MagLab lectures

Jan '18

Outline

0. Topics + scope of lectures

I. Entanglement entropies in field theories

   A. Causal domain

   B. Rindler space in vacuum

   C. 1 interval in CFT

   D. 1 interval in general thy

   E. 2 intervals in CFT

   F. Other topics

II. EEs in holographic theories

   A. Holography

   B. Ryu-Takayanagi formula

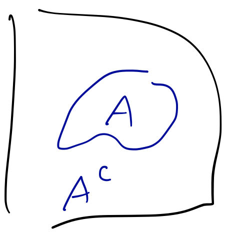
   C. Examples

   D. Properties

   E. Other topics

0. The topic of these lectures is entanglement entropies in quantum field theories. It has been understood gradually over the last 25 years that these quantities contain a wealth of physical info.

More specifically, given a QFT on a space, if we divide the space into a region A and its complement A^c



then (modulo certain subtleties that we'll ignore) by locality (commutation of separated observables) the Hilbert space factorizes

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{A^c}$$

Given a state ρ, we can then define

$$\rho_A := \mathrm{Tr}_{\mathcal{H}_{A^c}} \rho$$

$$S(A) := -\rho_A \,\mathrm{Tr} \ln \rho_A$$

(+ other quantities such as Rényi entropies + relative entropies)

Our basic aim will be to show how this quantity correlates w/

important aspects of the physics, such as criticality, correlation lengths, RG flows, finite temp, etc. I will then focus on theories w/ holographic duals, explaining the basics of holography and showing how entanglement is represented geometrically in these theories.

Throughout these lectures, I will limit myself to relativistic QFTs in 1+1 dimensions. This is because

- it focuses the discussion + allows me to be more concrete

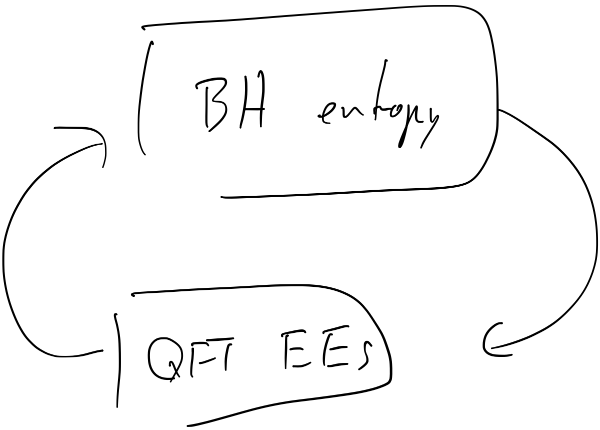
- most (but not all) of the important physics is already captured there
- my drawing skills are limited to 2d

Why is a high-energy theorist giving lectures at a CMT school?

This is an area of very fruitful interactions between the two fields.

In fact, one of the ways that entanglement entropies entered into CMT is from HET.

Originally this came from trying to understand black holes



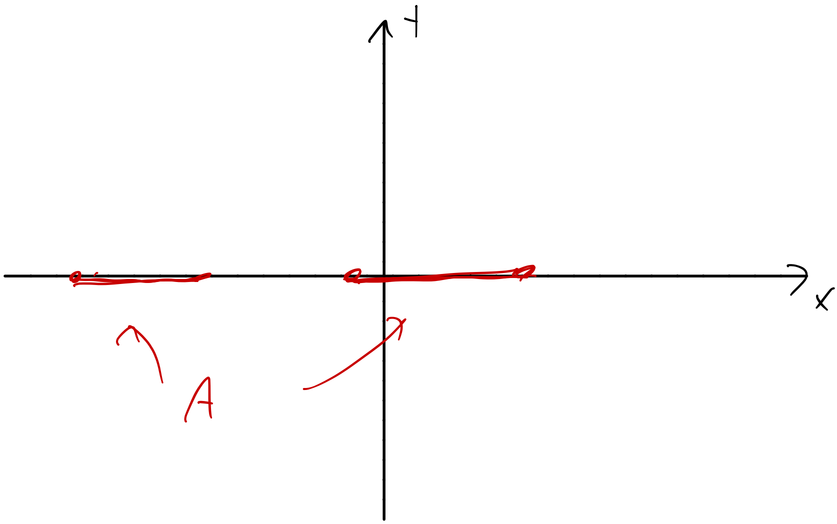
Key developments

'72	Bekenstein	BH entropy
'74	Hawking	
'75	Unruh	Rindler
	Bisognano-Wichmann	
'86	Sorkin	BH entropy $\stackrel{?}{=}$ EE
'93	Srednicki	area law
'94	Holzhey-Larsen-Wilczek	EE in 2d CFT
'03	Calabrese-Cardy	
'06	Ryu-Takayanagi	Holographic EE

I. Entanglement entropies in field theories

A. Causal domain

We consider a relativistic QFT in 1+1 d Minkowski space.

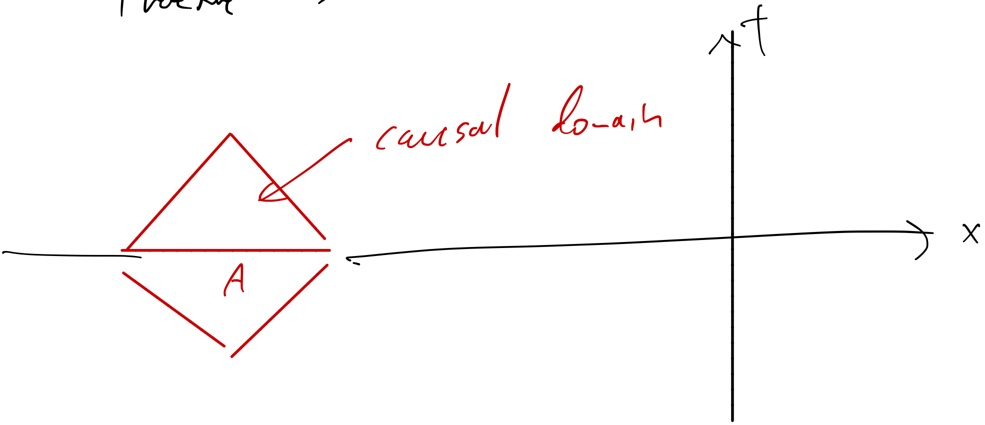


A spatial region A, say at $t=0$, is a set of intervals.

Consider a single interval. The exp. val. of op O_A in A is determined by ρ_A:

$$\langle O_A \rangle = \mathrm{tr}_{\mathcal{H}_A} (O_A \rho_A)$$

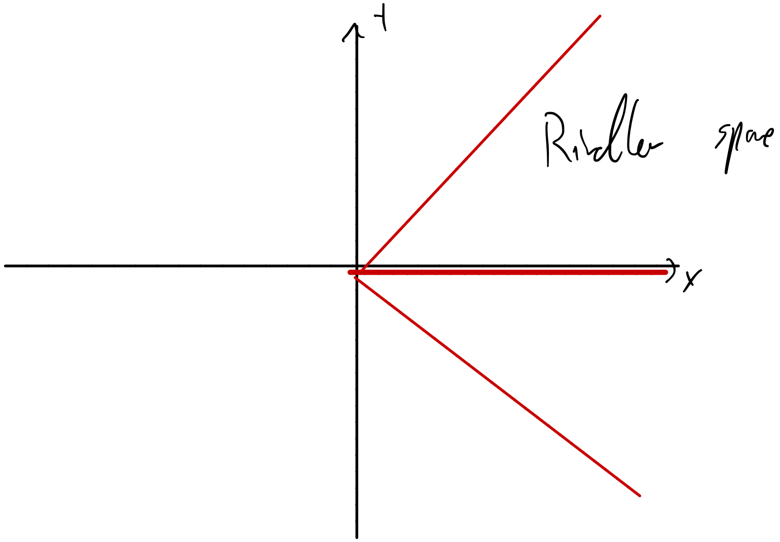
In fact, by Heisenberg EOM, an op in causal domain of A can be expressed in terms of ops in A, therefore is also determined by ρ_A.



Therefore, ρ_A and $S(A)$ are really associated not with A but w/ its causal domain.

B. Rindler space in vac.

Ex: $A = \{ t=0, \ x \geq 0 \}$



$\text{Domain} = \{ x \geq |t| \}$

Suppose $\rho = |0\rangle\langle 0|$

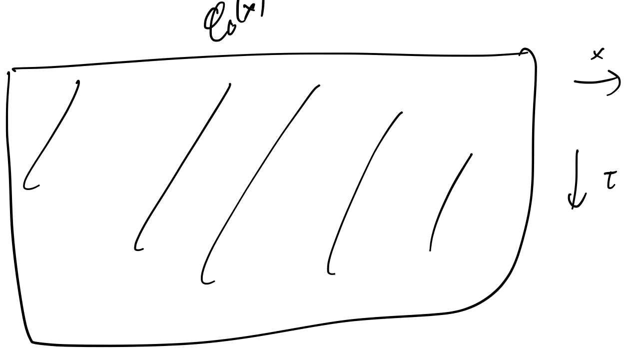
Pure but, because of entanglement, ρ_A is mixed

For an observer who stays within Rindler space, world appears mixed.

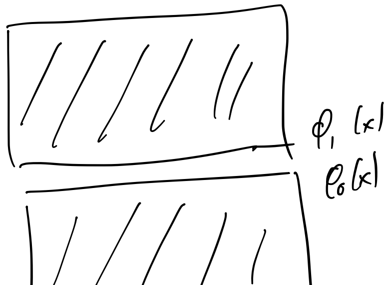
This case is simple enough that we can write down a closed-form expression for ρ_A

Also, the derivation illustrates a very useful technique using Euclidean path integrals.

Work in "position basis" for field, and let $\varphi_0(x)$ be a field config. Then

$\langle \varphi_0(x) | 0 \rangle \propto$ 

= Euclidean path int. on half-plane $\tau \leq 0$ w/ b.c. $\varphi(x, t=0) = \varphi_0(x)$

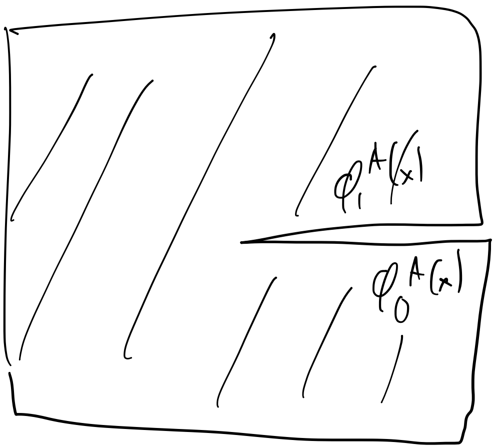
Then $\langle \varphi_0(x) | 0 \rangle \langle 0 | \varphi_1(x) \rangle \propto$ 

For configs $\varphi_0^A(x)$, $\varphi_1^A(x)$ on $x \geq 0$,

$$\langle \varphi_0^A(x) | \rho_A | \varphi_1^A(x) \rangle$$

$$= \int \mathcal{D}\varphi^{A^c}(x) \, \langle \varphi_0^A(x), \varphi^{A^c}(x) | \rho | \varphi_1^A(x) \varphi^{A^c}(x) \rangle$$

$\propto$

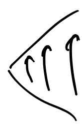


$$\Rightarrow \rho_A \propto e^{-2\pi K}$$

where K = gen. of Euclid. rot. about $x = \tau = 0$

= boost gen.

$= \int dx \, x T_{tt}(t=0, x)$



$\Rightarrow \rho_A$ is thermal w.r.t. boost gen!

Observer staying at proper distance $\ell = \sqrt{(x^1)^2 - (x^0)^2}$ from entangling surface sees temperature $\frac{1}{2\pi \ell}$

$\rightarrow$ Unruh effect

Close to entangling surface, fields are very hot — UV modes are decohered

Can use this to estimate $S(A)$ by adding up local entropies

Estimate of $S(A)$ for field of mass m

ent. density $s(T) \sim \begin{cases} T, & T > m \\ 0, & T < m \end{cases}$

$$\Rightarrow S(A) \sim \int_{\varepsilon}^{\infty} dx \; s(T)$$

UV cutoff $\longrightarrow \varepsilon$

$$\sim \int_{\varepsilon}^{1/m} dx \, \frac{1}{x}$$

$$\sim \ln \frac{1}{m\varepsilon}$$ UV divergent

Only fields out to distance

$$\xi = \frac{1}{m}$$

are entangled.

[Exer: Generalize this calculation to higher dim's.]

C. 1 interval in CFT

If we take limit $m \to 0$ in previous result we get IR divergence in addition to UV divergence. To cut it off consider finite interval of length L:

Ent. density of CFT is $s(T) = \frac{2\pi c}{6} T$

central charge

$\Rightarrow$ UV divergent part of EE is $\int_{\varepsilon} dx' \, s\left(\frac{1}{2\pi x'}\right) = -\frac{c}{6} \ln \varepsilon$

2 endpoints $\rightarrow -\frac{c}{3} \ln \varepsilon$

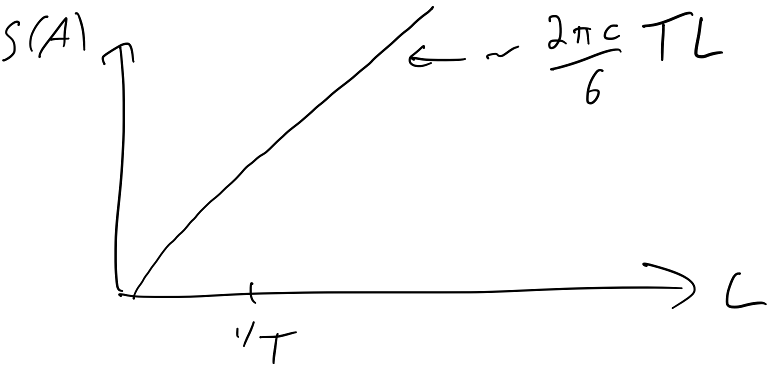
Entropy dimensionless $\Rightarrow S(A) = \frac{c}{3} \ln \frac{L}{\varepsilon}$

This result can be checked by an honest calculation using Euclidean path integral + CFT techniques.

(Holzhey - Larsen - Wilczek '94, Calabrese - Cardy '03)

If full system is at temp T, then we have instead

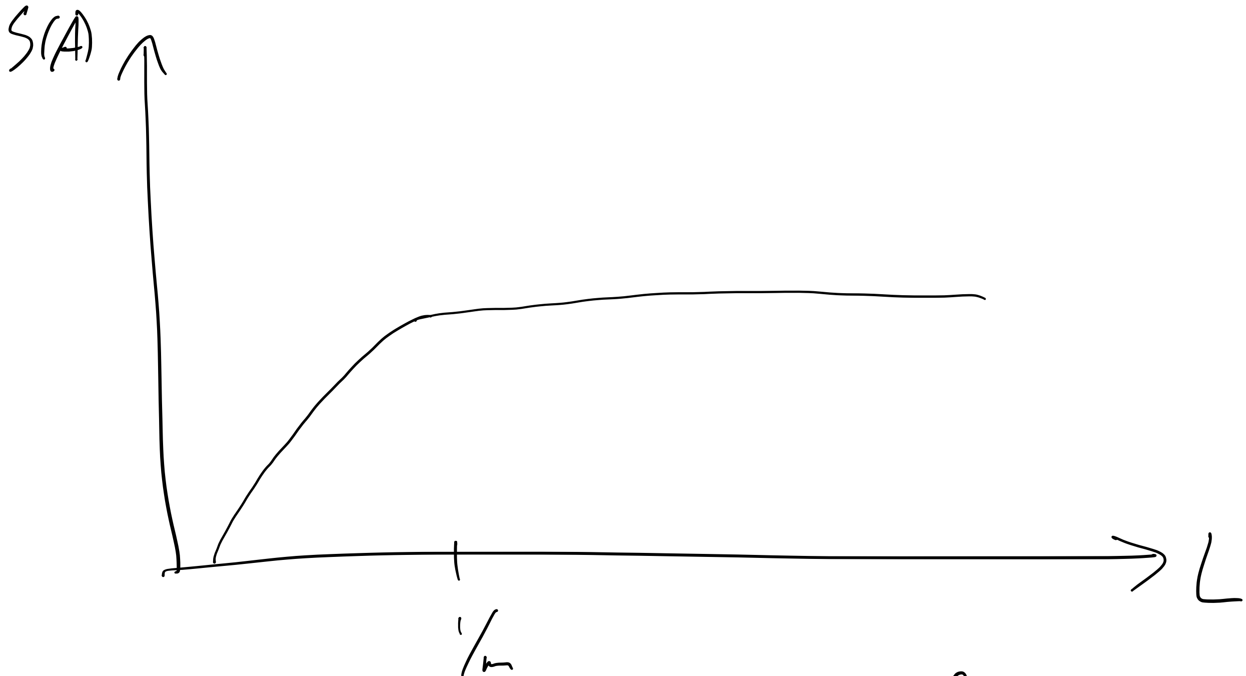
$$S(A) = \frac{c}{3} \ln \frac{\sinh(\pi T L)}{\pi T \varepsilon}$$



$\leftarrow \sim \frac{2\pi c}{6} T L$ (extensive thermal ent.)

D. 1 interval in general thy

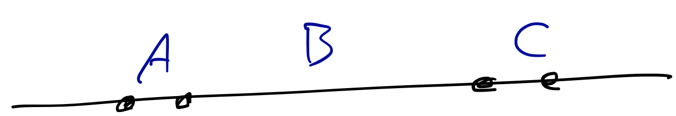
Based on Rindler discussion, expect entropy to saturate



(computed for free bosons + fermions semi-analytically by Casini-Huerta '07)

In all examples so far, $S(A)$ is concave func. of L

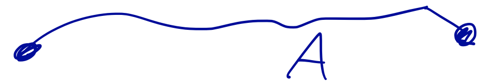
Reason is SSA:



$$0 \geq \left(S(ABC) - S(BC)\right) - \left(S(AB) - S(B)\right)$$

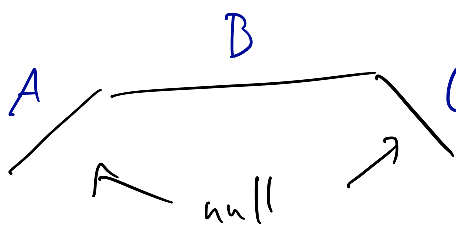
$$\xrightarrow[A,\,C\to 0]{} \frac{d^2 S(B)}{dL^2}$$

Using relativity, we can make a stronger statement



For any single interval, in vacuum, $S(A)$ is func. of proper distance between endpoints

Consider this config:



$$L_{AB} L_{BC} = L_B L_{ABC}$$

SSA $\Rightarrow$ S is concave func. of $\ln L$

$$\Rightarrow \frac{d}{dL} c(L) \leq 0 \text{ where } c(L) := 3L \frac{dS}{dL}$$

If theory has UV, IR fixed pts, then

$$c(L \to 0) = c_{uv}, \quad c(L \to \infty) = c_{IR} \Rightarrow c_{uv} \geq c_{IR}$$

Proof of Zamolodchikov C-theorem (Casini-Huerta '04)

Ingredients are same as Zamolodchikov's proof:

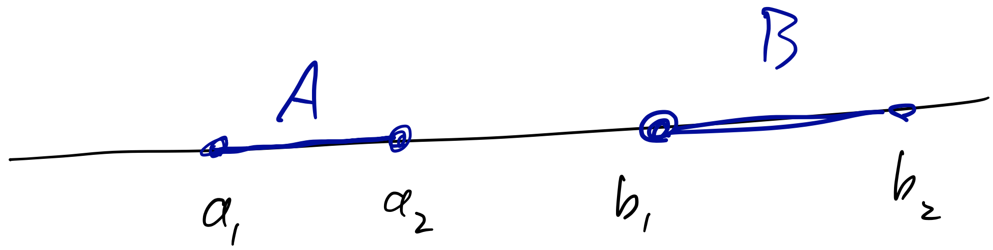
unitarity, locality, relativity

But proof seems very different

C-functions are different

E. 2 intervals in CFT

Now consider 2 separated intervals



Can consider mutual info:

$$I(A:B) := S(A) + S(B) - S(AB)$$

Quantifies correlations between A, B. However, very hard to compute $S(AB)$

Free massless Dirac fermion:

$$I(A:B) := S(A) + S(B) - S(AB)$$
$$= \frac{1}{2} \ln \frac{(b_2 - a_2)(b_1 - a_1)}{(b_2 - a_1)(b_1 - a_2)}$$

(Casini, Fosco, Huerta '05)

<u>Only</u> (non-topological) theory for which M.I. has been computed exactly (including free scalar!)

However, its qualitative features hold for any 2d CFT:

- finite ($\Leftarrow$ divergences are local on entangling surface)
- non-zero (else correlators would vanish)
- conformally invariant (because indep. of ε)

  $\Rightarrow$ func. of cross-ratio
- increases as func. of sizes of A, B (by SSA)

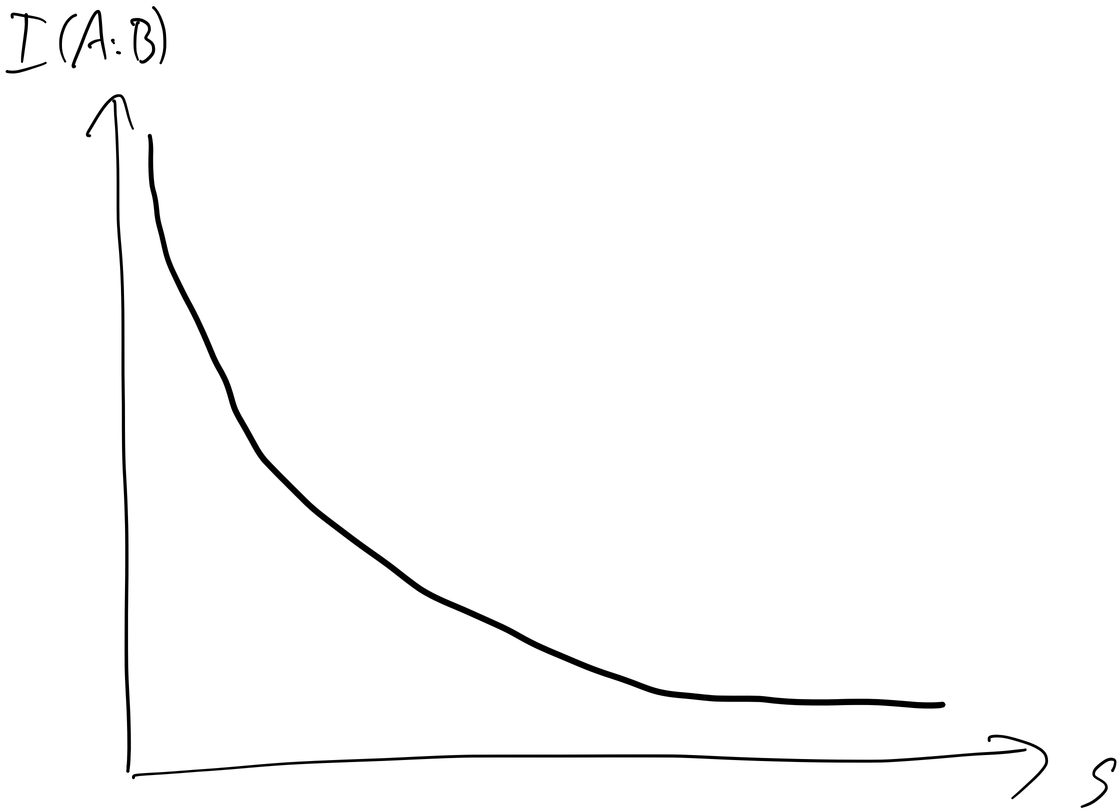
  $\Rightarrow$ decreases as func. of separation

  $$s := b_1 - a_2$$

  for fixed sizes $b_2 - b_1$, $a_2 - a_1$
- $\to \infty$ as $s \to 0$
- $\to 0$ as $s \to \infty$ like $s^{-4\Delta}$

  where Δ = dim of lightest non-trivial op

  (here $\Delta = \frac{1}{2}$)


I(A:B)
s

F. Other topics

2+1 d: topological EE, F for CFTs, F-theorem,

3+1 + higher
corners

time dependence: quenches etc.

Many other topics

II. EEs in holographic QFTs

Usually it's very difficult to compute EEs, even in free QFTs.

However, there is a class of theories where, in a certain limit it becomes easy because it becomes a classical geometry problem. These are the holographic theories.

A. Holography

Switch gears: Consider GR in 2+1 d (possibly w/ matter fields) with c.c. $\Lambda < 0$.

Write

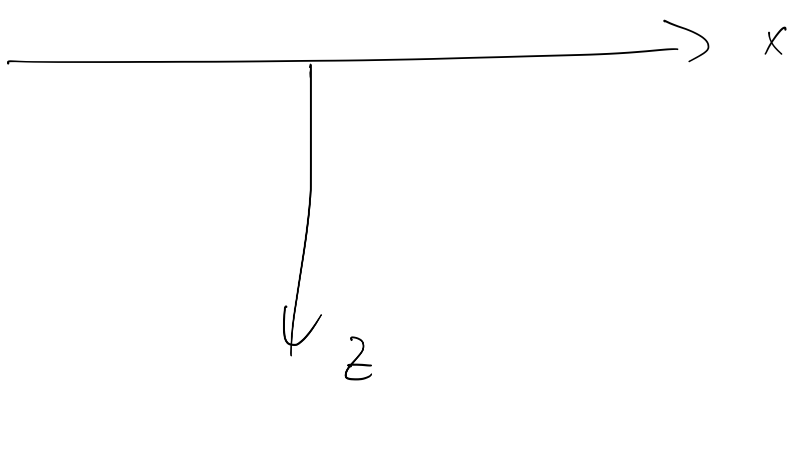
$$\Lambda = -\frac{1}{R^2}$$

R ← curvature length

Simplest vacuum sol'n is AdS_3:

$$ds^2 = \frac{R^2}{z^2}\left(-dt^2 + dx^2 + dz^2\right) \qquad (z > 0)$$

spatial section is hyperbolic plane



AdS_3 has an asymptotic bdy at $z=0$.

Grav. pot. $\to \infty$ there, so massive particles cannot reach it. Massless particles reach it in finite time; can impose b.c. so they reflect.

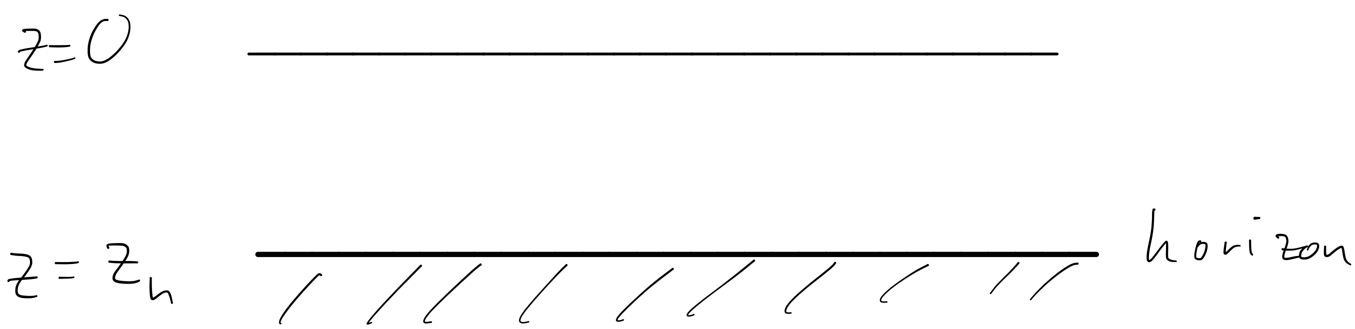
In GR, metric is dynamical, but we can impose a b.c. that it approach AdS_3 near $z=0$. Well-defined closed classical system.

For simple b.c. on matter, ground state is AdS_3.

An excited state is BTZ black hole:

$$ds^2 = \frac{R^2}{z^2}\left(-f(z)\,dt^2 + \frac{dz^2}{f(z)} + dx^2\right)$$

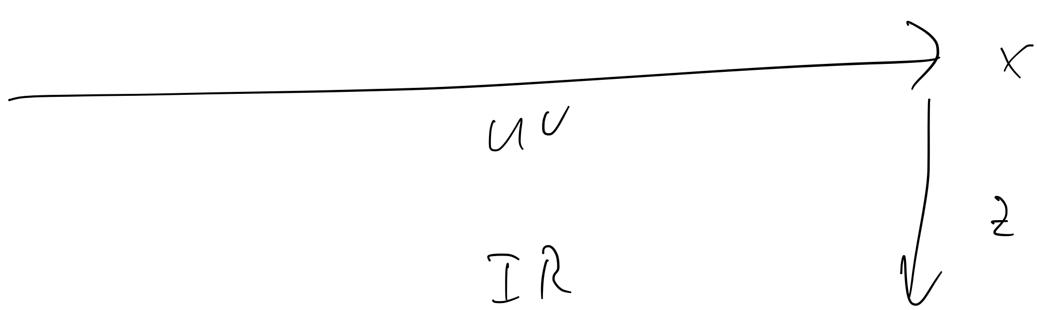
$$f(z) = 1 - \frac{z^2}{z_h^2}$$



Suppose our GR+matter system is the classical approx. to a quantum gravity theory, with $\ell_{Pl} = G_N \hbar \ll R$. Then, with AdS_3 b.c., we have a closed quantum system. It can be shown that this system is a 2d CFT

w/ $c = \dfrac{3R}{2\ell_{Pl}} \gg 1$

It is easy to see that it is strongly coupled. The classical GR + matter is a <u>collective description</u> of the large # of strongly-interacting fields. The map between the CFT + GR is non-local, but it is best to identify the spacetime where the CFT lives as the conformal bdy $z=0$ of the asympt. AdS_3 spacetime. Roughly speaking, the region near the bdy represents the UV of the CFT, regions far represent the IR



To impose a UV cutoff ε on the CFT, we would cut off the AdS_3 at $z=\varepsilon$.

$$S = \frac{1}{4 l_{pl}} \frac{R}{z_h} \stackrel{?}{=} \frac{2\pi c}{6} T = \frac{\cancel{2\pi}}{\cancel{6}} \frac{3R}{2 l_{pl}} \frac{1}{\cancel{2\pi} z_h} = \frac{R}{4 l_{pl} z_h}$$

$$ds^2 = \frac{R^2}{z^2}\left(-f(z)\, dt^2 + \frac{dz^2}{f(z)} + dx^2\right)$$

$$f(z) = 1 - \frac{z^2}{z_h^2} = \left(1-\frac{z}{z_h}\right)\left(1+\frac{z}{z_h}\right) \approx 2\underbrace{\left(1-\frac{z}{z_h}\right)}_{u}$$

$$du = -\frac{dz}{z_h} \qquad \frac{dz^2}{f(z)} = \frac{z_h^2}{f(z)} du^2 = z_h^2 \frac{du^2}{2u}$$

$$\frac{1}{2} \frac{du}{u^{1/2}} = d\left(u^{1/2}\right) \qquad = z_h^2\, 2\, d\rho^2$$

$$= d\rho \qquad 2\rho^2 d\tau^2 + 2 z_h^2 d\rho^2 = 2\left((\rho z_h)^2 \left(d\frac{\tau}{z_h}\right)^2 + d(z_h \rho)^2 \right)$$

$$\frac{\tau}{z_h} \sim \frac{\tau}{z_h} + 2\pi \qquad t \sim \tau + 2\pi z_h \qquad T = \frac{1}{2\pi z_h}$$

The BTZ black brane represents the thermal state of the CFT w/

$$T = \frac{1}{2\pi z_h}$$

The CFT may have a relevant op leading to an RG flow either to another CFT or a gapped thy. This is represented by changing the b.c. for a scalar s.t. the ground state is no longer AdS_3 but, at some z, either there is a domain wall to and AdS_3 w/ a different c.c. or space caps off:

RG flow w/ non-trivial IR fixed pt

$z = 0$

R_{UV} $\qquad$ c_{UV}

domain wall

$R_{IR} < R_{UV}$ $\qquad$ c_{IR}

RG flow to trivial thy:

end of space

at $z \sim \xi$

= corr. length

B. Ryu-Takayanagi formula

The Bekenstein-Hawking formula gives the entropy of a black hole in terms of the area of its event horizon:

$$S_{BH} = \frac{1}{4G_N} \text{area(horizon)}$$

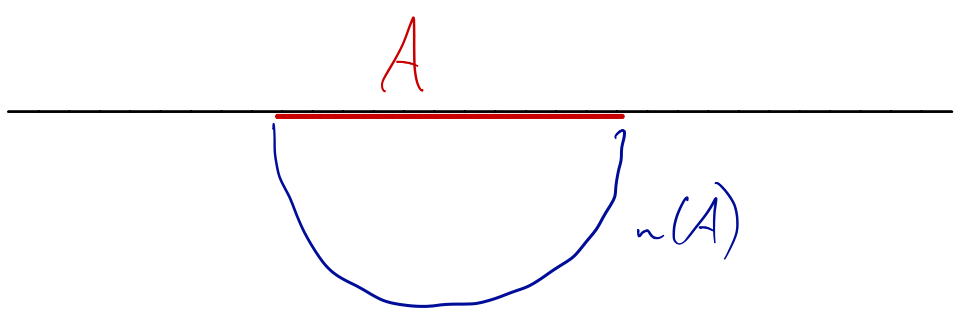
Inspired by this, RT '06 conjectured that the holographic EE of a region A is a

they is given by the area of the minimal surface $m(A)$ in bulk homologous to A:

$$S(A) = \frac{1}{4G_N} \text{area}(m(A))$$

In our case, $G_N = l_{Pl}$, the critical "surface" $m(A)$ is a geodesic connecting the endpoints of A, and its "area" is its length:

$$S(A) = \frac{1}{4 l_{Pl}} \text{length}(m(A))$$



As we'll see, this formula geometrizes all of the general features of EEs that we've discussed, and implies a few new ones that are special to holography

Since the characteristic scale of the geometry is R, the RT entropy is of order

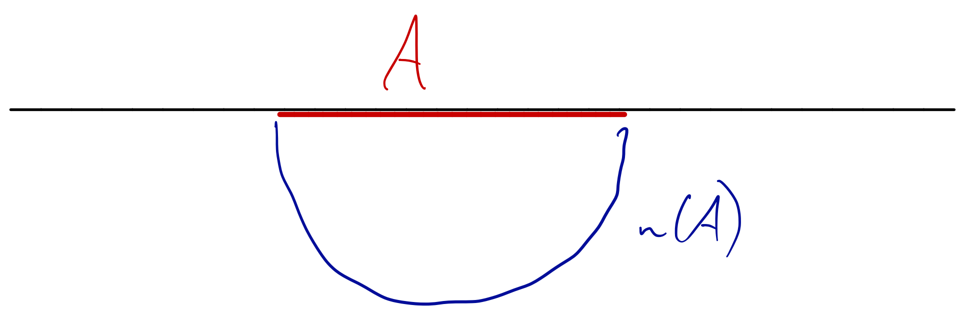
$$\frac{R}{l_{Pl}} \sim c$$

In addition there are subleading terms in $\frac{1}{c}$ that are not strictly geometrical

C. Examples

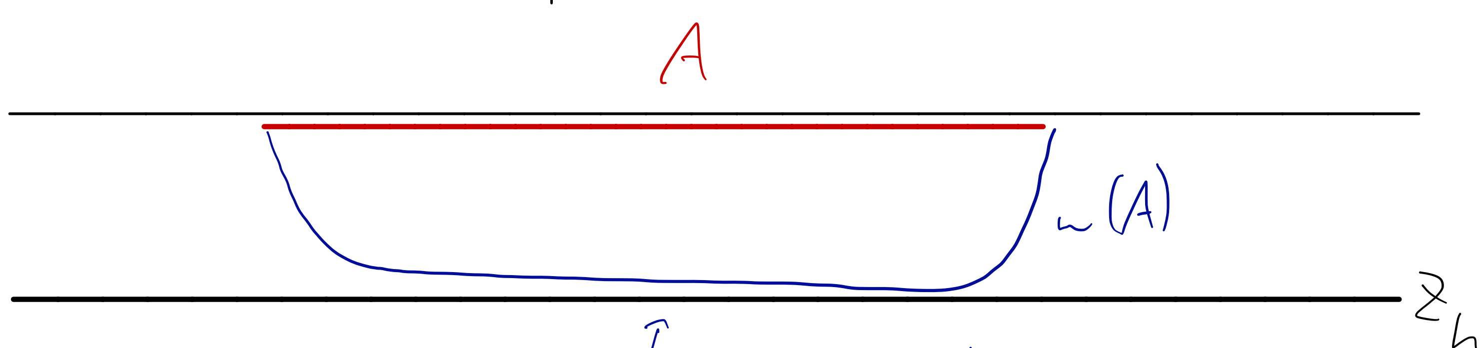
$$S(A) = \frac{1}{4 l_{Pl}} \text{length}(\gamma(A))$$

1) For 1 interval in ground state of CFT, easy to find geodesic in AdS_3 + calculate length, reproducing $S(A) = \frac{c}{3} \ln \frac{L}{\varepsilon}$

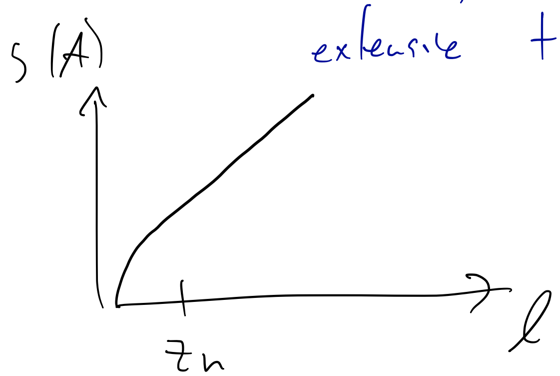


[Exercise]

2) At finite temp. for $L \gg z_h \sim \frac{1}{T}$,

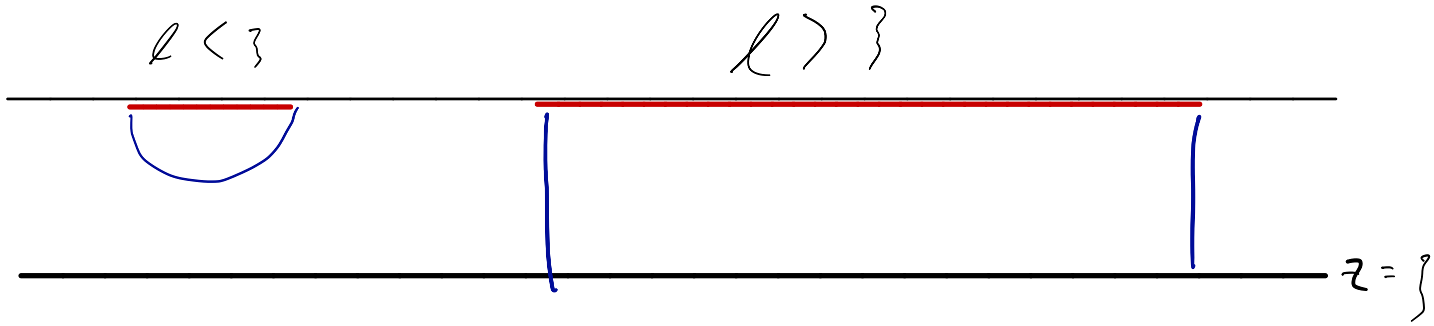


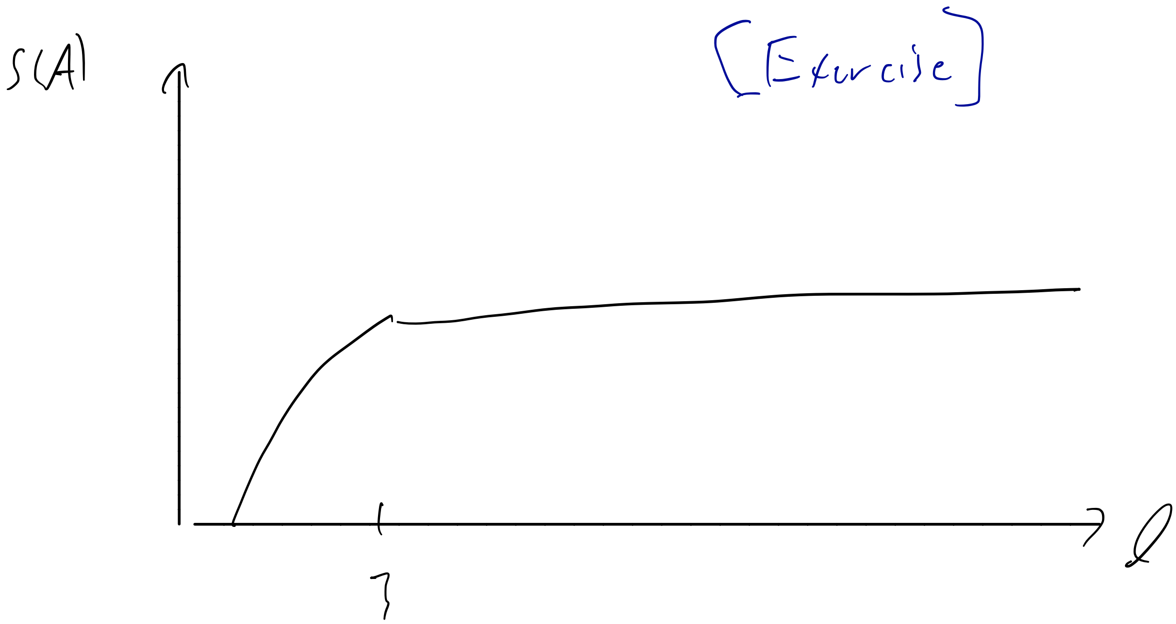
extensive thermal ent.



[Exercise]

3) In gapped thy



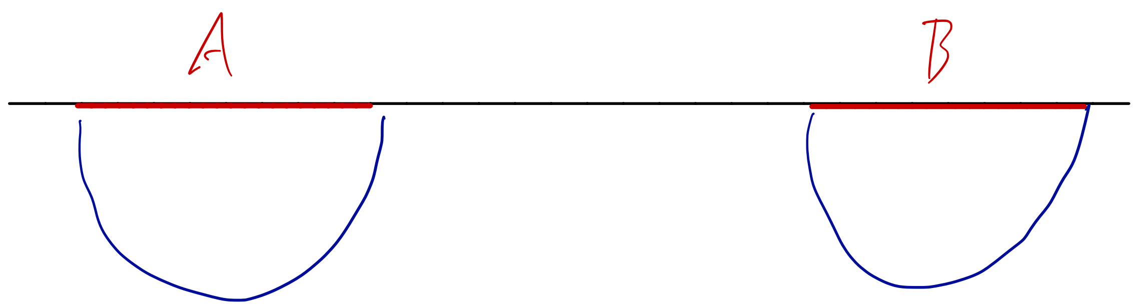


Here the saturation becomes sharp.

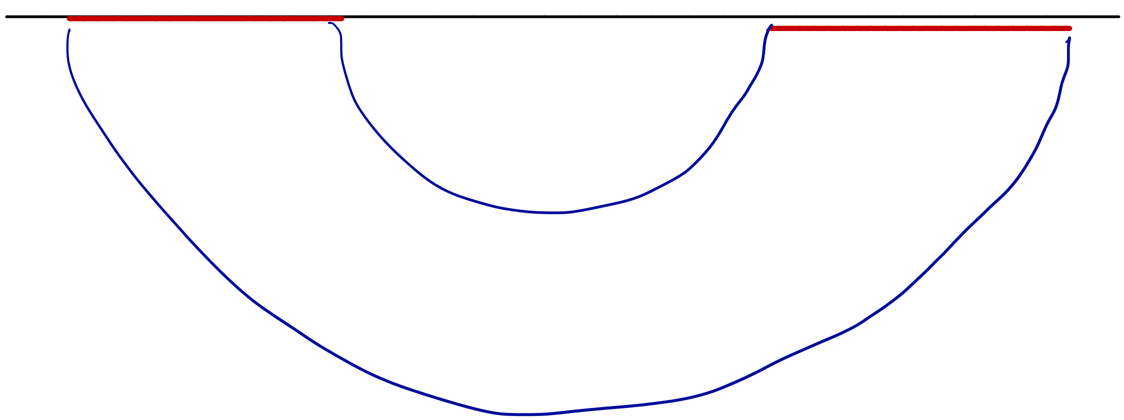
There is a phase transition, because $c \to \infty$ is a thermodynamic limit.

4) Mutual info

2 candidate min. surf. for AB:

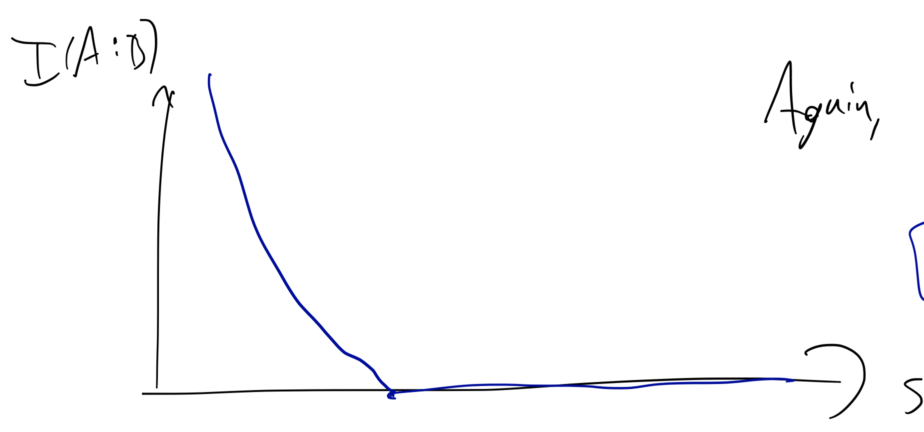


$m(AB) = m(A) \cup m(B)$



$m(AB) \neq m(A) \cup m(B)$

Both homologous to AB. One w/ shorter total length gives $S(AB)$



Again, phase transition

[Exercise]

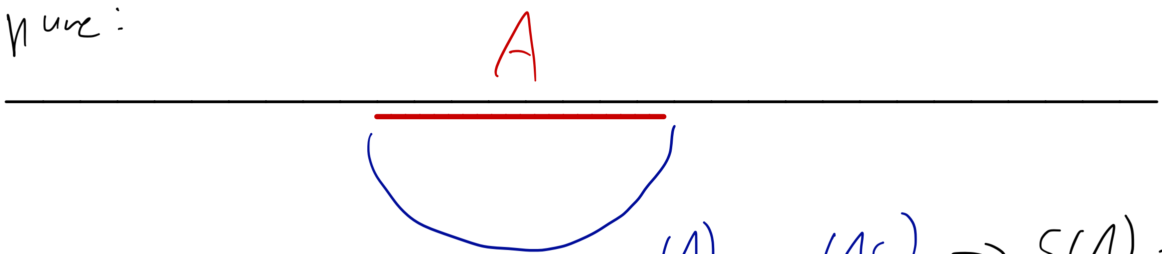
D. Properties

1) In a pure state, $S(A) = S(A^c)$

In a mixed state, not necessarily

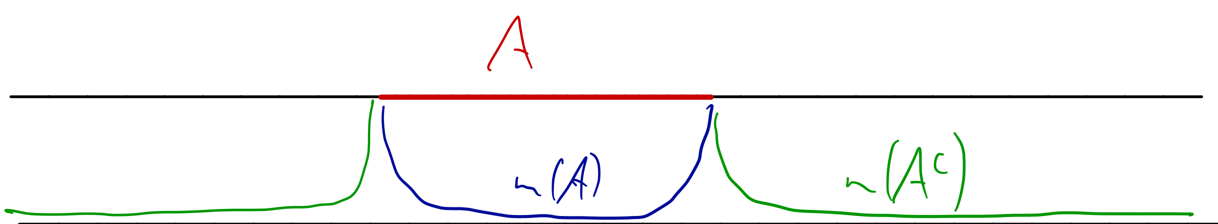
Obeyed by RT:

pure:



$m(A) = m(A^c) \Rightarrow S(A) = S(A^c)$

mixed:



$m(A) \neq m(A^c) \Rightarrow S(A) \neq S(A^c)$

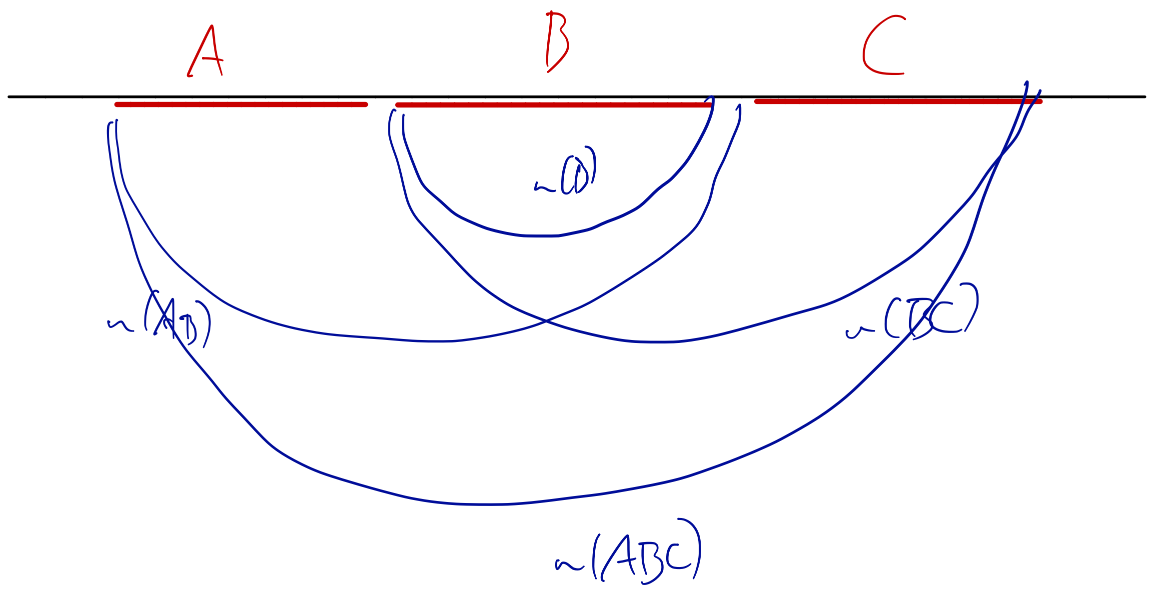
2) Subadditivity $S(AB) \leq S(A) + S(B)$

As we saw before, $m(A) \cup m(B)$ is always a candidate surface for AB, hence $S(AB) \leq \text{area}\,(m(A) \cup m(B))$

$$= \text{area}\,(m(A)) + \text{area}\,(m(B))$$

$$= S(A) + S(B)$$

3) Strong subadditivity $S(ABC) + S(B) \leq S(AB) + S(BC)$



In fact, all known general properties of EE are obeyed by RT.

There are also some that are special to holographic systems, e.g.

4) Superadditivity of mutual info

$$I(A:BC) \geq I(A:B) + I(A:C)$$

i.e. $$S(A)+S(B)+S(C)+S(ABC)$$

$$\leq S(AB)+S(AC)+S(BC)$$

[Proof: Exercise]

Ex:

Can have $I(A:B) = I(A:C) = 0$

but $I(A:BC) \neq 0$

This inequality is not true in general quantum systems (or even field theories)

E.g. $\rho_{ABC} = \frac{1}{2}\left(|000\rangle\langle 000| + |111\rangle\langle 111|\right)$

E. Many other directions & applications

- Bit threads
- time dependence (HRT)
- deriving Einstein eq.
- reconstructing bulk
- AdS/CMT, AdS/QCD
- Corrections
- Rényi's, relative entropies, ...

⋮