Quantum Hall transitions: history, recent developments, and challenges
Part I

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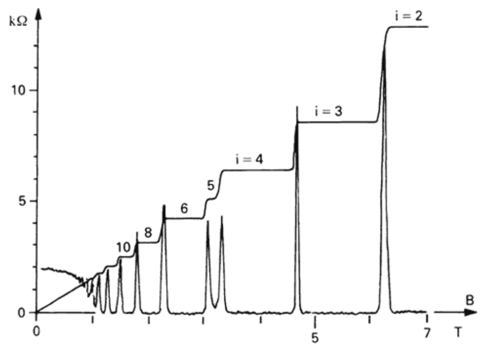
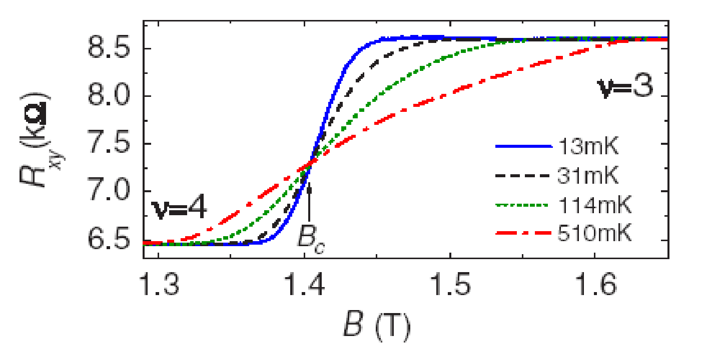
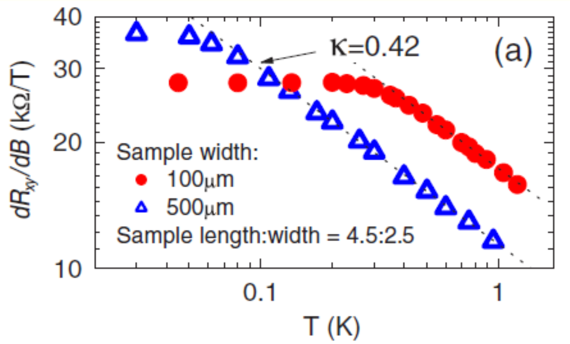
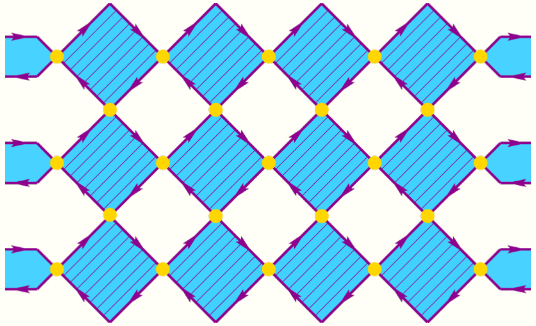
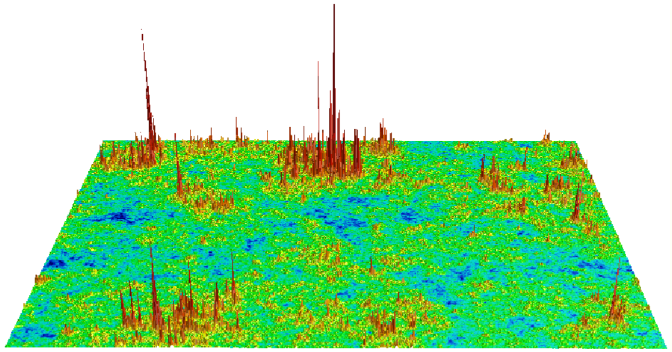
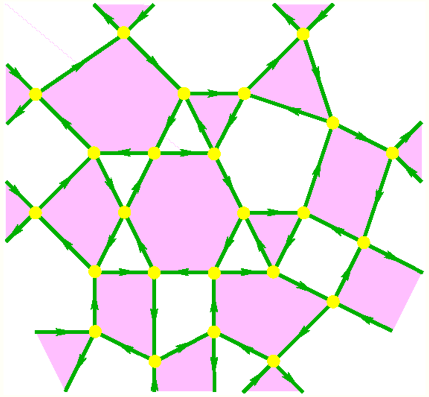




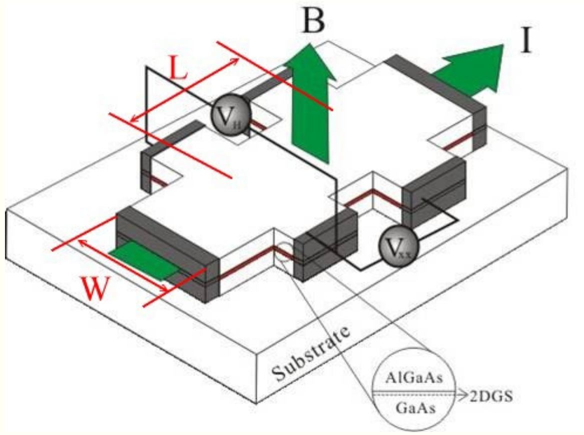


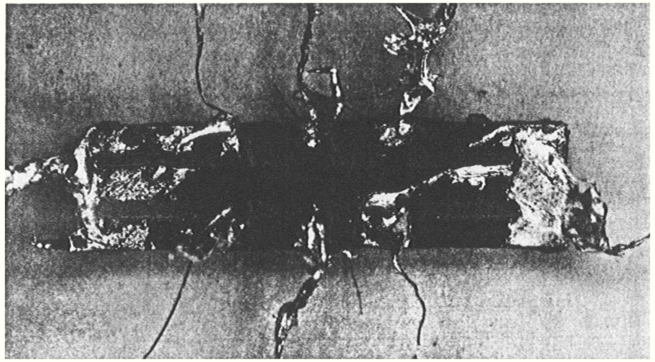


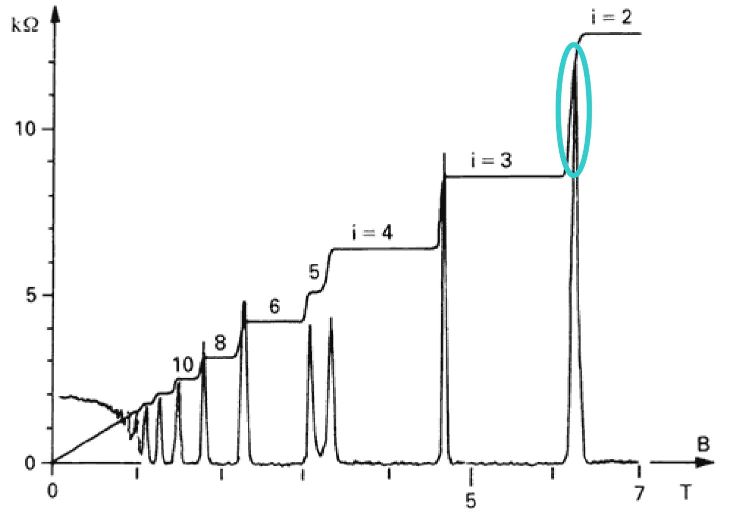




Quantum Hall effect



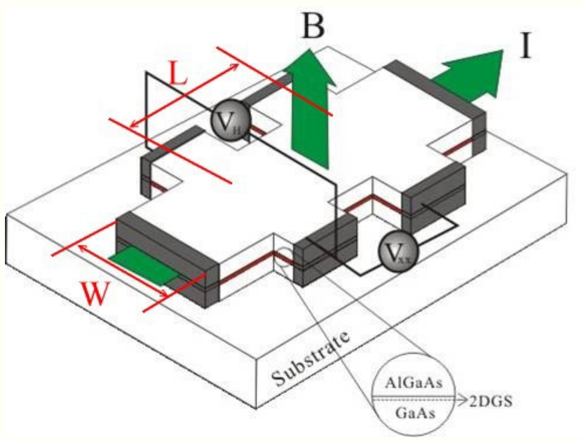


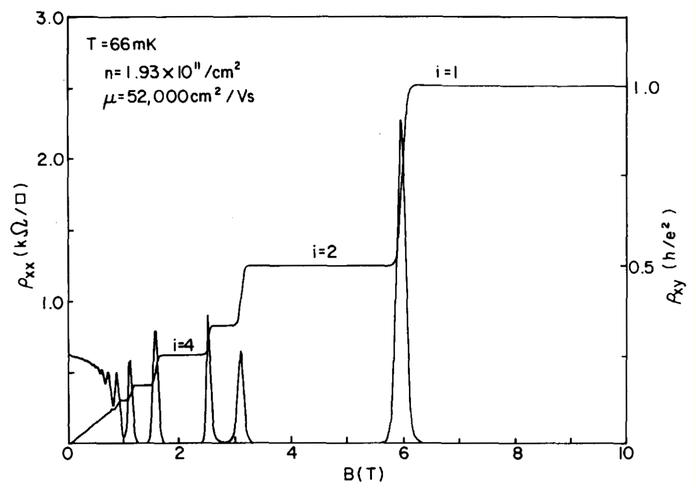


- 2D electron gas in strong magnetic field at low temperature
- Hall resistance (resistivity) shows quantized plateaus
- Universality: localization and topology
- Transitions between plateaus also show universal features

Quantum Hall experiments

- Experiments on low-mobility $Al_xGa_{1-x}As/GaAs$ heterostructures
  - Integer plateaus in the Hall resistivity and peaks in the longitudinal resistivity





H. P. Wei, unpublished

$$\rho_{xy} = \frac{1}{i}\frac{h}{e^2},$$

i – an integer

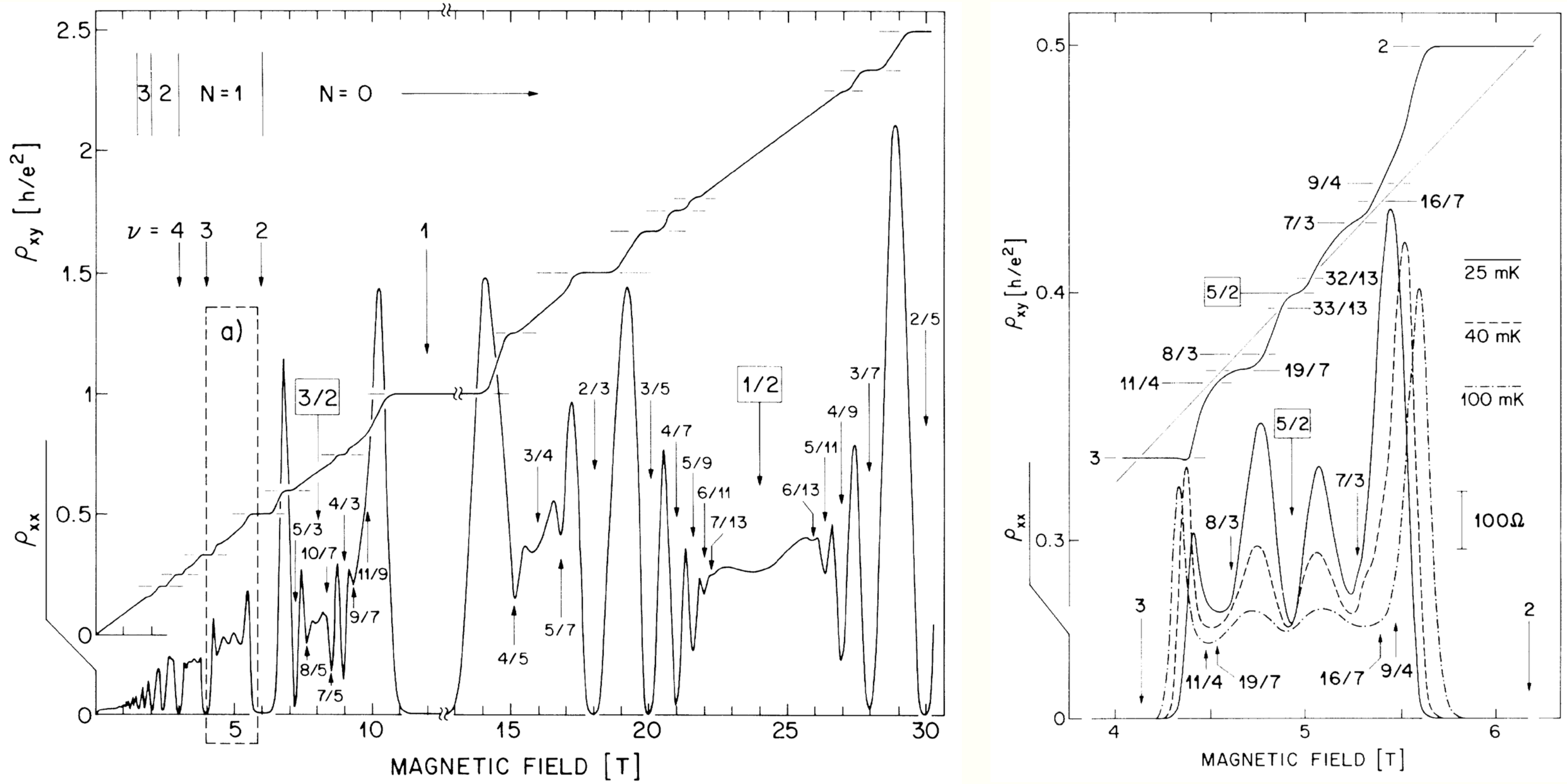
  - The role of disorder
  - Fundamental constants determine the "resistance quantum" or the von Klitzing constant

$$R_K = h/e^2 = 25\,812.80745...\Omega$$

  - Resistivity and resistance have the same units in 2D

Quantum Hall experiments

- Experiments on high-mobility $Al_xGa_{1-x}As/GaAs$ heterostructures ($\mu = 1.3 \times 10^6 \frac{cm^2}{V \cdot s}$)
  - Fractional plateaus. Notice changes with temperature



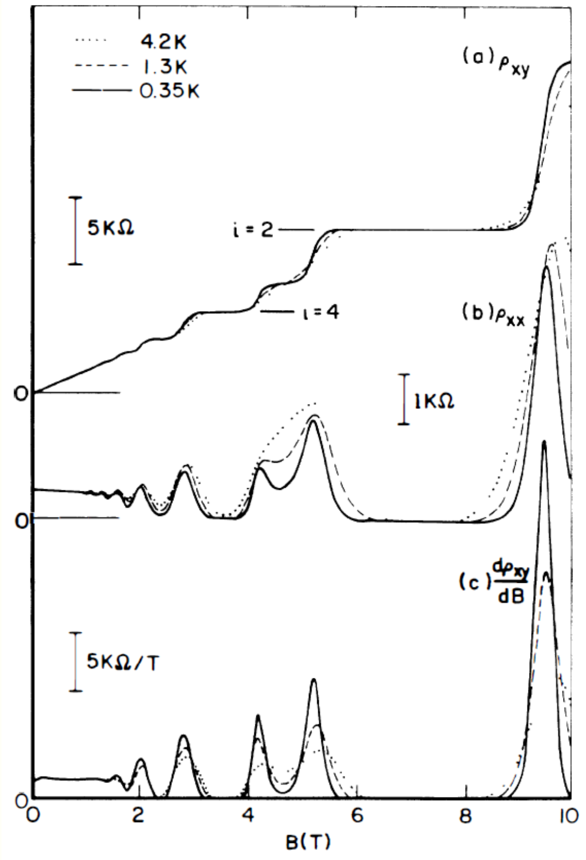
$$\rho_{xy} = \frac{1}{\nu}\frac{h}{e^2},$$

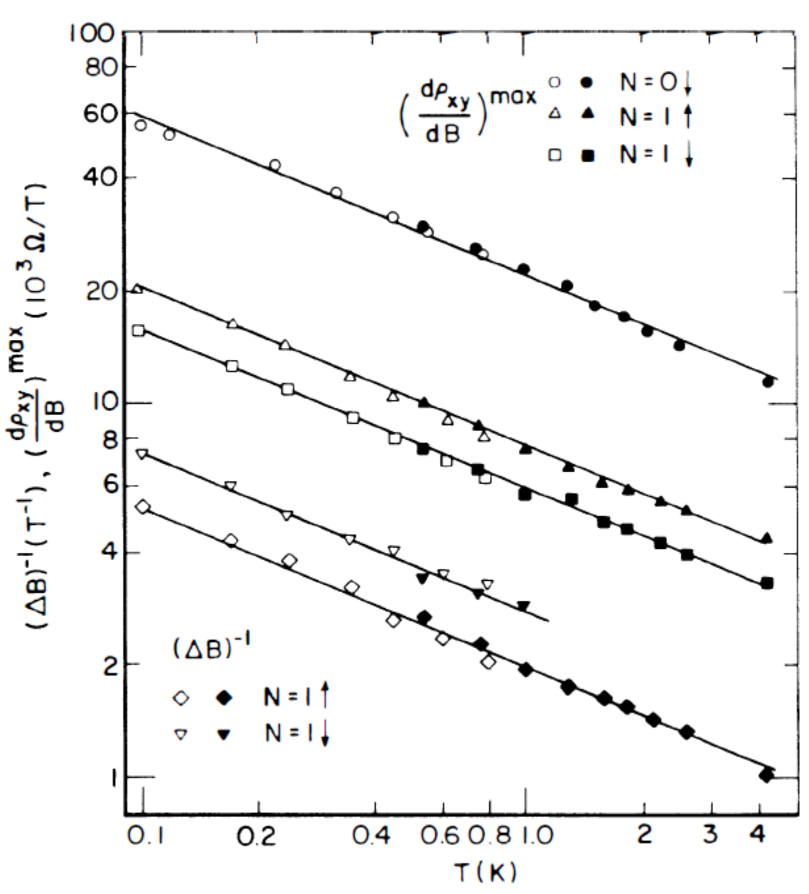
$$\nu - \text{a (simple) fraction}$$

R. Willet et al. PRL, 59, 1776 (1987)

IQH transitions: experiments

- $In_xGa_{1-x}As/InP$ heterostructures: temperature dependence





$$\left(\frac{dR_{xy}}{dB}\right)^{\max} \sim T^{-\kappa}, \quad \Delta B \sim T^{\kappa},$$

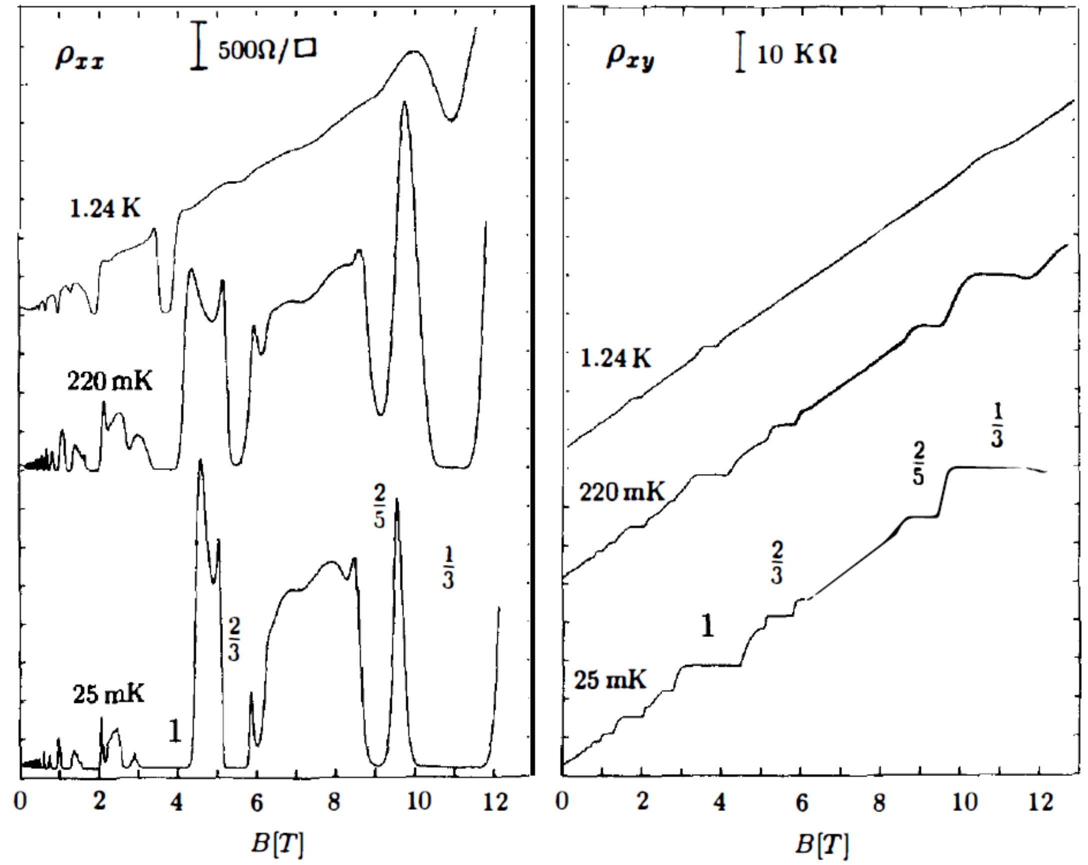
$$\kappa \approx 0.42 \pm 0.04$$

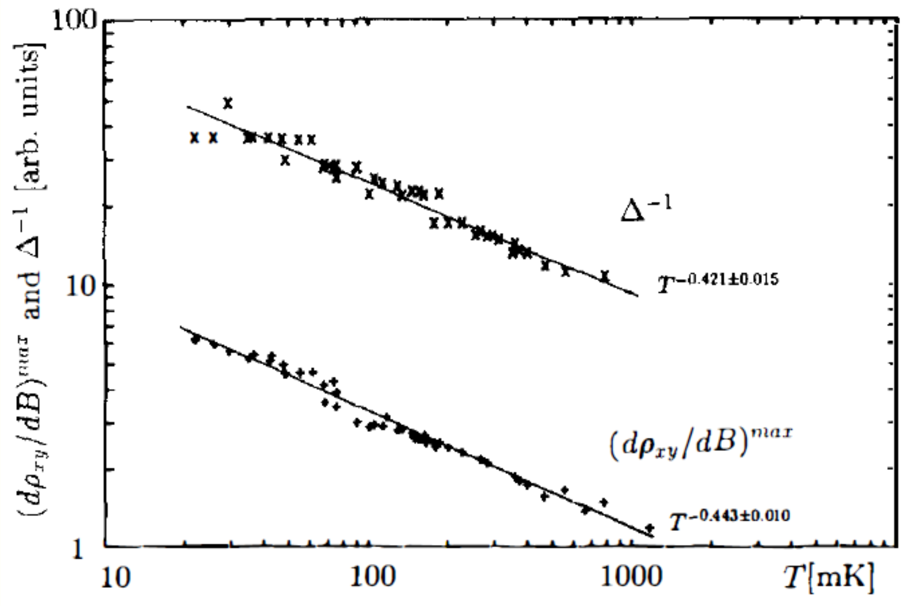
H. P. Wei et al. *“Experiments on Delocalization and Universality in the Integral Quantum Hall Effect”* PRL 61, 1294 (1988)

- The widths and steepness of transitions scale as powers of temperature

FQH transitions: experiments

- $Al_xGa_{1-x}As/GaAs$ heterostructures: direct transition between





$$\left(\frac{dR_{xy}}{dB}\right)^{\max} \sim T^{-\kappa}, \quad \Delta B \sim T^{\kappa},$$

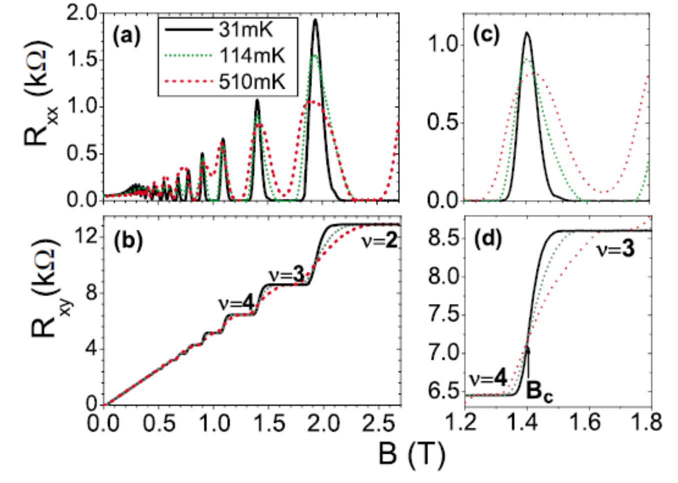
$$\kappa \approx 0.43 \pm 0.02$$

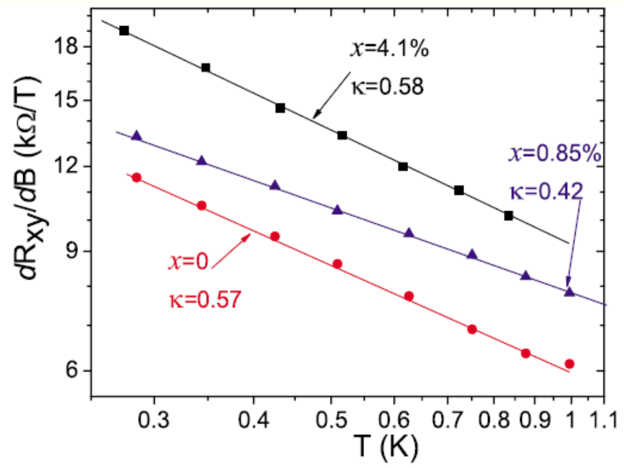
L. Engel et al, *"Critical Exponent in the Fractional Quantum Hall Effect"* Surf. Sci. 229, 13 (1990)

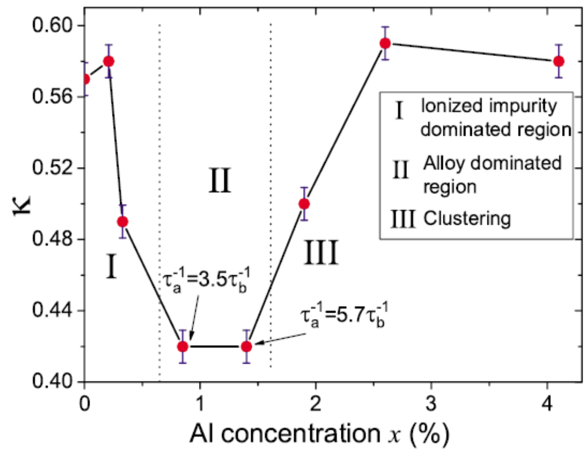
- The widths and steepness of transitions scale as powers of temperature

Universality?

- $Al_xGa_{1-x}As/Al_{0.33}Ga_{0.67}As$ heterostructures





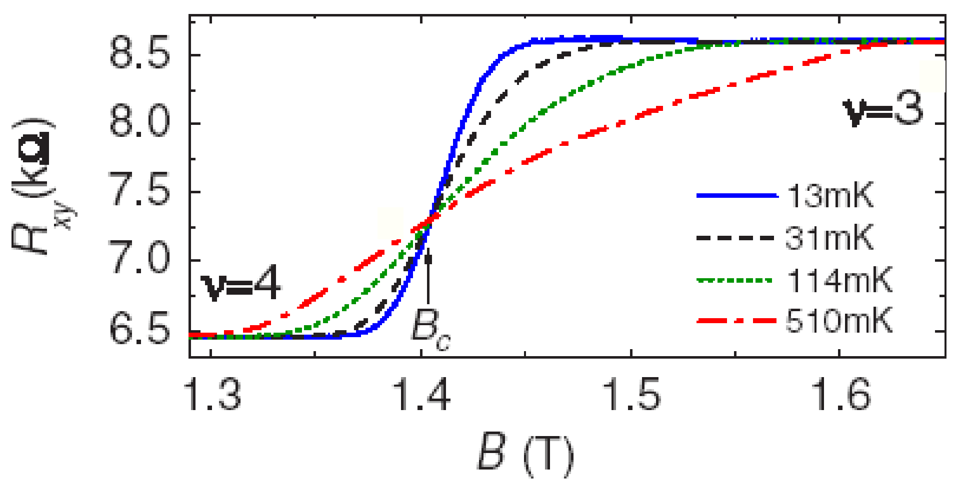


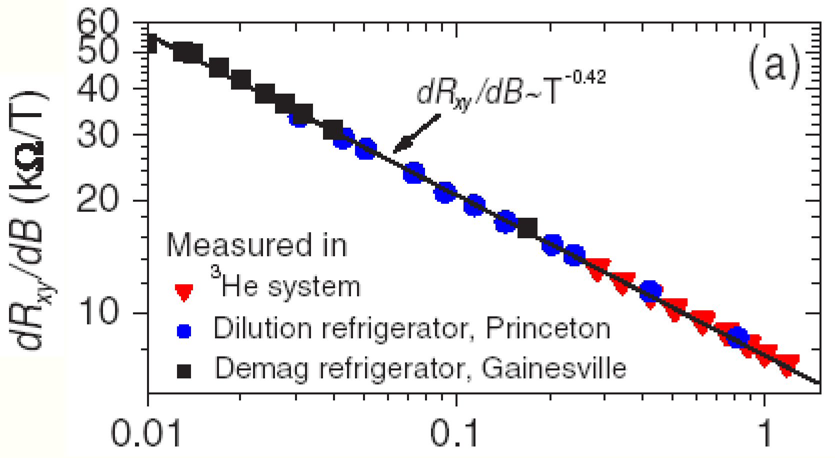
W. Li et al. "*Scaling and Universality of Integer Quantum Hall Plateau-to-Plateau Transitions*" PRL 94, 206807 (2005)

- Exponent κ depends on disorder?
- Universal scaling observed for short-range (alloy) disorder
- The issue of universality keeps coming up

Universal critical scaling near IQH transition

- $Al_xGa_{1-x}As/Al_{0.33}Ga_{0.67}As$ heterostructures with $x = 0.85\%$





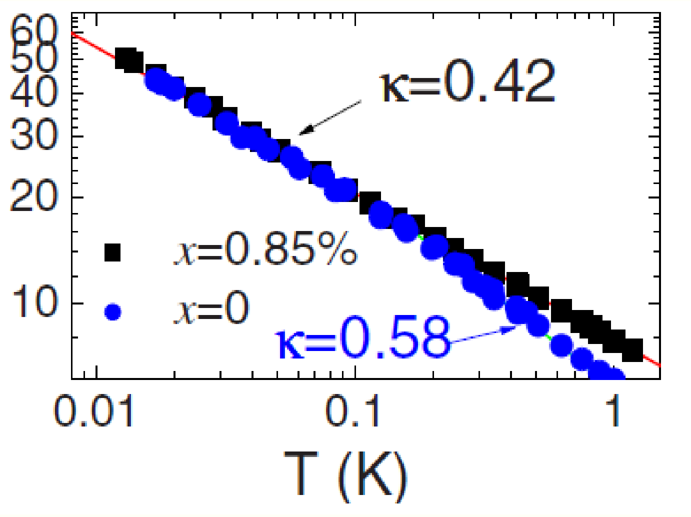
W. Li et al. "*Scaling in Plateau-to-Plateau Transition: A Direct Connection of Quantum Hall Systems with the Anderson Localization Model*" PRL 102, 216801 (2009)

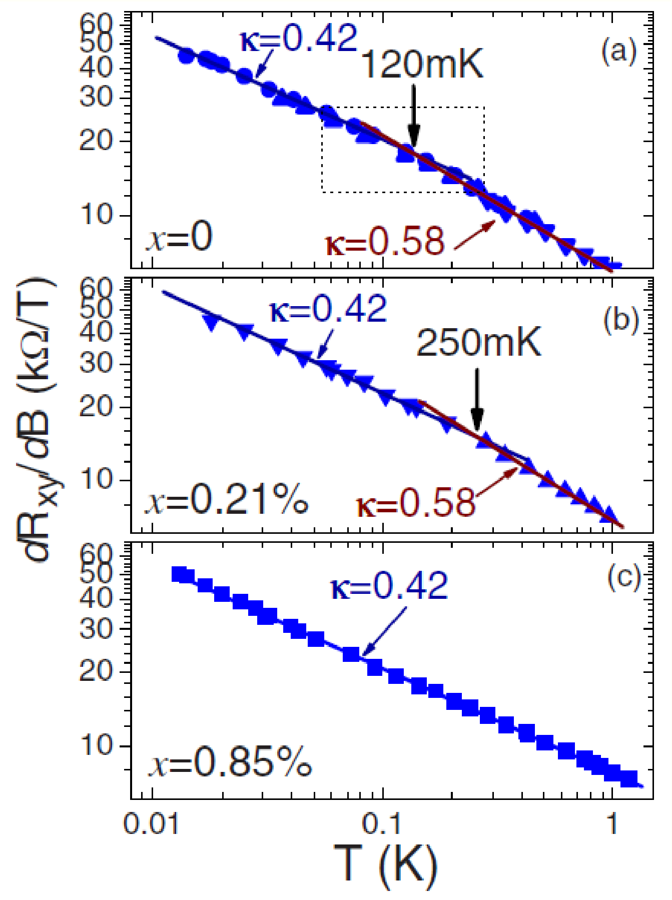
- "Perfect scaling through two full decades of temperature from 1.2 K down to 12 mK"

$$\left.\frac{dR_{xy}}{dB}\right|_{B_c} \sim T^{-\kappa}, \quad \kappa = 0.42 \pm 0.01$$

Crossover to universal critical scaling near IQH transition

- $Al_xGa_{1-x}As/Al_{0.33}Ga_{0.67}As$ heterostructures with different x





W. Li et al. "*Crossover from the nonuniversal scaling regime to the universal scaling regime in quantum Hall plateau transitions*" PRB 81, 033305 (2010)

Universal critical scaling near IQH and FQH transitions

- QH transitions in trilayer graphene

S. Kaur et al., Nature Communications (2024)

Universality of quantum phase transitions in the integer and fractional quantum Hall regimes

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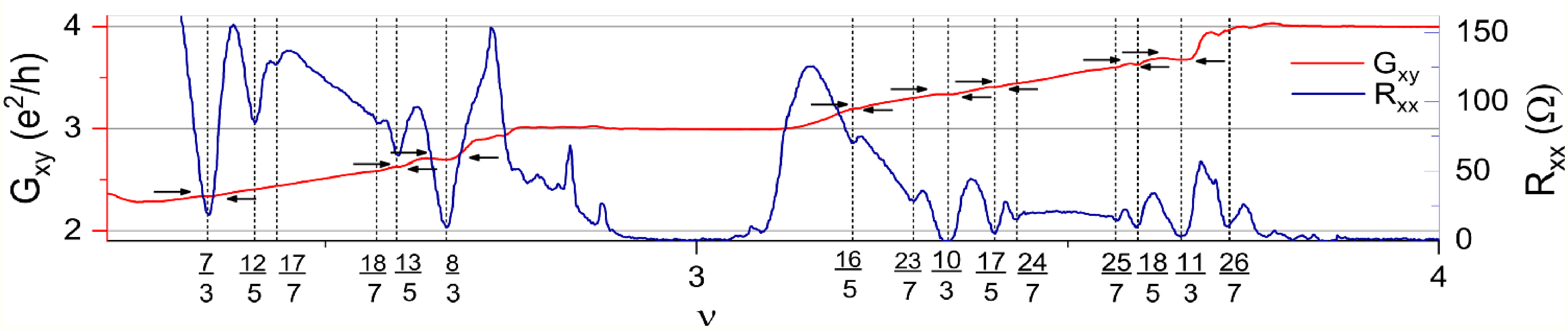
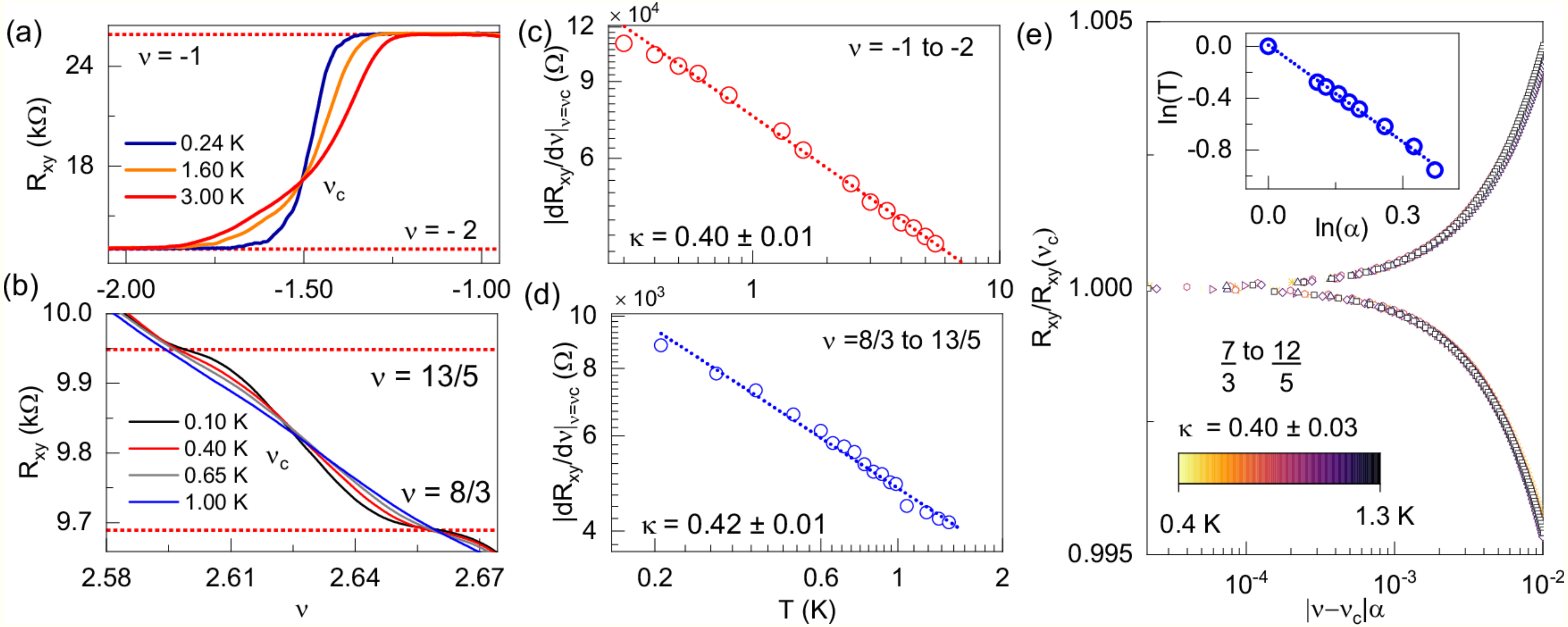
Simrandeep Kaur[1,8], Tanima Chanda[1,8], Kazi Rafsanjani Amin[2,8], Divya Sahani[1], Kenji Watanabe[3], Takashi Taniguchi[4], Unmesh Ghorai[5], Yuval Gefen[6], G. J. Sreejith[7] & Aveek Bid[1]



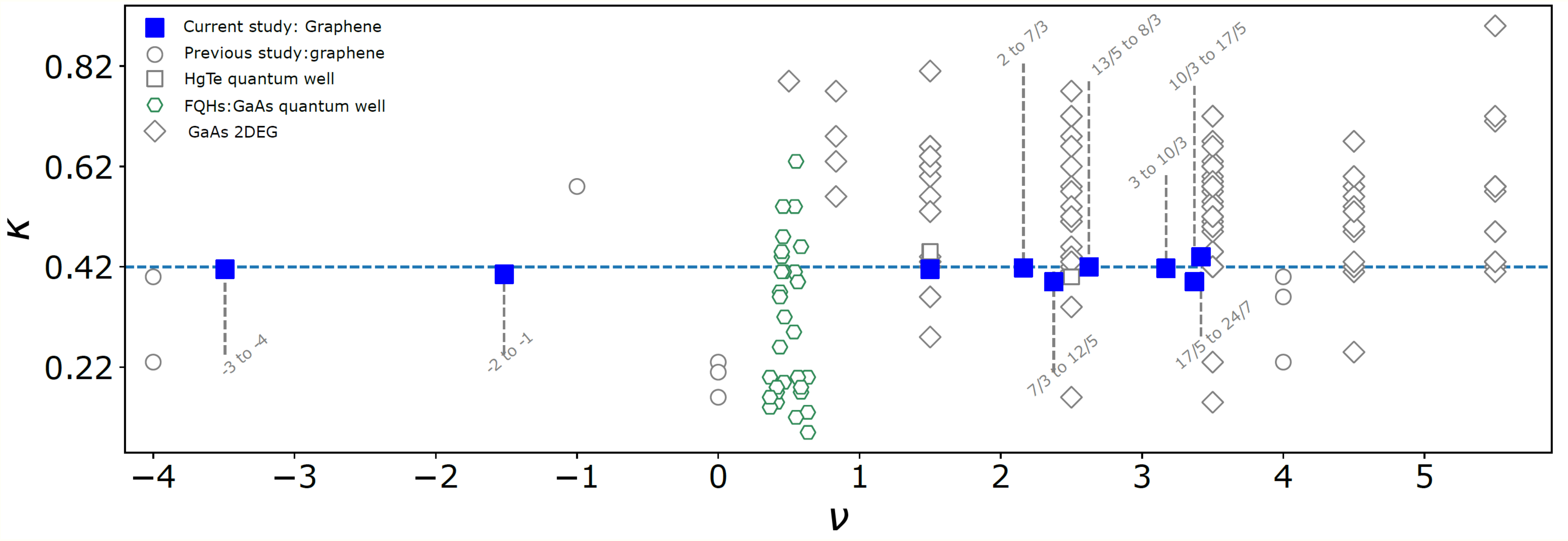
Universal critical scaling near IQH and FQH transitions

- QH transitions in trilayer graphene



Universal critical scaling near IQH and FQH transitions

- QH transitions in trilayer graphene: comparison with previous studies



Basic theoretical picture: quantum phase transitions at $T = 0$

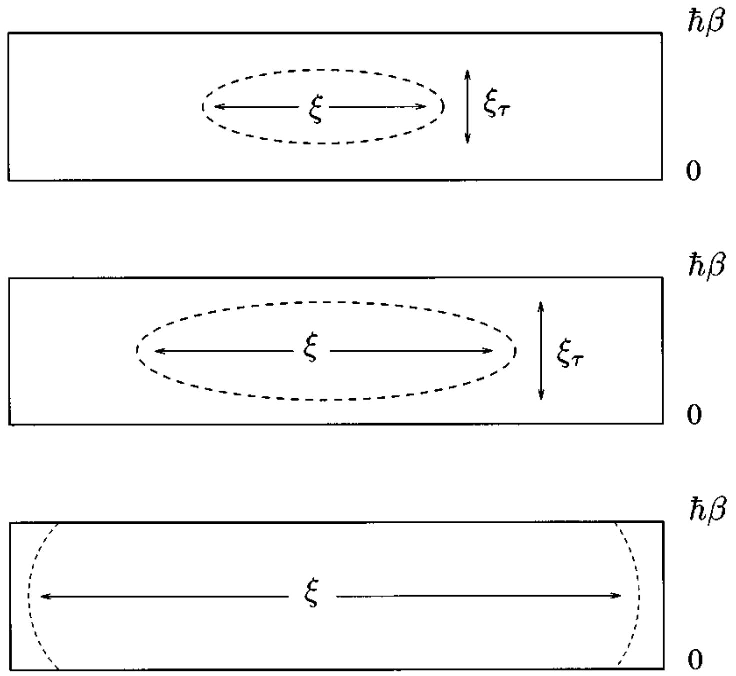
- Quantum phase transitions ($\hbar = k_\mathrm{B} = 1$)
  - Change in the ground state upon tuning a parameter K (coupling constant) in the Hamiltonian
  - Diverging (correlation) length scale near the critical coupling K_c, as $\delta = K - K_c \to 0$: $\xi \sim |\delta|^{-\nu}$
  - Diverging (correlation) time scale near the critical coupling K_c, as $\delta = K - K_c \to 0$: $\xi_\tau \sim \xi^z \sim |\delta|^{-z\nu}$
  - Closing gap near the critical coupling K_c, as $\delta = K - K_c \to 0$: $\Delta \sim |\delta|^{z\nu}$
  - Dynamical scaling exponent z relates temporal and spatial scales
- The diverging scales determine the scaling forms of measured observables
  - Scaling at $T = 0$ close to K_c $$O(k, \omega, K) = \xi^{\Delta_O} F_O(k\xi, \omega\xi_\tau)$$
  - Scaling dimension Δ_O of the observable O is usually different from its naïve (engineering) dimension
- Theory goals: compute critical exponents ν, z, dimensions of observables Δ_O, and scaling functions

Scaling at $T \neq 0$

- Finite-size scaling
  - $T > 0$ imposes a finite length $L_\tau = 1/T$ in the (imaginary) time direction

$$O(\omega, T, K) = \xi^{\Delta_O} F_O(\omega\xi_\tau, L_\tau/\xi_\tau) = L_\tau^{\Delta_O/z} f_O(\omega/T, \delta/T^{1/z\nu})$$

  - Temperature cuts off coherent quantum fluctuations at $\omega \sim T$
- Dephasing: finite T destroys coherence of quantum fluctuations
  - Two scaling regimes for frequency scaling separated by $T_\omega \sim \omega$
  - By the dynamic scaling, there is a “dephasing” length $L_\phi \sim L_\tau^{1/z} \sim T^{-1/z}$ beyond which fluctuations are classical
  - Dephasing rate $\xi_\tau^{-1} \sim T$
- Caveat: relevance of interactions (more later)

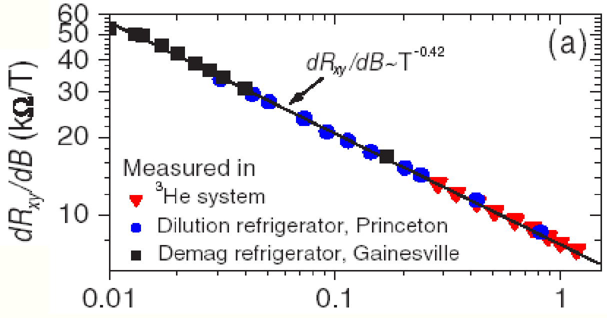


Quantum critical scaling at QH transitions

- Longitudinal and Hall resistivities $\rho_{L,H}$ have zero scaling dimension

$$\rho_{L,H}(\omega, T, B) = f_{L,H}(\omega/T, \delta/T^{1/z\nu}), \qquad \delta = \frac{B - B_c}{B_c}$$

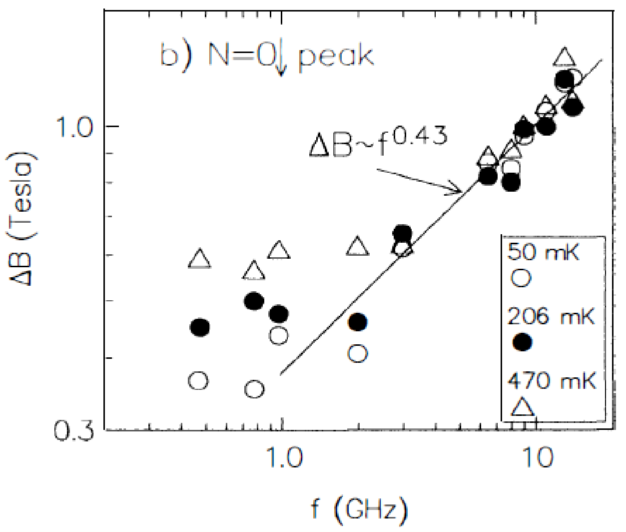
- DC transport, temperature scaling



$$\rho_{L,H}(T, B) = f_{L,H}(\delta/T^{1/z\nu}) \quad \Rightarrow \quad \Delta B^{-1}, \left.\frac{d\rho_H}{dB}\right|_{B_c} \sim T^{-\kappa}, \quad \kappa = \frac{1}{z\nu}$$

- Frequency scaling for $\hbar\omega \gg k_B T$

$$\rho_{L,H}(\omega, B) = f_{L,H}(\delta/\omega^{1/z\nu}) \quad \Rightarrow \quad \Delta B \sim \omega^{\kappa}, \quad \kappa = \frac{1}{z\nu}$$



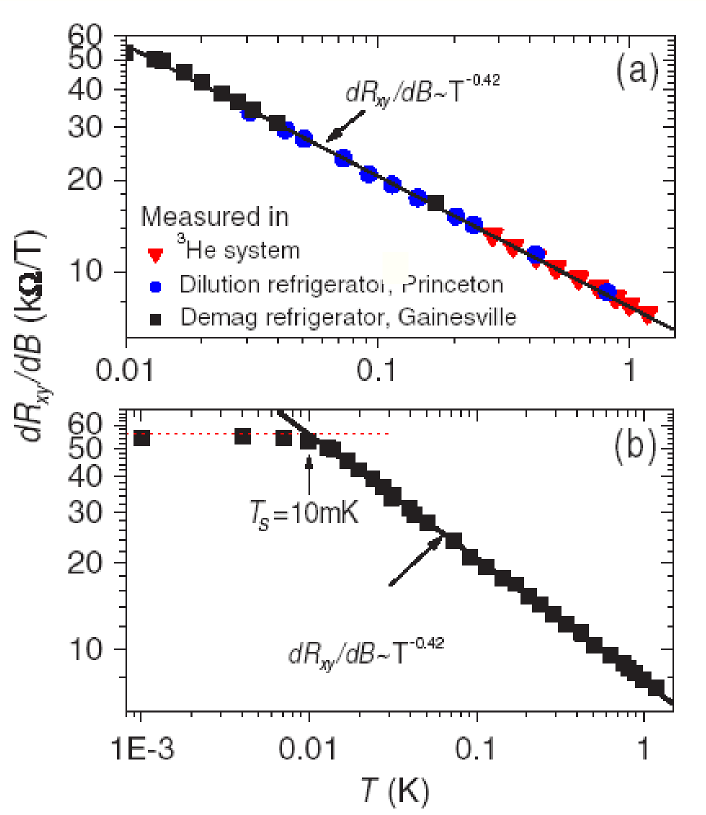
L. Engel et al, *"Microwave Frequency Dependence of Integer Quantum Hall Effect: Evidence for Finite-Frequency Scaling"* PRL 71, 2638 (1993)

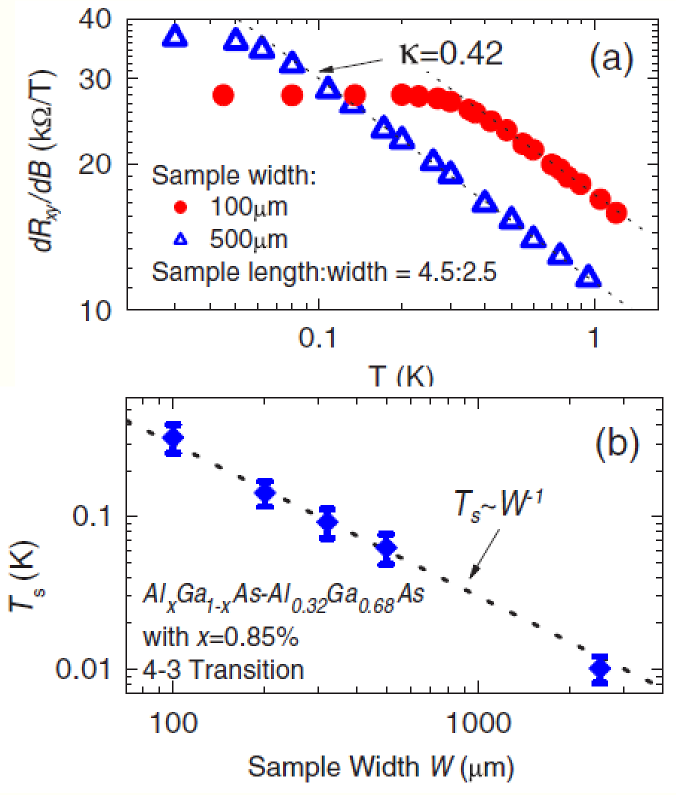
- Is it possible to separately measure z and/or ν?

Universal critical scaling near IQH transition

- Experiments at very low temperature

W. Li et al, 102 (2009); PRB 81 (2010)





- Saturation of scaling when $L_\phi \sim T^{-1/z} \approx W$ (sample width)

- Localization length exponent and dynamical exponent $\nu \approx 2.38, \quad z = 1$

Nonlinear critical current scaling

- Theoretical scaling picture
  - Finite electric field E introduces finite length and time scales l_E and l_E^z
  - At $T = 0$ these scales replace the "dephasing" scales
  - Relate $eEl_E \sim \hbar l_E^{-z}$. This gives $l_E \sim E^{-1/(z+1)}$
  - Scaling form of nonlinear resistivities

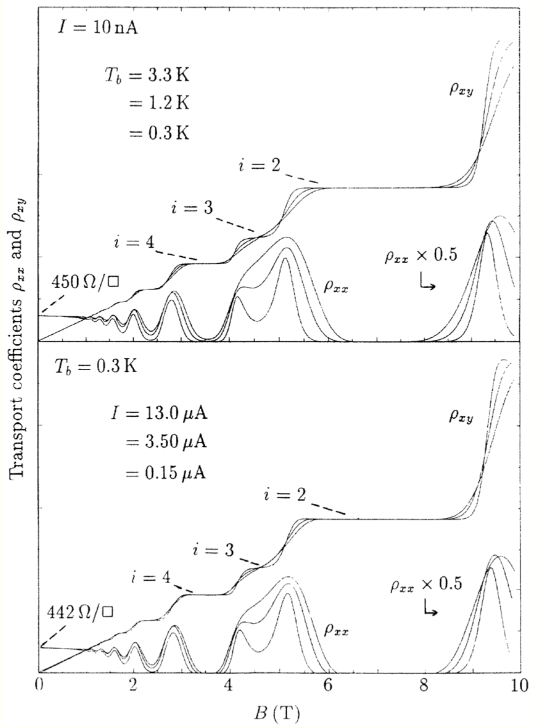
$$\rho_{L,H}(\omega, T, B) = g_{L,H}(\delta/T^{1/z\nu}, l_E/\xi) = \tilde{g}_{L,H}(\delta/T^{1/z\nu}, \delta/E^{1/(z+1)\nu})$$

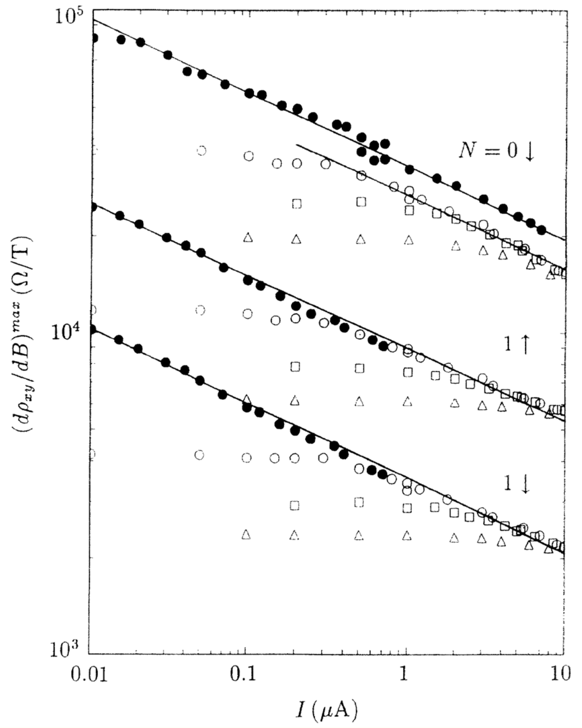
  - Can separately determine $z\nu$ and $(z+1)\nu$
  - Two scaling regimes separated by $T_0(E) \sim l_E^{-z} \sim E^{z/(z+1)}$
  - Current scaling for $T \ll T_0(E)$ (high E)

$$\rho_{L,H}(E, B) = f_{L,H}(\delta/E^{1/(z+1)\nu}) \quad \Rightarrow \quad \Delta B^{-1}, \frac{d\rho_H}{dB}^{\max} \sim E^{-b}, \quad b = \frac{1}{(z+1)\nu}$$

Nonlinear critical current scaling

- Experiments on $In_xGa_{1-x}As/InP$ heterostructures





$$b = \frac{1}{(z+1)\nu} \approx 0.23 \pm 0.02$$

H. P. Wei et al. *"Current scaling in the integer quantum Hall effect"* PRB 50, 14609 (1994)

- All data is consistent with $\nu \approx 2.4$, and $z = 1$

Nonlinear critical current scaling

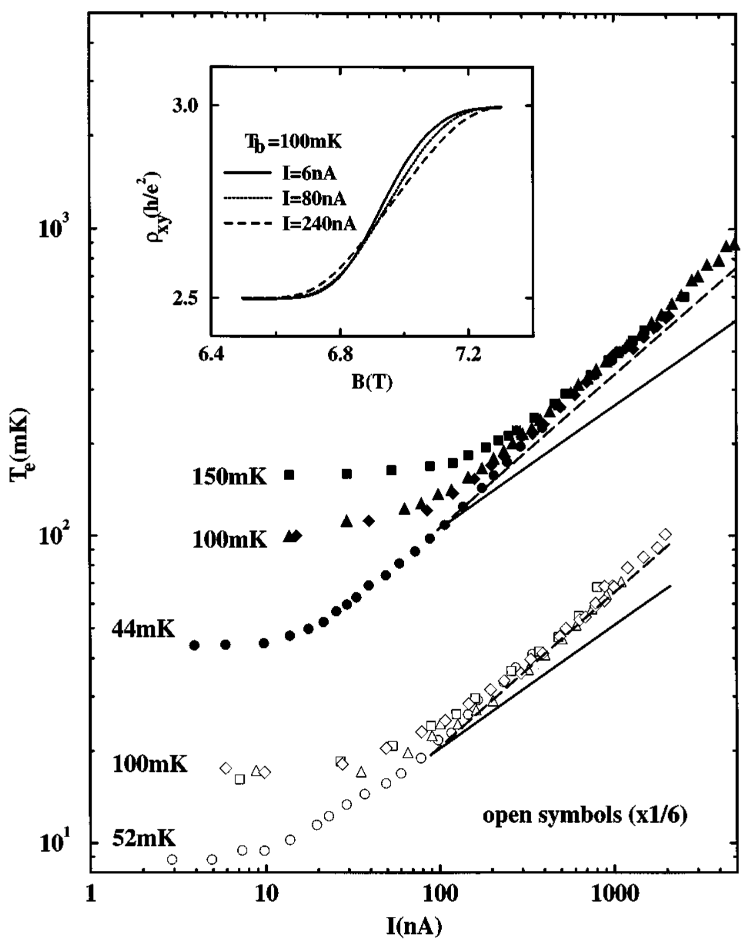
- Experiments on $Al_xGa_{1-x}As/GaAs$ heterostructures

- Crossover temperature $T_0(E) \sim E^{z/(z+1)}$ as the effective electronic temperature

$$a = \frac{z}{z+1} \approx 0.49 \pm 0.03$$

- All data is consistent with $\nu \approx 2.4$, and $z = 1$

E. Chow et al *"Experiments on inelastic scattering in the integer quantum Hall effect"* PRL 77, 1143 (1996)



Relevance of interactions and dephasing

- Mesoscopic physics point of view: start with noninteracting or weakly-interacting quasiparticles (electrons) subject to disorder and strong magnetic field
- This system in 2D has an (Anderson localization) integer quantum Hall transition – an interesting and important problem in itself, and I will focus on it later
- Then add various interactions (electron-electron, electron-phonon, etc.) and treat them perturbatively
- This leads to various dephasing mechanisms with dephasing rates $\tau_\phi^{-1} \sim T^p$ and lengths $l_\phi \sim T^{-1/z_T}$, with $z_T = 2/p \neq z$ and a mechanism-specific exponent $p \geq 1$
- Whether this picture survives at the quantum critical point depends on the (ir)relevance of interactions at the non-interacting fixed point and is determined by an RG treatments of interactions
- It turns out that the (ir)relevance of interactions depends on their range:
  - Short-range interactions are irrelevant, long-range (Coulomb) interactions are relevant

D.-H. Lee and Z. Wang PRL 76, 4014 (1996)

Short-range interactions and critical scaling

D.-H. Lee and Z. Wang PRL 76, 4014 (1996)
Z. Wang et al. PRB 61, 8326 (2000)
I. S. Burmistrov et al. Ann. Phys. 326, 1457 (2011)

- Temperature scaling at Anderson transitions cannot be explained within the single-particle picture
- Short-range (irrelevant) interactions nontrivially modify the critical scaling with temperature

$$\rho_{L,H}(\omega, T, B) = f_{L,H}(T\xi^{z_T}, \omega\xi^{z}) \quad \Rightarrow \quad \Delta B \sim \min(\omega^{1/z\nu}, T^{1/z_T\nu})$$

- The new exponents are determined by the scaling dimension $-\alpha$ of the interaction strength:

$$p = 1 + \frac{2\alpha}{z}, \quad z_T = \frac{2}{p} = \frac{2z}{z+2\alpha}$$

- In turn, α is determined by the (multifractal) scaling of a certain correlator of critical wave functions of the disordered non-interacting system

$$M_{jk} = \int d\mathbf{r}_1 d\mathbf{r}_2 K_{jk}(\mathbf{r}_1, \mathbf{r}_2) U(\mathbf{r}_1 - \mathbf{r}_2), \quad K_{jk}(\mathbf{r}_1, \mathbf{r}_2) = |\psi_j(\mathbf{r}_1)\psi_k(\mathbf{r}_2) - \psi_j(\mathbf{r}_2)\psi_k(\mathbf{r}_1)|^2$$

Composite fermions and universality of FQH transitions

J. K. Jain, S. A. Kivelson, and N. Trivedi *"Scaling Theory of the Fractional Quantum Hall Effect"* PRL 64, 1297 (1990)
S. Pu, G. J. Sreejith, and J. K. Jain *"Anderson Localization in the Fractional Quantum Hall Effect"* PRL 128, 116801 (2022)

- Composite fermions (CFs) theory maps FQH transitions of electrons to IQH transitions of CFs
- Attaching $2m$ vortices to electrons at integer filling ν_{CF} gives a variational FQH state at the Jain fractions

$$\nu = \frac{\nu_{\mathrm{CF}}}{2m\nu_{\mathrm{CF}} \pm 1}$$

- Including disorder (for $m = 1$), the IQH transition $1 \to 2$ maps to FQH transition $1/3 \to 2/5$
- More in J. Jain's lectures

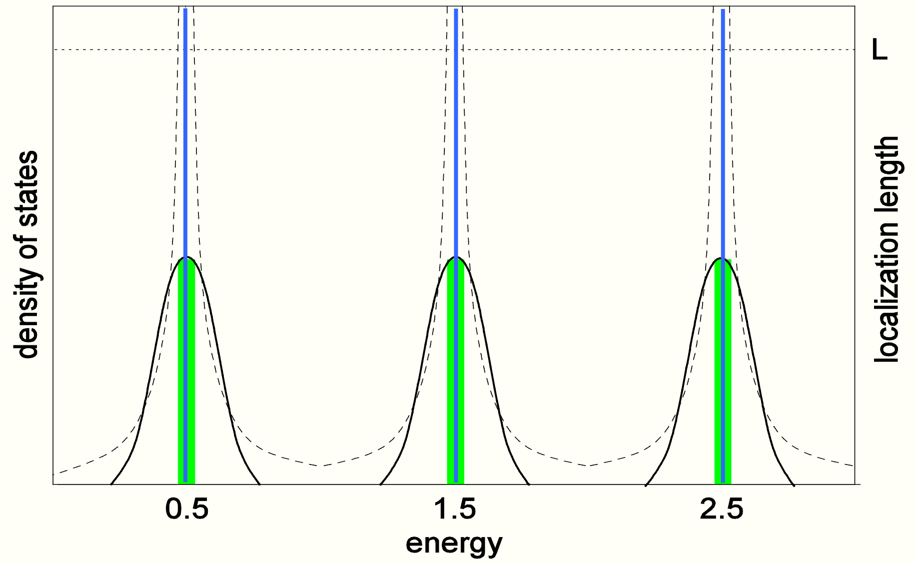
IQH and localization in strong magnetic field

- Single electron in a magnetic field and a random potential

$$H = \frac{1}{2m}\Big(-i\hbar\nabla + \frac{e}{c}\mathbf{A}\Big)^2 + U(\mathbf{r})$$

- Without disorder: Landau levels
- Disorder broadens the levels and localizes most states
- Extended states near E_c (green)
- IQH transition upon varying E_F or B
- Diverging scale is the localization length

$$\xi(E) \propto |E - E_c|^{-\nu}$$



- An Anderson (localization-delocalization) transition: a non-interacting quantum phase transition
- DOS is smooth (non-singular) across the transition, which implies $z = 2$
- All observables are random, a complete theory would describe their distributions

Symmetries and AZ classes

A. Altland, M. Zirnbauer '96

Conventional (Wigner-Dyson) classes

	T	spin rot.	chiral	p-h	symbol
GOE	+	+	−	−	AI
GUE	−	+/−	−	−	A
GSE	+	−	−	−	AII

- Integer quantum Hall effect (→ A)

Chiral classes

	T	spin rot.	chiral	p-h	symbol
ChOE	+	+	+	−	BDI
ChUE	−	+/−	+	−	AIII
ChSE	+	−	+	−	CII

Bogoliubov-de Gennes classes

	T	spin rot.	chiral	p-h	symbol
	+	+	−	+	CI
	−	+	−	+	C
	+	−	−	+	DIII
	−	−	−	+	D

- Spin quantum Hall effect (exact results) (→ C)
- Thermal quantum Hall effect (→ D)

Methods, models, and recent results in theory of Anderson transitions

- No small parameter, no perturbation theory. Expect conformal field theory (CFT) with $c = 0$
- A lot of intuition comes from network models amenable to numerics — J. T. Chalker, P. D. Coddington `88
- Recent advances include
  - High-precision numerics (irrelevant operators)
    K. Slevin, T. Ohtsuki `09
    W. Nuding, A. Klümper, A. Sedrakyan `15
    F. Evers et al., T. Vojta et al., R. Roemer et al.… `18-`25
  - Field theory: non-linear sigma model, symmetry analysis of multifractal (MF) wave functions
    N. Charles, IAG, J. F. Karcher, A. W. W. Ludwig, A. D. Mirlin, M. R. Zirnbauer `11-`24
  - Constraints from conformal symmetry on MF spectra
    R. Bondesan, D. Wieczorek, M. R. Zirnbauer `14-`19
    J. Padayasi, IAG `23
  - Mapping to classical models, statistical mechanics and CFT
    E. Bettelheim, IAG, A. W. W. Ludwig `12
    IAG, J. F. Karcher, A. D. Mirlin `22
  - Random networks and quantum gravity
    H. Topchyan, IAG, W. Nuding, A. Klümper, A. Sedrakyan `17-`25
    A. Mukherjee, IAG, V. Kazakov `25
    E. Bettelheim, IAG, E. F. M. Ramirez `25