Fractional Statistics and Interferometry

Lecture by B. I. Halperin, Harvard University

at the Winter School *on*

“Correlations in flat bands: From the FQHE to Moiré”

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Outline

- Definitions of Fractional Statistics
- Application to Fractional Quantized Hall States
- Measurements using a Fabry-Perot interferometer
- Effects of Coulomb interactions in the compressible regime.
- Interferometry with non-Abelian statistics

Fractional Statistics: Definition from Geometric Phase

- Based on microscopic wavefunction for electronic ground state with several quasiparticles at specified positions.
- Assume energy gap to all excited states.
- Calculate geometric phase accumulated when you interchange two identical quasiparticles adiabatically around a closed path C.
- Subtract phase that would be accumulated by moving a single quasiparticle around the same path. Difference is the statistical phase.

Fractional Statistics

Definition via Effective Wave Functions

- Define an effective wave function and effective Hamiltonian to describe behavior of quasiparticles, after electrons have been eliminated from problem.
- Effective Hamiltonian should gives energies, equations of motion for quasiparticles.
- Description assumes at least some quasiparticles are free to move.

Features of the effective wave function

- There is a gauge freedom for the choice of phase of the effective wave function.
- For QPs with fractional statistics, if you want effective Hamiltonian to contain only short-range interactions, you must use multivalued wave functions.
- Interchanging two identical QPs around a counterclockwise path multiplies the effective wave function by a phase factor $e^{i\theta}$ that is not 1 or -1.
- Moving one particle all the way around another is topologically equivalent to two interchanges. Picks up a phase $e^{2i\theta}$.

Alternate gauge choice.

- For particles with fractional statistics, If you want to use single-valued effective wave functions (bosonic or fermionic) the effective Hamiltonian must include long-distance interactions via a Chern-Simons gauge field.

- Attaches θ / π Chern-Simons flux quanta to each particle.

- If you move one particle adiabatically around another, you pick up a phase factor $e^{2i\theta}$.

Non-Abelian Statistics

- System with N well-separated localized quasiparticles has a ground state “degeneracy” that grows exponentially with N. (For finite separations, degeneracy is split, but splittings fall off exponentially with separation).
- Interchanging two identical quasiparticles produces a unitary transformation in the Hilbert space. For multiple interchanges, result depends on the order of interchanges. Determined by the topology of the braiding of world lines. (Representation of the braid group.)

Theorem

- Quasiparticles with fractional charge must exhibit fractional (or non-Abelian) statistics.
- Follows from gauge invariance. (Cf. argument in review by D.E. Feldman and B.I. Halperin, Rep. Prog. Phys. **84**, 076501 (2021))
- Example: At $\nu = 1/3$, quasiparticles with charge $\pm e/3$ will have statistical angle $\theta_{stat} = \pi/3$
- For Jain states $\nu = \frac{p}{2p+1}$, $\theta_{stat} = \frac{2p-1}{2p+1}\pi$
- How can you see fractional statistics directly in an experiment ?

Interference experiments and fractional statistics using edge states of FQHE

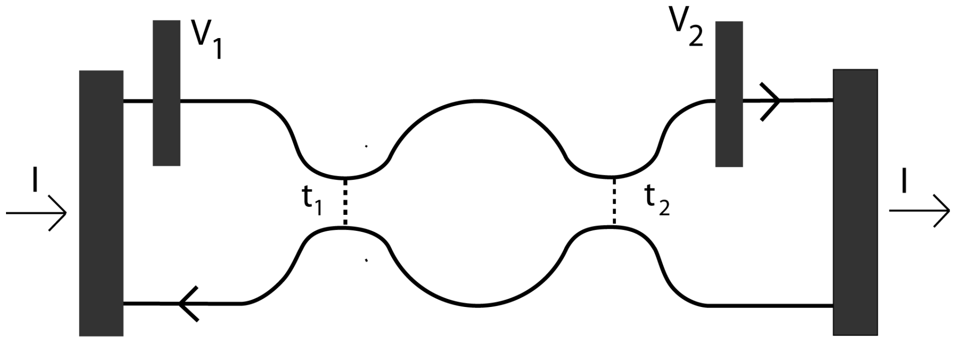
Effects of fractional statistics may be manifest, in an interference experiment.

Basic idea dates back to Kivelson (1990).

"Hall bar" geometry with two narrow constrictions. Current is carried by chiral edge states. Particles can tunnel from one edge to the other at the constrictions, leading to backscattering.

Backscattered particles increase the measured resistance. Interference between particles scattered at different constrictions can give oscillations in measured resistance as a function of parameters.

ν = 1/3: Weak backscattering at constrictions



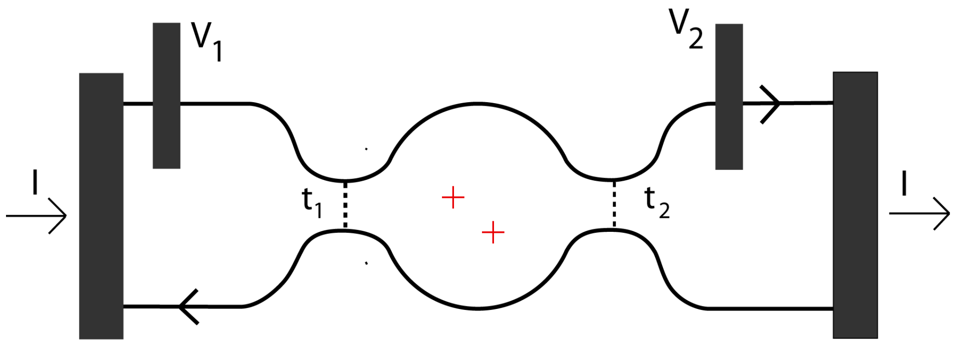
Basic interference process: $R \propto |t_1 + t_2 e^{i\varphi}|^2$.

For particle of charge e/3, expect $\varphi = 2\pi\, AB/(3\Phi_0)$.

For small changes in A or B, φ will change by 2π when $\delta A\, B = 3\,\Phi_0$, or $\delta B\, A = 3\,\Phi_0$

Area period δA = area that encloses one electron.

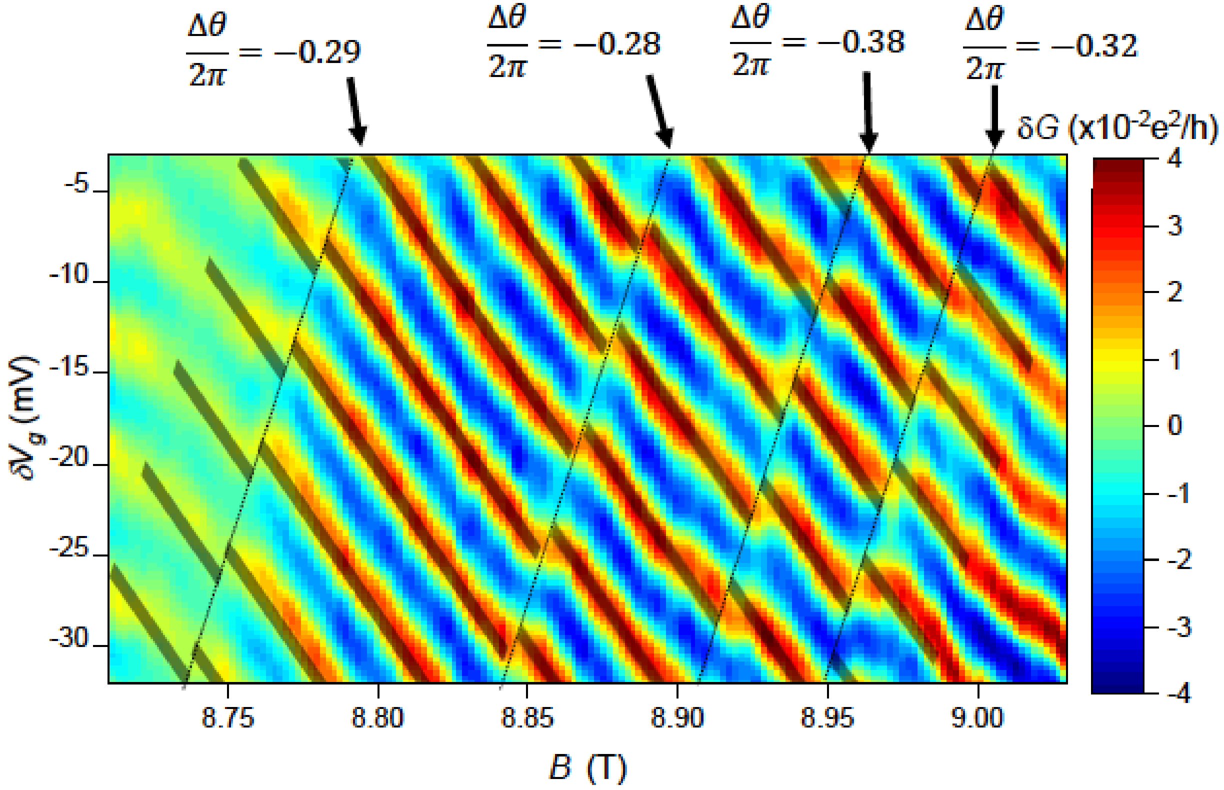
ν = 1/3: Effects of quasiparticles



What happens to interference pattern if you add quasiparticles or quasiholes to the interior of sample??

Addition of a quasiparticle or quasihole should shift the interference pattern by phase $\pm 2\theta_{stat} = \pm 2\pi/3$, i.e. by 1/3 of a period, assuming that area of interference loop does not change.

Results: Nakamura , Liang, Gardner, Manfra, (2020)



Phase jumps occur when an e/3 quasiparticle enters or leaves the interferometer area.

Why did it take 30 years?

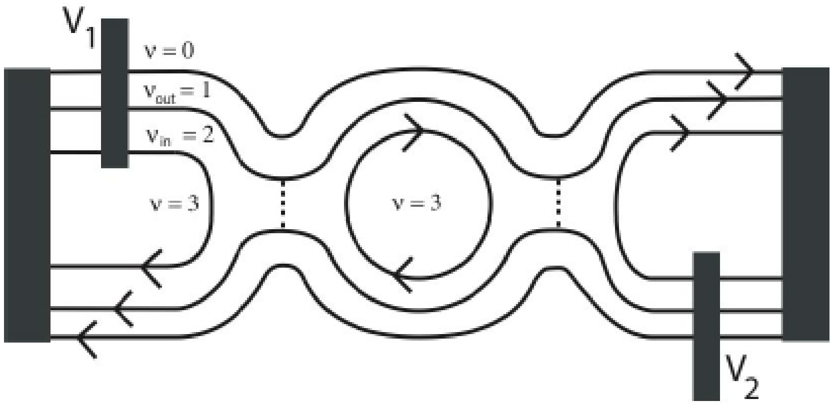
- Generally difficult to control the number of QPs inside the interferometer area.
- Most experiments have been done with interferometer in a compressible regime, E_F in a region of localized states, not a true energy gap. N_e fluctuates thermally.
- Also, localized QPs can affect edge states through Coulomb interactions, as well as statistical phase; can complicate interpretation of interferometer phase.
- Nakamura et al constructed a sample where Coulomb interaction is very well screened and other properties well controlled.
- Operate with E_F in a true energy gap, small density of impurity states.

Fabry–Perot Interferometers in the Compressible Regime

- Coulomb interactions can be very important, even in the integer case, and can lead to a qualitative change in the interference pattern.
- Generalize discussion to consider cases with multiple co-propagating edge modes.

Multiple edge channels

Integer case $\nu = 3$ is illustrated



We shall consider only integer states or fractional states with co-propagating edge modes.

Fractional states can have counter-propagating edge modes. Scattering between edge modes can obliterate interference effects.

Aharonov-Bohm behavior. Predicted for integer case without Coulomb interactions

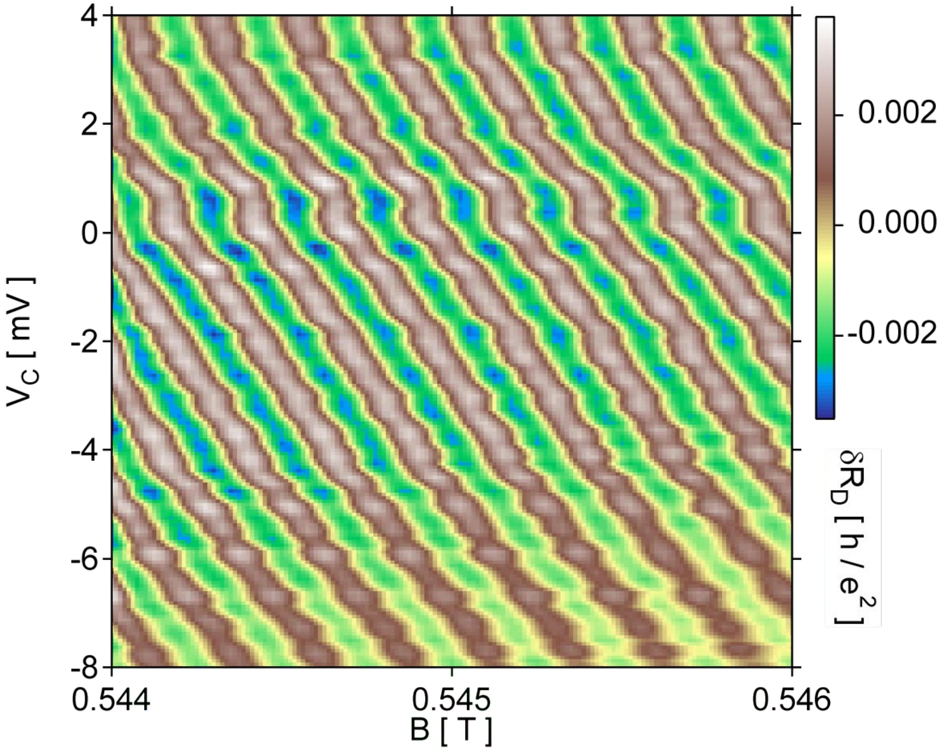
Lines of constant phase have negative slope

Interference periods

$\delta B A = \Phi_0$

$\delta A B = \Phi_0$

Seen Experimentally in Some Samples

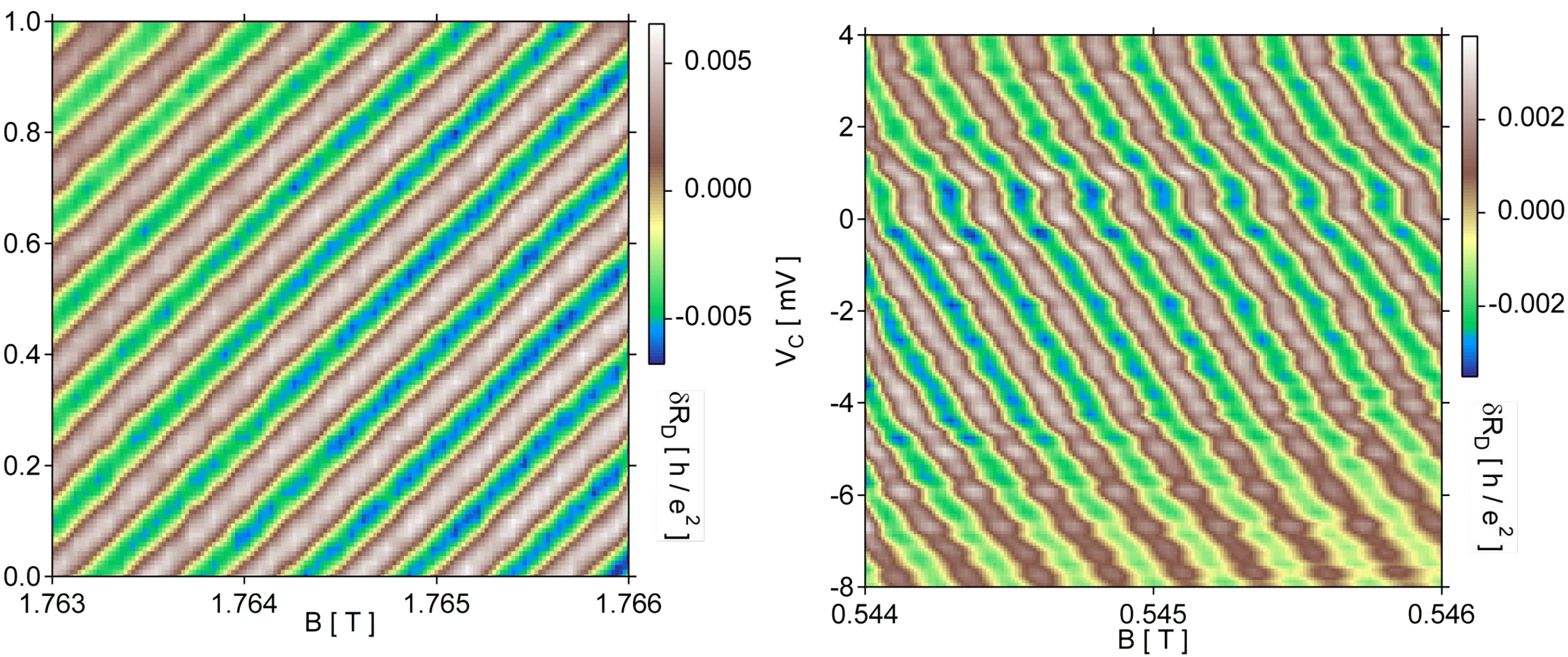


Yiming Zhang et al., 2009: Experimental results

Two types of behavior

Measure in a 2D plane of *B* and V_G

"Coulomb Dominated" "Aharonov-Bohm"



CD stripes have positive slope, and field period may be different than AB

Explanation

Because of Coulomb interactions, actual area enclosed by orbits are not precisely determined by geometry of the defining electrodes.

When the number of localized charges changes. Coulomb interactions cause the edge state area to respond in a direction that tends to keep fixed the total number of electrons in the interferometer

As one varies B or V, the requirement that localized electrons inside the interferometer region be an integer leads to oscillations oscillations in the resistance with a different period than AB period.

Calculation

IQHE or Jain State, weak backscattering, innermost edge mode; tunneling qp is elementary charge: Interferometer phase has form

$$\theta = 2\,\theta_{\rm stat}\; N_L + \frac{Bq}{\hbar} A\,, \quad N_L \text{ is an integer}$$

Want: $<e^{i\theta}>_T \sim \iint dA\, dN_L\; e^{i\theta}\; e^{-E/T} \sum_n \delta(N_L - n)$

$$E \sim K_A\, \delta A^2 + K_L\, \delta N_L^2 + K_{AL}\, \delta A\, \delta N_L$$

δA, δN_L are deviations of A, N_L from values A^0, N_L^0 that would minimize energy if we ignore the integer constraint.

Results

Want: $\langle e^{i\theta} \rangle_T \sim \iint dA\, dN_L\, e^{i\theta}\, e^{-E/T} \sum_n \delta(N_L - n)$

$E \sim K_A\, \delta A^2 + K_L\, \delta N_L^2 + K_{AL}\, \delta A\, \delta N_L$

If we ignore integer constraint, we find
$\langle e^{i\theta} \rangle_T \sim \exp[i\, \theta(A^0, N_L^0)]\, \exp[-T/E_0]$.

Phase $\theta(A^0, N_L^0)$ will vary linearly with B, V_G, if $\Delta B, \Delta V_G$ are small

Integer constraint induces additional modulations with a different period, different combination of B, VG

Consequences

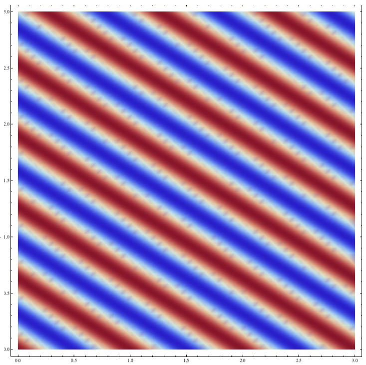
We find that AB and CD stripes should both exist in general. (Also higher harmonics of both).

Depending on parameters, one will be generally. dominant over the other at realistic temperatures.

In some cases, AB and CD will be visible simultaneously, giving rise to a “checkerboard” pattern.

Intermediate Coupling, Lower T

V_G ↑



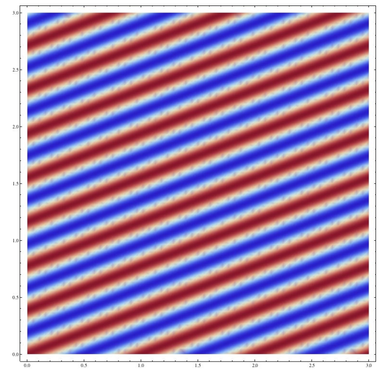
B →

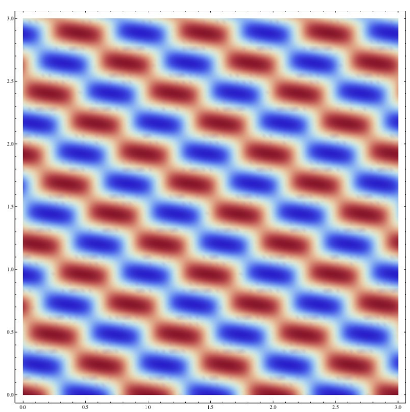
AB

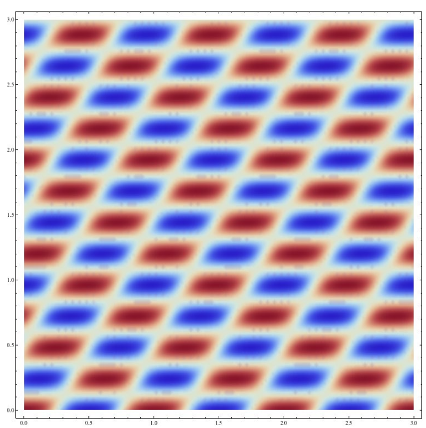
Re $\langle e^{i\phi} \rangle$

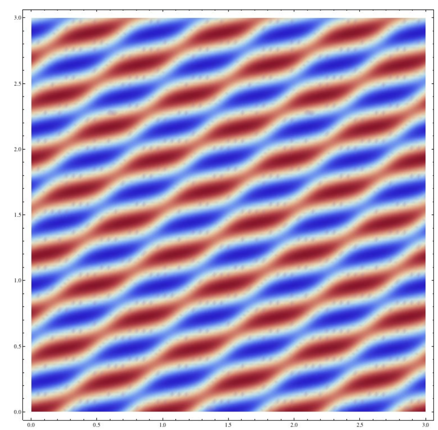
$T \neq 0$

CD







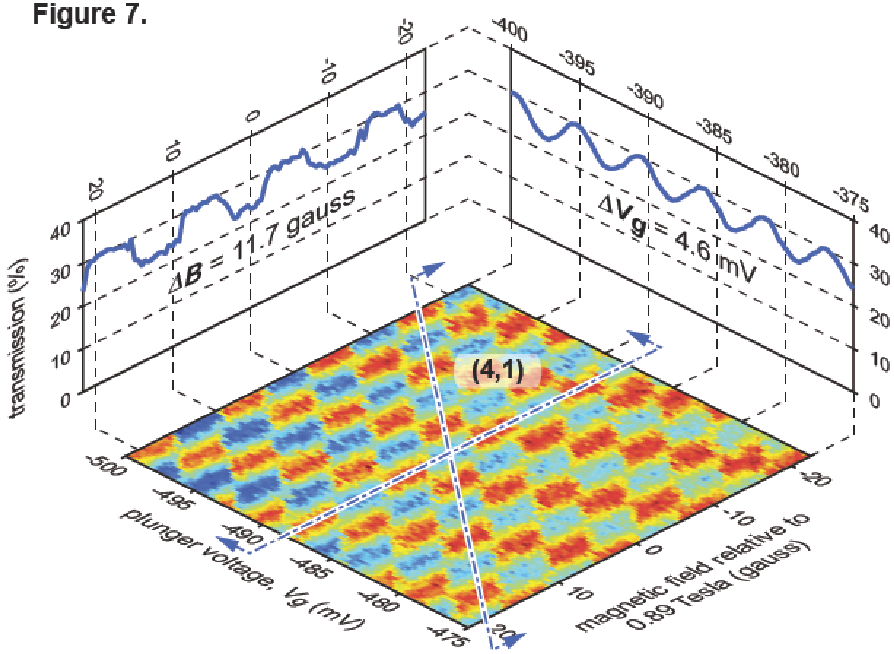


($\nu_c = 2 - \varepsilon$; $f_T = 1$) Calculations by B. Rosenow

Checker-board pattern -- Experiments

From Ofek et al., PNAS 2011

Figure 7.



Flux Period: CD Regime

For integer QHE: In CD regime (strong Coulomb interactions) flux period is given by

$$\delta B A = \Phi_0 / (\nu - 1)$$

For $\nu = 1$: CD stripes are horizontal (interferometer phase is independent of B).

Also true for $\nu = 1/3$, even for weak Coulomb interactions.

Fabry-Perto Interferometer with <u>non-Abelian</u> statistics

Ising Anyons

- In the simplest type of non-Abelian statistics, the ground state of N quasiparticles has dimension $2^{N/2}$.
- Such QPs are called "Ising Anyons".
- It has been proposed that the FQH state at $\nu = 5/2$ has QPs with charge e/4 that are Ising anyons.

Interferometry with Ising Anyons

- Interference period, as well as the phase, depends on number of anyons inside the interferometer.
- If there are an odd number of e/4 QPs in the interior, bringing another e/4 QP around the interferometer will change the quantum state, not just multiply by a phase factor.
- Therefore interference signal associated with e/4 particles will be absent.
- e/4 interference can occur if there is an even number of QPs in the interior.

Experimental results at $\nu = 5/2$

- Experiments by Willett and collaborators show evidence for even-odd alternation in interference period depending on number of e/4 QPs inside interferometer.
- Also at $\nu = 7/2$.
- But there are questions about microscopic details.
- Evidence is statistical, and there could be other explanations for observations.

The precise nature of the FQH State at ν = 5/2 is an outstanding puzzle.

- Numerical results on finite systems support either the Pfaffian state (Moore & Read, 1991) or its particle-hole conjugate (anti-Pfaffian.
- Experiments (Bannerjee, et al, 2018) point to a totally different state (PH-Pfaffian).
- Experiments measure thermal conductance, carried by edge states, quantized and topologically protected.

In any case, the precise nature of the FQH State at { = 5/2 remains an outstanding puzzle.

Credits: Effects of Coulomb Interactions

Reference: B. I. Halperin, Ady Stern, Izhar Neder, and Bernd Rosenow, *Theory of the Fabry-Pérot Quantum Hall Interferometer,* Phys. Rev. B 83, 155440 (2011)

Motivated by experiments at Harvard by Yiming Zhang, Doug McClure, Angela Kou, and C. M. Marcus, who also contributed to the theoretical picture. Also, experiments at Weizmann, discussed by N. Ofek, A. Bid, M. Heiblum, A. Stern, V. Umansky, and D. Mahalu (PNAS, 2010)

Related earlier work: B. Rosenow and B. I. Halperin, (PRL 2007).

Applications to ν=1/3 experiments of Nakimura *et al*. discussed by Rosenow and Stern (PRL 2020), Feldman and Halperin (Re. Prog. Phys. 2021)

Historical remarks

Fabry-Perot interference experiments in the integer regime date back to late 1980s: van Wees, et al; Simmons et al.; etc.

Many technical improvements since then. Include use of back gates to vary density by Goldman group. Introduction of 2D color maps of resistance in B – V_G plane, by Marcus and Heiblum groups.

Theoretical work, recognizing the importance of Coulomb interactions in interference experiments , and of fractional statistics for FQHE, also date back to late 1980's and early 1990's: Jain, Kivelson, Patrick Lee, Goldman and Su. Important paper by Chamon, Freed, Kivelson, Su, and Wen (1997).

Theory described above is a refinement of these ideas.

Fractional Statistics: Theory History

- Leinaas and J. Myrheim, (1977); Wilczek, (1982).

Showed fractional statistics to be possible in principle in 2D, and described their general properties.

- Halperin, (1984); Arovas, Schreiffer, Wilczek, (1984)

Argued that quasiparticles in Fractional Quantized Hall states should show fractional statistics. => Existence in the real world.

Halperin argument was based on effective wave functions.

Arovas et al. calculated geometric phase for Laughlin holes.