Lecture on quantum entanglement in condensed matter systems

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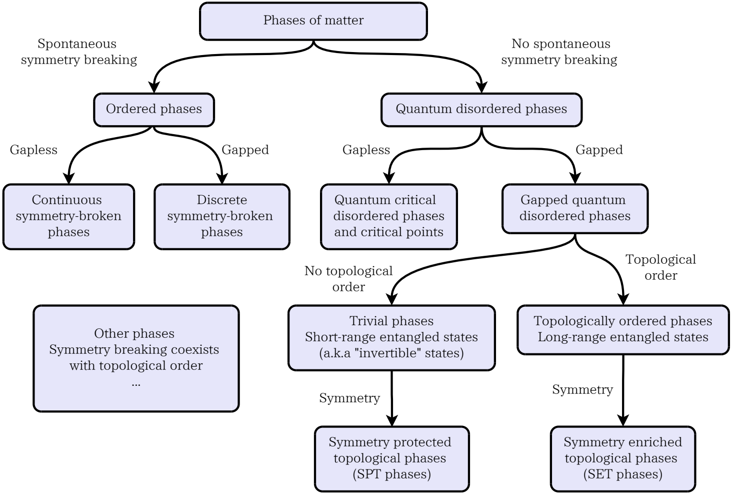
Overview

- Quantum entanglement as an "order parameter"
  - SPT phases (free systems)
  - (1+1)d CFTs
  - Perturbed CFTs
  - (2+1)d topologically ordered phases
  - ...
- Developing theoretical/computational tools:
  - DMRG, MPS, PEPS, MERA, and other tensor networks
- Other applications – ETH and many-body localization, thermalization and chaos in dynamical systems, etc.
- Applications to physics of spacetime

Phases of matter



Entanglement and entropy of entanglement

- (0) States of your interest, e.g., $\rho_{tot} = |\Psi\rangle\langle\Psi|$.
- (i) Bipartition Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$.
- (ii) Partial trace:

$$\rho_A = \mathrm{Tr}_B |\Psi\rangle\langle\Psi| = \sum_j p_j |\psi_j\rangle_A \langle\psi_j|_A \quad (\sum_j p_j = 1) \tag{1}$$

- (iii) von Neumann Entanglement entropy:

$$S_A = -\mathrm{Tr}_A \left[\rho_A \ln \rho_A\right] = -\sum_j p_j \ln p_j \tag{2}$$

- (iv) Entanglement spectrum $\rho_A \propto \exp(-H_e)/Z$:

$$\{\xi_i\} \quad \text{where} \quad p_i =: \exp(-\xi_i)/Z \tag{3}$$

- Mutual information:

$$I_{A:B} \equiv S_A + S_B - S_{A\cup B} \tag{4}$$

- Rényi entropy:

$$R_A^{(q)} = \frac{1}{1-q} \ln(\mathrm{Tr}\,\rho_A^q). \tag{5}$$

  Note that $S_A = \lim_{q\to 1} R_A^{(q)}$. $\{R_A^{(q)}\}_q$ = entanglement spectrum.

- The Rényi mutual information:

$$I_{A:B}^{(q)} \equiv R_A^{(q)} + R_B^{(q)} - R_{A\cup B}^{(q)} \tag{6}$$

- Other entanglement measures, e.g., entanglement negativity.

Some key properties

- If ρ_{tot} is a pure state and $B = \bar{A}$, $S_A = S_B$.
- If ρ_{tot} is a mixed state (e.g., $\rho_{\mathrm{tot}} = e^{-\beta H}$), $S_A \neq S_B$ even when $B = \bar{A}$,
- If $B = \emptyset$, $S_A = S_{\mathrm{thermal}}$.
- Subadditivity:

$$S_{A+B} \leq S_A + S_B. \tag{7}$$

  i.e., the positivity of the mutual information:
  $I_{A:B} = S_A + S_B - S_{A+B} \geq 0$.

- Strong subadditivity

$$S_B + S_{ABC} \leq S_{AB} + S_{BC} \tag{8}$$

  By setting $C = \emptyset$, we obtain the subadditivity relation.

ES in non-interacting systems

- Consider the ground states $|GS\rangle$ of free (non-interacting) systems, and bipartitioning $\mathcal{H} = \mathcal{H}_L \otimes \mathcal{H}_R$.
- When $\rho_{tot} = |GS\rangle\langle GS|$ is a Gaussian state, H_e is quadratic [Pesche (02)].

$$H_e = \sum_{I,J\in L} \psi_I^\dagger K_{IJ} \psi_J, \quad I = \mathbf{r}, \sigma, i, \dots \tag{9}$$

- H_e can be reconstructed from 2pt functions: $C_{IJ} := \langle GS|\psi_I^\dagger \psi_J|GS\rangle$.

$$C = \begin{pmatrix} C_L & C_{LR} \\ C_{RL} & C_R \end{pmatrix}, \quad C_{RL} = C_{LR}^\dagger. \tag{10}$$

- Correlation matrix is a projector:

$$C^2 = C, \quad Q^2 = 1 \quad (Q_{IJ} := 1 - 2C_{IJ}). \tag{11}$$

- Entanglement Hamiltonian:

$$H_e = \sum_{I,J\in L} \psi_I^\dagger K_{IJ} \psi_J, \quad K = \ln[(1 - C_L)/C_L]. \tag{12}$$

E.g. the integer quantum Hall effect

- A prototype of topological phases
- Characterized by quantized Hall conductance $\sigma_{xy} = (e^2/h) \times (\text{integer})$.
- Gapped bulk, gapless edge
- Robust against disorder and interactions
- Chiral edge states in ES

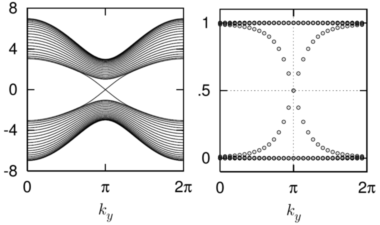
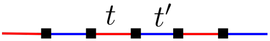


Figure: Physical v.s. entanglement spectra of a Chern insulator [SR-Hatsugai (06)]

E.g. the SSH model

- 1d lattice fermion model:

$$H = t \sum_i \left(a_i^\dagger b_i + h.c. \right) + t' \sum_i \left(b_i^\dagger a_{i+1} + h.c. \right) \tag{13}$$



- Phase diagram:

- Physical spectrum, entanglement spectrum, entanglement entropy.

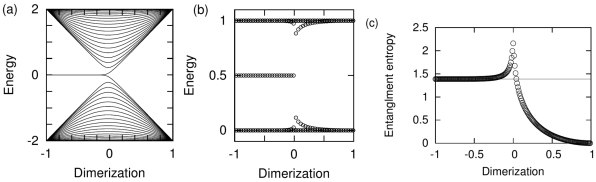
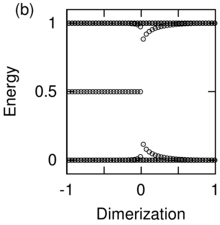


Figure: [SR-Hatsugai (06)]

Symmetry-protected degeneracy in ES

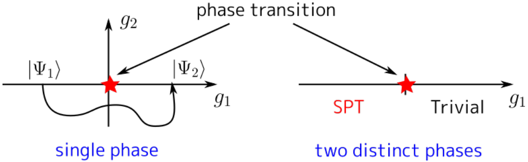
- Robust zero mode in ES; 2-fold degeneracy for each level.



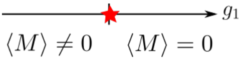
- $S_A = A \log \xi / a_0 + \log 2$
- Degeneracy is symmetry-protected; Symmetry: $a_i \to a_i^\dagger$, $b_i \to -b_i^\dagger$. (Class D or AIII/BDI topological insulator)
- Symmetry-protected degeneracy is an indicator of symmetry-protected topological (SPT) phases. [Pollmann-Berg-Turner-Oshikawa (10)]

Symmetry-protected topological phases (SPT phases)

- "Deformable" to a trivial phase (state w/o entanglement) in the absence of symmetries.
- (Unique ground state on any spatial manifold – "invertible")
- But sharply distinct from trivial state, once symmetries are enforced.



- Example: SSH model, time-reversal symmetric topological insulators, the Haldane phase
- Symmetry-breaking paradigm does not apply: no local order parameter



Entanglement spec. and non-spatial symmetry

- How about symmetry ?
- Corr. matrix inherits symmetries of the Hamiltonian

$$\psi_I \to U_{IJ}\psi_J, \quad H_{phys} \to U^\dagger H_{phys} U = H_{phys}, \\ Q \to U^\dagger Q U = Q \tag{14}$$

- Non-spatial symmetry, the sub block of corr. matrix inherits symmetries:

$$Q_L \to U^\dagger Q_L U = Q_L \tag{15}$$

So does the entanglement Hamiltonian. This may result in degeneracy in the ES.

Another example

- Spin-1 Antiferromagnetic spin chain

$$H = \sum_j \boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1} + U_{zz} \sum_j (S_j^z)^2 \tag{16}$$

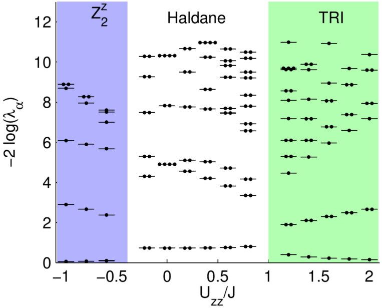
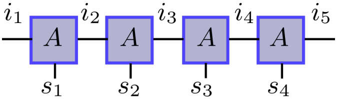


Figure: [Pollmann-Berg-Turner-Oshikawa (10)]

- Matrix product state representation:

$$\Psi(s_1, s_2, \cdots) = \sum_{\{i_n=1,\cdots\}} A^{s_1}_{i_1 i_2} A^{s_2}_{i_2 i_3} A^{s_3}_{i_3 i_4} \cdots \quad s_a = -1, 0, 1$$



- Symmetry action: for $g, h \in$ Symmetry group, we have $U(g)$ acting on physical Hilbert space:

$$U(g)U(h) = U(gh)$$
$$U(g)^{s'}_{s} A^{s} = V^{-1}(g) A^{s'} V(g) e^{i\theta_g} \quad (17)$$

- Symmetry acts on the "internal" space projectively:

$$V(g)V(h) = e^{i\alpha(g,h)} V(gh) \quad (18)$$

[Chen et al (11), Pollmann et al (10-12), Schuch et al (11)]

(Entanglement spec)^2 and SUSY QM

- From $C^2 = C$:

$$\begin{aligned} C_L^2 - C_L &= -C_{LR}C_{RL}, \\ Q_L C_{LR} &= -C_{LR} Q_R, \\ C_{RL} Q_L &= -Q_R C_{ReL}, \\ C_R^2 - C_R &= -C_{RL} C_{LR} \end{aligned} \tag{19}$$

- Introduce:

$$\mathcal{S} = 1 - \begin{pmatrix} Q_L^2 & 0 \\ 0 & Q_R^2 \end{pmatrix}, \quad \mathcal{Q} = \begin{pmatrix} 0 & 2C_{LR} \\ 0 & 0 \end{pmatrix}, \quad \mathcal{Q}^\dagger = \begin{pmatrix} 0 & 0 \\ 2C_{RL} & 0 \end{pmatrix}. \tag{20}$$

- SUSY algebra

$$\begin{aligned} &[\mathcal{S}, \mathcal{Q}] = [\mathcal{S}, \mathcal{Q}^\dagger] = 0, \\ &\{\mathcal{Q}, \mathcal{Q}^\dagger\} = \mathcal{S}, \quad \{\mathcal{Q}, \mathcal{Q}\} = \{\mathcal{Q}^\dagger, \mathcal{Q}^\dagger\} = 0 \end{aligned} \tag{21}$$

Entanglement spec. and spatial symmetries

- L/R = "fermionic"/"bosonic" sector; $C_{L,R}$ intertwines the two sectors:

$$\mathcal{H}_L \underset{C_{RL}}{\overset{C_{LR}}{\rightleftarrows}} \mathcal{H}_R \tag{22}$$

- Spatial symmetry $\mathcal{O}$: choose bipartitioning s.t.

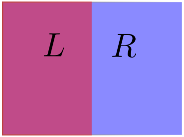
$$\mathcal{O} : \mathcal{H}_L \longleftrightarrow \mathcal{H}_R \tag{23}$$

$$O = \begin{pmatrix} 0 & O_{LR} \\ O_{RL} & 0 \end{pmatrix}, \quad O_{LR} O^{\dagger}_{LR} = O_{RL} O^{\dagger}_{RL} = 1 \tag{24}$$

- Symmetry of entanglement Hamiltonian:

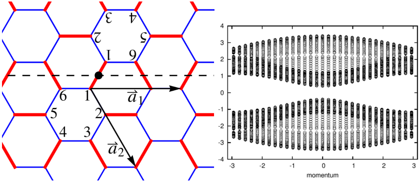
$$Q_L C_{LR} O^{\dagger}_{LR} = C_{LR} O^{\dagger}_{LR} Q^{*}_{L} \tag{25}$$

[Turner-Zhang-Vishwanath (10), Hughes-Prodan-Bernevig (11), Fang-Gilbert-Bernevig (12-13), Chang-Mudry-Ryu (14)]

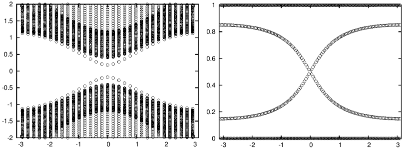


Graphene with Kekule order

- Kekule distortion in graphene



- Degeneracy protected by inversion



- Entanglement spec. is more useful than physical spec.

Short notes: Conformal field theory in (1+1)d

- Scale invariance in (1+1)d $\rightarrow$ conformal symmetry (Polchinski)
- Conformal symmetry is infinite dimensional. Holomorphi-anti-holomorphic factorization
- Infinite symmetry generated by stress energy tensor

$$T(z) = \sum_{n=-\infty}^{+\infty} L_n z^{-n-2}, \quad \bar{T}(\bar{z}) = \sum_{n=-\infty}^{+\infty} \bar{L}_n \bar{z}^{-n-2}, \tag{26}$$

- Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m,-n} \tag{27}$$

- Characterized by a number c "central charge" (among others)

- Structure of the spectrum: "tower of states":

$$
\begin{aligned}
&|h, N; j\rangle \otimes |\bar{h}, \bar{N}; \bar{j}\rangle, \\
&L_0|h, N; j\rangle = (h + N)|h, N; j\rangle. \\
&\bar{L}_0|\bar{h}, \bar{N}; \bar{j}\rangle = (\bar{h} + \bar{N})|\bar{h}, \bar{N}; \bar{j}\rangle.
\end{aligned} \tag{28}
$$

- In other words:

$$
\mathcal{H} = \bigoplus_{h,\bar{h}} n_{h,\bar{h}} \mathcal{V}_h \otimes \overline{\mathcal{V}}_{\bar{h}}, \tag{29}
$$

$n_{h,\bar{h}}$: the number of distinct primary fields with conformal weight $(h, \bar{h})$.
(For simplicity, we only consider the diagonal CFTs with $n_{h,\bar{h}} = \delta_{h,\bar{h}}$.)

Central charge

- $c =$ Weyl anomaly; at critical points, there are emergent scale invariance, but this emergent symmetry is broken by an anomaly.
- $c \simeq$ (number of degrees of freedom)
- c shows up in free energy and specific heat, etc:

$$c_V = \frac{\pi c}{3 v \beta} \tag{30}$$

  Note: v is non-universal.
- Can be extracted from the entanglement entropy scaling:

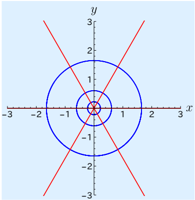
$$S_A = \frac{c}{3} \log R + \cdots \tag{31}$$

- RG monotone. (Zamolodchikov c-function; entropic c-function)

Radial and angular quantization

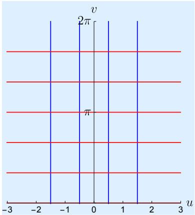
- $w(z) = \log z$

$z = x + iy$



$$w(z) = \log z$$

$w = u + iv$



- CFT on a plane $\leftrightarrow$ CFT on a cylinder
- Radial evolution $\leftrightarrow$ Hamiltonian
- Angular evolution (Entanglement or Rindler Hamiltonian) $\leftrightarrow$ Hamiltonian with boundary

Radial flow – Finite size scaling

- CFT on a cylinder of circumference L

$$H = \frac{1}{2\pi}\int_0^L dv\, T_{uu}(u_0, v) = \frac{1}{2\pi}\oint_{C_w} dw\, T(w) + \text{(anti-hol)} \tag{32}$$

- Conformal map: cylinder $\to$ plane $w = \frac{L}{2\pi}\log z$

$$\oint_{C_w} dw\, T(w) = \oint_{C_z} dz\, \frac{dw}{dz}\left(\frac{2\pi}{L}\right)^2\left[z^2 T(z) - \frac{c}{24}\right] = \oint_{C_z} dz\left(\frac{L}{2\pi}\right)\left[zT(z) - \frac{c}{24}\frac{1}{z}\right] \tag{33}$$

- CFT Hamiltonian on a cylinder can be written in terms of dilatation operator $L_0 + \bar{L}_0$ on a plane:

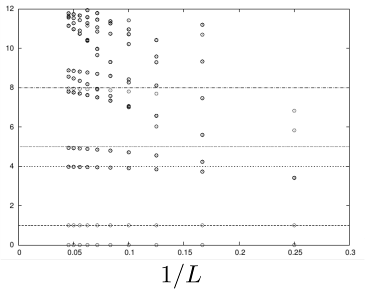
$$H = \frac{2\pi}{L}\left(L_0 + \bar{L}_0 - \frac{c}{24}\right) \tag{34}$$

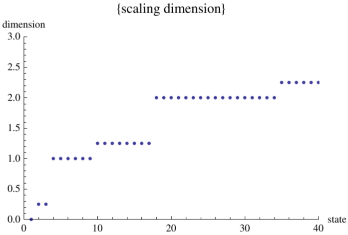
- Gives relation between stress tensor (on z-plane) to a "physical" Hamiltonian on a finite cylinder.
- Level spacing scales as $1/L$.
- Levels are equally spaced (within a tower)
- The $c/24 \times 1/L$ part allows us to determine c (numerically). (the extensive part $A \times L$ has to be subtracted.)
- Degeneracy $\to$ full identification of the theory

Radial flow – Numerics

- XX model: $H = \sum_j \left(S_j^x S_{j+1}^x + S_j^y S_{j+1}^y \right)$

$$\frac{E - E_{GS}}{E_1 - E_{GS}}$$



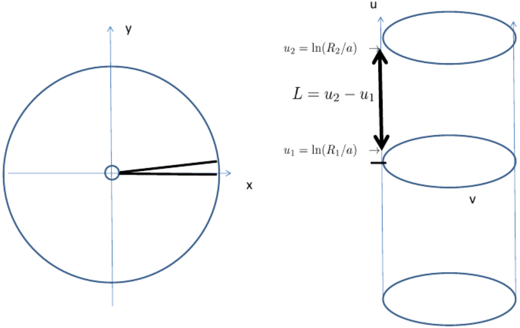


c =1 compactified free boson at R= sqrt(2)

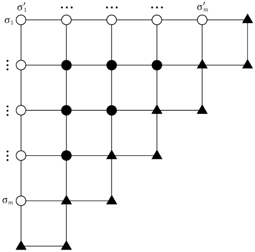
- For a given tower, all levels are equally spaced.
- Level spacing scales as $1/L$.

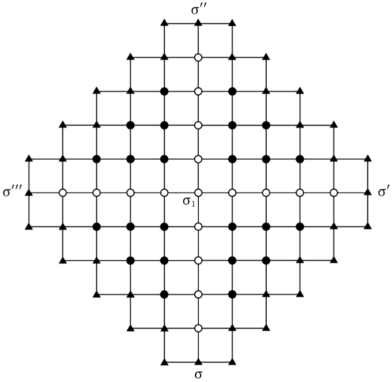
Angular flow

$$z = (x + iy) = \exp(w) = \exp(u + iv)$$



- Corner transfer matrix $A_{\sigma|\sigma'}$ and partition function $Z = \mathrm{Tr}\, A^4$





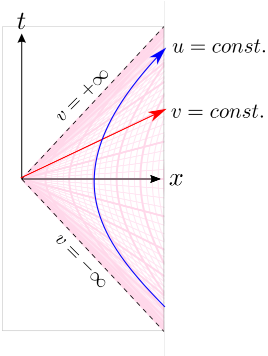
[Baxter (80's); Figures:Wikipedia]

Angular flow = Entanglement (Rindler) Hamiltonian

- In Euclidean signature, $z = x + iy = e^w = e^{u+iv}$ maps the complex z-plane to a cylinder.
- In Minkowski signature: $(t, x) \to (u, v)$ (Rindler coordinate):

$$x = e^u \cosh v,$$
$$t = e^u \sinh v.$$

- In the Rindler coord., the half of the 2d spacetime is inaccessible ("traced out").
- Radial evolution in the complex z-plane $\to$ u-evolution in the cylinder
- Angular evolution in the complex z-plane $\to$ v-evolution in the cylinder = entanglement (or Rindler) Hamiltonian



[Figures: Wikipedia]

Rindler Hamiltonian

- Constant u trajectories = World-lines of observer with constant acceleration a where $a = 1$ in our case.
  Accelerated observer in Minkowski space = Static observer in Rindler space

- Unruh effect: Vacuum is observer dependent. Observer in an accelerated frame (Rindler observer) sees the vacuum of the Minkowski vacuum as a thermal bath with Unruh temperature

$$T = \frac{a}{2\pi} = \frac{1}{2\pi} \tag{35}$$

- This is due to a "Rindler horizon" and inability to access the other part of spacetime. Rinder coordinates covers with metric

$$ds^2 = e^{2au}(-dv^2 + du^2) \tag{36}$$

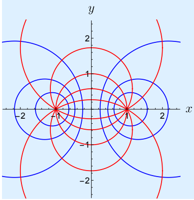
  only covers $x > |t|$ (the right Rindler wedge).

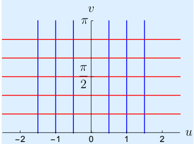
- Left Rindler wedge is defined by

$$x = e^u \cosh v,$$
$$t = -e^u \sinh v.$$

Entanglement Hamiltonian for finite interval

- $w(z) = \ln(z+R)/(z-R)$





- Entanglement hamiltonian on finite interval $[-R, +R] \to$ Hamiltonian with boundaries
- Transforming from strip to plane:

$$H = \int du\, T_{vv}|_{v_0=\pi} = \int_{-R}^{+R} dx\, \frac{(x^2 - R^2)}{2R} T_{yy}|_{y=0} \qquad (37)$$

- Entanglement spec: $1/\log(R)$ scaling

E.g., Casini-Huerta-Myers (11), Cardy-Tonni (16)

SSH chain

- Entanglement spectrum of CFT GS: $H^E = const. \frac{L_0}{\log(R/a)}$

$$H = t \sum_i \left(a_i^\dagger b_i + h.c. \right) + t' \sum_i \left(b_i^\dagger a_{i+1} + h.c. \right) \tag{38}$$

with $t = t'$

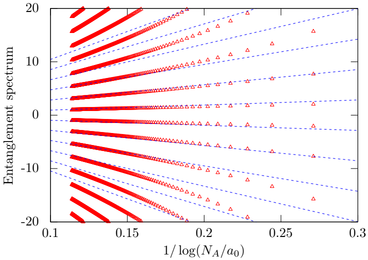


Figure: [Cho-Ludwig-Ryu (16)]

Numerics

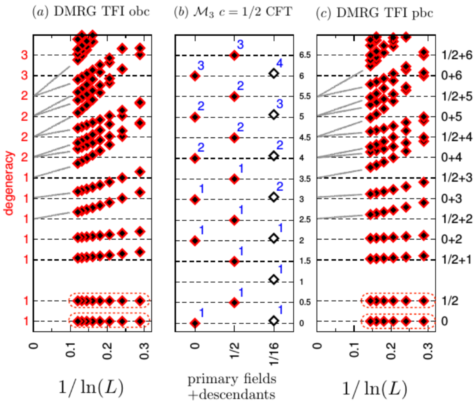


Figure: [Lauchli (13)]

Remarks:

- What is an analogue of the radial direction?
- It is related to the so-called sine-square deformation (SSD).
  [Gendiar-Krcmar-Nishino (09), Hikihara-Nishino (11), ...]
- Evolution operator:

$$H = \int_0^{\pi} dv\, T_{uu}(u_0, v) = r_0^2 \int_0^{2\pi} d\theta\, \frac{\cos\theta + \cosh u_0}{\sinh u_0}\, T_{rr}(r, \theta) \tag{39}$$

- In the limit $R \to 0$,

$$H \sim \int_0^{L} ds\, \sin^2\left(\frac{\pi s}{L}\right) T_{rr}\left(\frac{L}{2\pi}, \frac{2\pi s}{L}\right) \tag{40}$$

[Ishibashi-Tada (15-16); Okunishi (16); Wen-Ryu-Ludwig (16)]

Perturbed CFT

- Add a relevant perturbation

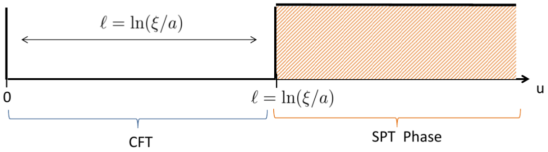
$$S = S_* + g \int d^2 z \, \phi(z, \bar{z}) \tag{41}$$

and go into a massive phase; Consider the entanglement Hamiltonian for half space.

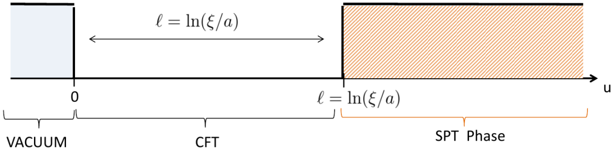
- The above conformal map leads to an exponentially growing potential

$$S_* + g \int_{u_1}^{u_2} du \int_0^{2\pi} dv \, e^{yu} \, \Phi(w, \bar{w}) \tag{42}$$

with length scale $\log(\xi/a)$.



- Entanglement spectrum for gapped phases is given by a CFT with boundaries (Boundary CFT in short) of a nearby CFT



Partition function:

$$Z_{AB} = \mathrm{Tr}_{AB}\, e^{-H_e} \tag{43}$$

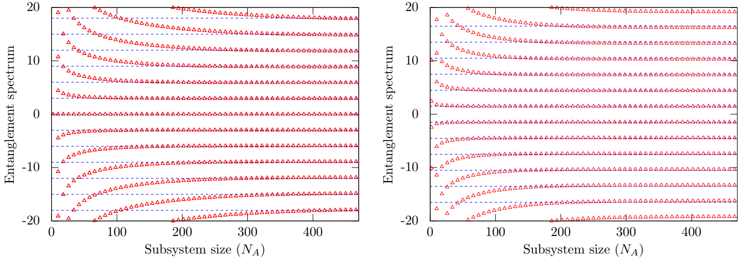
Here, $A =$ vacuum and $B = SPT$. ["RG domain wall" idea:]

- Spectrum is given by half of the full CFT:

$$H_e = const. \frac{L_0}{\log(\xi/a)}$$

Numerics: SSH model

- Spectrum depends on type of boundaries (type of SPTs): There is *symmetry-protected degeneracy* in the topological phase.



BCFT and SPT

- Entanglement spectrum for gapped phases is given by BCFT
- When the gapped phase is an SPT, the topological invariant can also be computed from BCFT. [Cho-Shiozaki-Ryu-Ludwig (16)]
- Switching space and time,

$$Z = \mathrm{Tr}\, e^{-\beta/\ell L_0} = \langle A|e^{-\ell/\beta(L_0+\bar{L}_0)}|B\rangle \tag{44}$$

  we introduce boundary states $|A\rangle$ and $|B\rangle$:

$$(L_n - \bar{L}_{-n})|B\rangle = 0, \quad \forall n \in \mathbb{Z} \tag{45}$$

- From $|B\rangle$, the corresponding SPT phase can be identified by the phase

$$g|B\rangle_h = \varepsilon_B(g|h)|B\rangle_h, \quad g, h \in G$$

  where $|B\rangle_h$ is the boundary state in h-twisted sector. This phase is called the discrete torsion phase $\varepsilon_B(g|h) \in H^2(G, U(1))$.

Boundary states as gapped states

- Conformally invariant boundary states, $(L_n - \bar{L}_{-n})|B\rangle = 0$.
- Boundary states $|B\rangle$ do not have real-space correlations:

$$\langle B|e^{-\epsilon H}\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)e^{-\epsilon H}|B\rangle/\langle B|e^{-2\epsilon H}|B\rangle$$

  where $x_1, \cdots, x_n$ refer n different spacial positions. In the limit $\epsilon \to 0$ with $x_i \neq x_j$ the correlation function factorizes and does not depend on $x_i - x_j$.
- Boundary states represent a highly excited state within the Hilbert space of a gapless conformal field theory and can be viewed as gapped ground states. [Miyaji-Ryu-Takayanagi-Wen (14), Cardy (17), Konechny (17)]

Free fermion example

- A massive free massive Dirac fermion in (1+1)d:

$$H = \int dx \left[-i\psi^\dagger \sigma_z \partial_x \psi + m\psi^\dagger \sigma_x \psi \right], \quad \psi = (\psi_L, \psi_R)^T$$

- The ground state of this Hamiltonian is given by

$$|GS\rangle = \exp\left[\sum_{k>0} \frac{m}{\sqrt{m^2+k^2}+k} \left(\psi^\dagger_{Lk}\psi_{Rk} + \psi^\dagger_{R-k}\psi_{L-k} \right) \right] |G_L\rangle \otimes |G_R\rangle$$

  where $\psi_{L,Rk}$ is the Fourier component of $\psi_{L,R}(x)$, and $|G_{L,R}\rangle$ is the Fock vacuum of the left- and right-moving sector. In the limit $m \to \infty$ $(m/(v_F k) \to \infty)$, $|GS\rangle$ reduces to the boundary states of the free massless fermion theory.

More details

- SPT phases in (1+1)d are classified by group cohomology $H^2(G, U(1))$. [Chen-Gu-Liu-Wen (02)] Recall:

$$V(g)V(h) = e^{i\alpha(g,h)}V(gh) \tag{46}$$

- CFT context: Discrete torsion phases in CFT [Vafa (86) ...] and in BCFT [Douglas (98) ...].
- Discrete torsion phases and entanglement spectrum (symmetry-protected degeneracy):
  Twisted partition function:

$$Z_{AB}^h = \mathrm{Tr}_{\mathcal{H}_{AB}}\left[\hat{h}e^{-\beta H_{AB}^{open}}\right]$$

  vanishes when $A \neq B$. (symmetry-enforced vanishing of partition function).
  Exchange time and space, $Z_{AB}^h = {}_h\langle A|e^{-\frac{\ell}{2}H^{closed}}|B\rangle_h$ and insert g to show

$$[\varepsilon_B(g|h) - \varepsilon_A(g|h)]Z_{AB}^h = 0$$

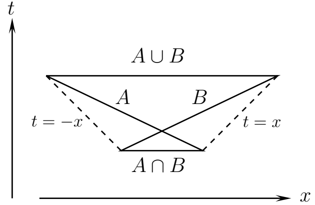
- Entropic c-function [Casini (04)] :

$$c_E(R) := 3R\frac{dS(R)}{dR} \tag{47}$$

- At critical points, $c_E = c$ (central charge).
- From strong subadditivity:

$$S_A + S_B \geq S_{A\cap B} + S_{A\cup B} \tag{48}$$

can argue that S is concave w.r.t. $\log R$:

- 

$$2S(\sqrt{rR}) \geq S(R) + S(r) \tag{49}$$

Taking the limit: $r \to R$:

$$\frac{c_E'(R)}{3} = S'(R) + RS''(R) \leq 0 \tag{50}$$

Remark: F-theorem

- Is there an analogue of c and c-theorem in (2+1)d? (No weyl anomaly in (2+1)d)
- EE of a disc D of radius R [Ryu-Takayanagi (06), Myers-Sinha (10)]:

$$S_D(R) = \alpha \frac{2\pi R}{\epsilon} - F(R) \tag{51}$$

  F at the critical point is a universal constant. C.f. topological entanglement entropy.
- F is related to the partition functions on a sphere S^3, $F = -\log Z(S^3)$ [Casini-Huerta-Myers (11)].
  F-theorem: [Jafferis et al (11), Klebanov et al (11)]:

$$F_{UV} \geq F_{IR}$$

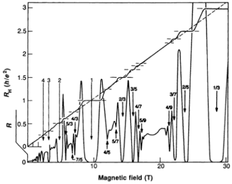
- Entropic $\mathcal{F}$ function: [Liu-Mezei (13)]

$$\mathcal{F}(R) = \left(R\frac{\partial}{\partial R} - 1 \right) S_D(R) \tag{52}$$

  $\mathcal{F}(R)|_{CFT} = F$ and $\mathcal{F}'(R) \leq 0$ [Casini-Huerta (12)]
- Applications [Grover (12), ...] Stationarity ?

Topological phases of matter

- Topologically ordered phases: phases which support anyons ($\simeq$ support topology dependent ground state degeneracy)
- E.g., fractional quantum Hall states,



$\mathbb{Z}_2$ quantum spin liquid, etc.

- Quantum phases which are not described by the symmetry-breaking paradigm. (I.e., Landau-Ginzburg type of theories)
- Instead, characterized by properties of anyons (fusion, braiding, etc.) (I.e., topological quantum field theories)

Algebraic theory of anyons

- (Bosonic) topological orders are believed to be fully characterized by a unitary modular tensor category (UMTC).
- (i) Finite set of anyons $\{1, a, b, \ldots\}$ equipped with quantum dimensions $\{1, d_a, d_b, \ldots\}$ $(d_a \geq 1)$. Total quantum dimension D:

$$D = \sqrt{\sum_a d_a^2} \tag{53}$$

- (ii) Fusion $a \times b = \sum_c N_{ab}^c c$.
- (iii) The modular T matrix, $T = \mathrm{diag}\,(1, \theta_a, \theta_b, \ldots)$ where $\theta_a = \exp 2\pi i h_a$ is the self-statistical angle of a with h_a the topological spin of a.
- (iv) The modular S matrix encodes the braiding between anyons, and given by ("defined by")

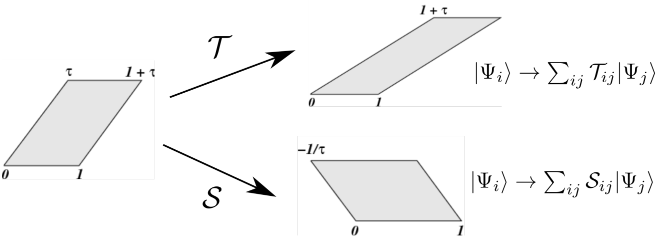
$$S_{ab} = \frac{1}{D} \sum_c N_{ab}^c \frac{\theta_c}{\theta_a \theta_b} d_c. \tag{54}$$

Chiral central charge

- There may be topologically ordered phases with the same braiding properties, but different values of c, the chiral central charge of the edge modes.
- Albeit the same braiding properties, they cannot be smoothly deformed to each other without closing the energy gap.
- Topological order is conjectured to be fully characterized by (S, T, c)

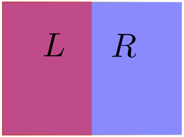
Ground states and S and T

- Ground state degeneracy depending on the topology of the space (topological ground state degeneracy), related to the presence of anyons [Wen (90)]
- Ground state degeneracy on a spatial torus, $\{|\Psi_i\rangle\}$.
- S and T are extracted from the transformation law of $\{|\Psi_i\rangle\}$ [Wen (92)]



Topologically ordered phases and quantum entanglement

- Consider: the reduced density matrix ρ_A obtained from a ground state $|GS\rangle$ of a topologically-ordered phase by tracing out half-space.



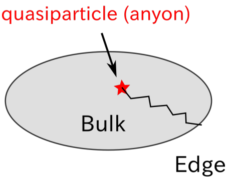
$$\rho_A \propto \mathrm{Tr}_R\, e^{-\epsilon H} |B.S.\rangle\langle B.S.| e^{-\epsilon H}$$

[Qi-Katsura-Ludwig (12), Fliss et al. (17), Wong (17)]

- Different (Ishibashi) boundary states correspond to different ground states
- With this explicit form of the reduced density matrix, various entanglement measures can be computed: [Wen-Matsuura-SR]
  - the entanglement entropy
  - the mutual information
  - the entanglement negativity

Bulk-boundary correspondence

- Bulk anyon $\leftrightarrow$ twisted boundary conditions at edge:



- Bulk wfn $|\Psi_i\rangle \leftrightarrow$ boundary partition function χ_i
- Bulk S and T matrices acting on $|\Psi_i\rangle$ on *spatial torus*
  $\leftrightarrow$
  S and T matrices acting on boundary partition function χ_i on *spacetime torus* [Cappelli (96), ...]

$$\chi_a\left(e^{-\frac{4\pi\beta}{l}}\right) = \sum_{a'} \mathcal{S}_{aa'}\chi_{a'}\left(e^{-\frac{\pi l}{\beta}}\right) \qquad (55)$$

- Conformal BC: $L_n|b\rangle = \overline{L}_{-n}|b\rangle$ ($\forall n \in \mathbb{Z}$)
- Ishibashi boundary state:

$$|h_a\rangle\!\rangle \equiv \sum_{N=0}^{\infty} \sum_{j=1}^{d_{h_a}(N)} |h_a, N; j\rangle \otimes \overline{|h_a, N; j\rangle} \quad (56)$$

- Topological sector dependent normalization (regularization):

$$|\mathfrak{h}_a\rangle\!\rangle = \frac{e^{-\epsilon H}}{\sqrt{\mathfrak{n}_a}} |h_a\rangle\!\rangle \quad \text{so that} \quad \langle\!\langle \mathfrak{h}_a|\mathfrak{h}_b\rangle\!\rangle = \delta_{ab}. \quad (57)$$

- More generically, one can consider a superposition $|\psi\rangle = \sum_a \psi_a |\mathfrak{h}_a\rangle\!\rangle$
- Reduced density matrix:

$$\begin{aligned}\rho_{L,a} &= \mathrm{Tr}_R(|\mathfrak{h}_a\rangle\!\rangle\langle\!\langle \mathfrak{h}_a|) \\ &= \sum_{N,j} \frac{1}{\mathfrak{n}_a} e^{-\frac{8\pi\epsilon}{l}(h_a + N - \frac{c}{24})} |h_a, N; j\rangle\langle h_a, N; j|. \end{aligned} \quad (58)$$

Some details

- Trance of the reduced density matrix:

$$\mathrm{Tr}_L\,(\rho_{L,a})^n = \frac{1}{\mathfrak{n}_a^n}\chi_a\big(e^{-\frac{8\pi n\epsilon}{l}}\big) = \frac{\chi_a\big(e^{-\frac{8\pi n\epsilon}{l}}\big)}{\chi_a\big(e^{-\frac{8\pi\epsilon}{l}}\big)^n} \tag{59}$$

- Modular transformation

$$\begin{aligned}\chi_a\big(e^{-\frac{8\pi n\epsilon}{l}}\big) &= \sum_{a'}\mathcal{S}_{aa'}\chi_{a'}\big(e^{-\frac{\pi l}{2n\epsilon}}\big)\\ &\to \mathcal{S}_{a0}\times e^{\frac{\pi c l}{48 n\epsilon}} \quad (l/\epsilon\to\infty),\end{aligned} \tag{60}$$

i.e., only the identity field I, labeled by "0" here, survives the limit.

- Hence, in the thermodynamic limit $l/\epsilon\to\infty$:

$$\mathrm{Tr}_L\,(\rho_{L,a})^n = \frac{\sum_{a'}\mathcal{S}_{aa'}\chi_{a'}\big(e^{-\frac{\pi l}{2n\epsilon}}\big)}{\Big[\sum_{a'}\mathcal{S}_{aa'}\chi_{a'}\big(e^{-\frac{\pi l}{2\epsilon}}\big)\Big]^n} \to e^{\frac{\pi c l}{48\epsilon}\left(\frac{1}{n}-n\right)}(\mathcal{S}_{a0})^{1-n}, \tag{61}$$

- Final result:

$$S_L^{(n)} = \frac{1+n}{n} \cdot \frac{\pi c}{48} \cdot \frac{l}{\epsilon} - \ln \mathcal{D} + \frac{1}{1-n} \ln d_a^{1-n}$$
$$S_L^{\mathsf{vN}} = \frac{\pi c}{24} \cdot \frac{l}{\epsilon} - \ln \mathcal{D} + \ln d_a \quad (62)$$

where

$$\mathcal{S}_{a0} = d_a/\mathcal{D} \quad (63)$$

is the quantum dimension.

Lessons

- Entanglement cut may be more useful than physical cut.
- Entanglement and universal information of many-body systems.
- Entanglement can tell the direction of the RG flow.
- Entanglement and spacetime physics
- Entanglement has a topological interpretation in particular in topological field theories.
- ...