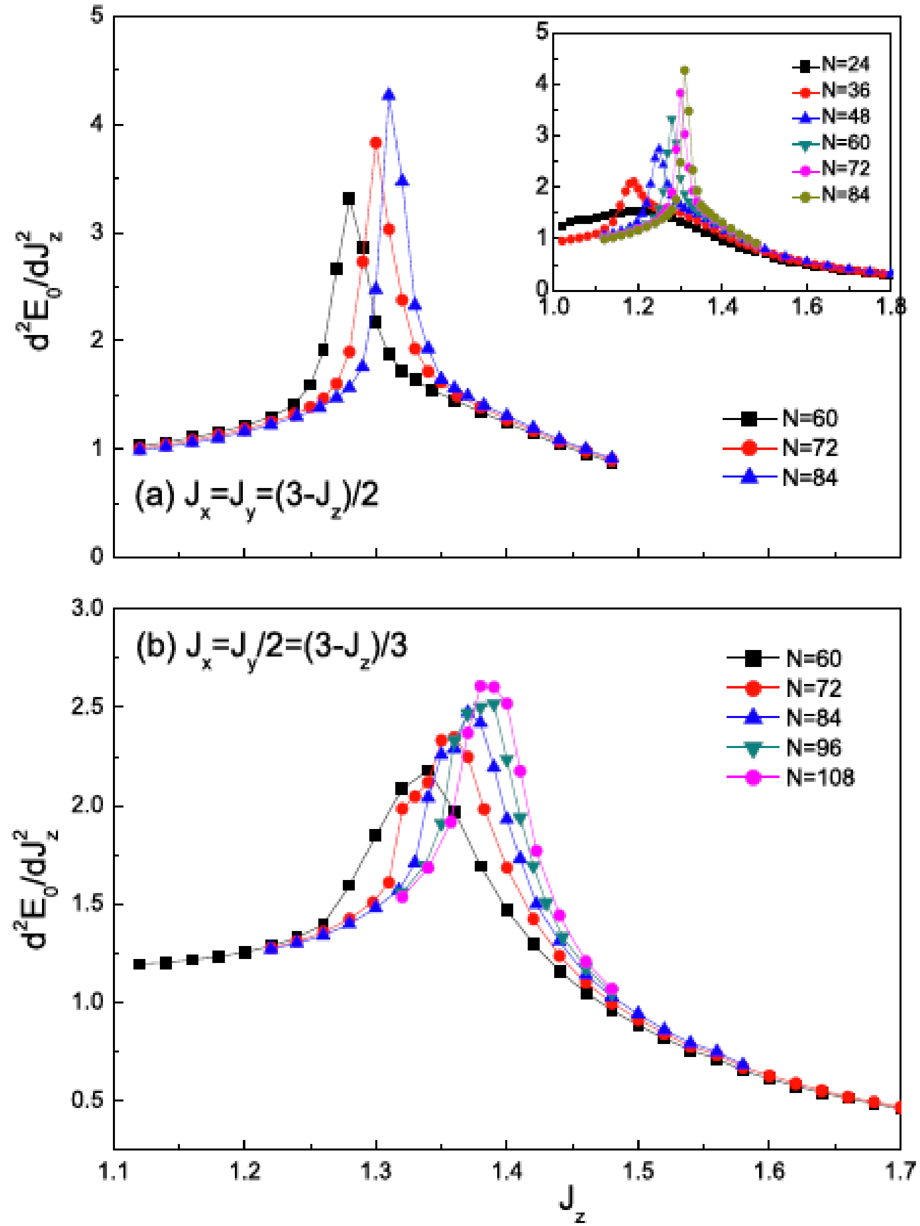
- Cylinder with isotropic coupling $J_x = J_y = J_z > 0$
- DMRG calculation finds a gapped phase.
- Indication of phase transition when J_x/J_z is tuned.



Phase boundary

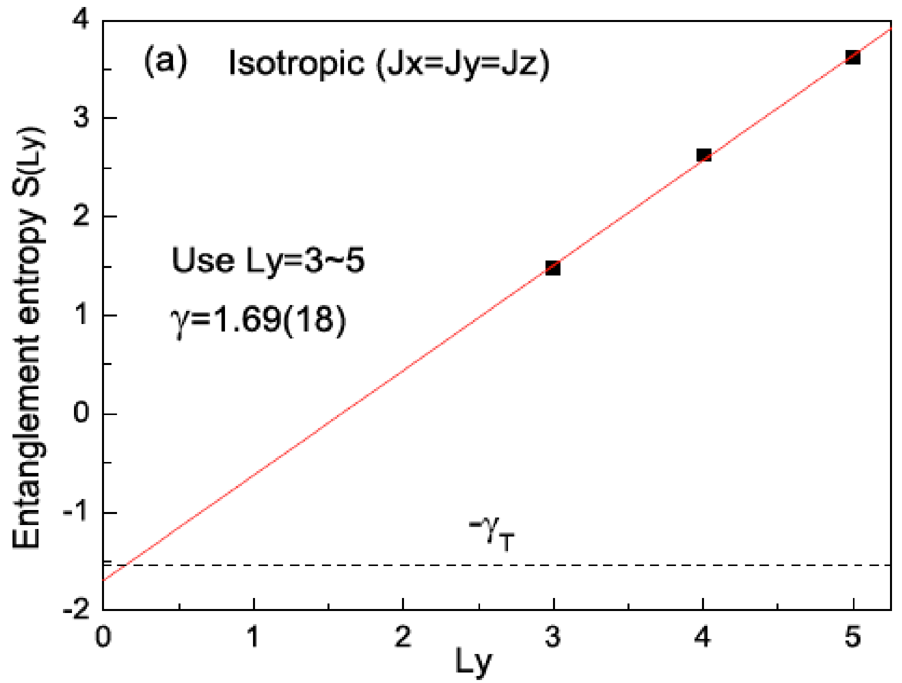
- Indication of phase transition when anisotropy is introduced. (Calculation for 3 chains)





Topological entropy

- Topological entanglement entropy $S = \alpha L - \gamma$
- We obtain a topological entropy $\gamma = 1.69 \pm 0.18$
- The topological entropy of the Z_3 parafermion TQFT is $D = \sqrt{3(1+\phi^2)}$, $\phi = (\sqrt{5}+1)/2$, $\gamma = \log D \simeq 1.19$,



- Probably there is a large finite size effect. Possible solution by optimizing the model.

Summary of the second lecture

- Genons can be realized in experimentally accessible bilayer FQH states
- Tunneling and quantum interference measurements can probe the parafermion zero modes and non-Abelian qubits of genons.
- Genons can be realized in topological nematic states by lattice dislocation.
- Genons can be used as “slave particles” to construct semi-solvable Z_n generalizations of Kitaev model, with possibly a non-Abelian topological phase. More numerical works are required for understanding this model.

Summary

- Twist defects can be defined in topologically ordered states with topological symmetry.
- Genons as branchcut defects, which are also genus generators
- Non-Abelian genons can be defined even in an Abelian theory. Genons in Halperin states can have Majorana statistics and it's generalization.
- Genons can be realized in a wide range of systems, such as bilayer FQH, FCI, graphene pentagon defect.
- In Abelian states, all point and line defects can be classified and their topological properties can be calculated.

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