TALLAHASSEE WINTER SCHOOL 2018: ENTANGLEMENT IN MANY BODY STATES

Outline:

1. Brief Review of entanglement entropy and its scaling

2. Entanglement in Gaussian states:

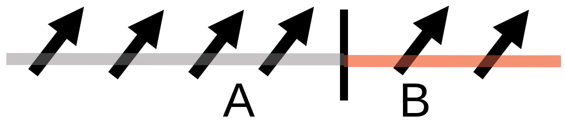
*** Fermions**

*** Bosons**

"What do you mean, 'a quantum fluctuation?' Didn't we discuss cause and effect?"

3. Entanglement and locality: a new quantum phase transition from area law to volume law

REDUCED DENSITY MATRIX ENTROPY



$$H = H_A \otimes H_B$$

$$|\Psi\rangle_{AB} = \sum_{i=1}^{\dim H_A} \sum_{j=1}^{\dim H_B} C_{ij} \, |u_i\rangle_A \, |v_j\rangle_B$$

$$C_{ij} = U_{ik} \lambda_k V^{+}{}_{kl}$$

$$|\Psi\rangle_{AB} = \sum_{i=1}^{N} \lambda_i \, |\xi_i\rangle_A \, |\zeta_i\rangle_B$$

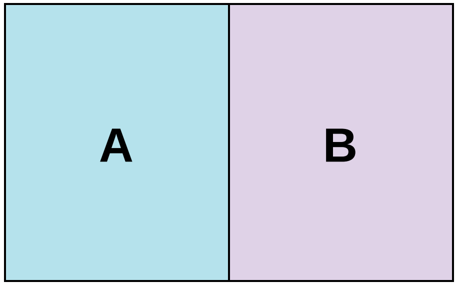
Schmidt decomposition

(note single index)

N is called the Schmidt number

REVIEW OF ENTANGLEMENT ENTROPY:

Reduced density matrix:



$$\rho_A = Tr_B \rho = \sum_{i=1}^{N} |\lambda_i|^2 |\xi_i\rangle_A \langle \xi_i|$$

More proper definition for the reduced density matrix:

$$\langle \psi_{AB} | \hat{O}_A | \psi_{AB} \rangle = Tr_A \left(\rho_A \hat{O}_A \right)$$

Different characterizations of the entropy in the state:
Schmidt number, entropy, Renyi entropy, many more

ENTANGLEMENT ENTROPY:

$$\boxed{S_A = -Tr\,\rho_A \log \rho_A} \text{ where } \rho_A = Tr_B\,\rho$$

Examples:

$S_A(|\uparrow\uparrow\rangle) = 0$

and for the singlet state

$S_A\left(\frac{|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle}{\sqrt{2}}\right) = 1$

EE is a **"non-local"** version of correlation functions

EE may be **non-vanishing even at zero temperature**

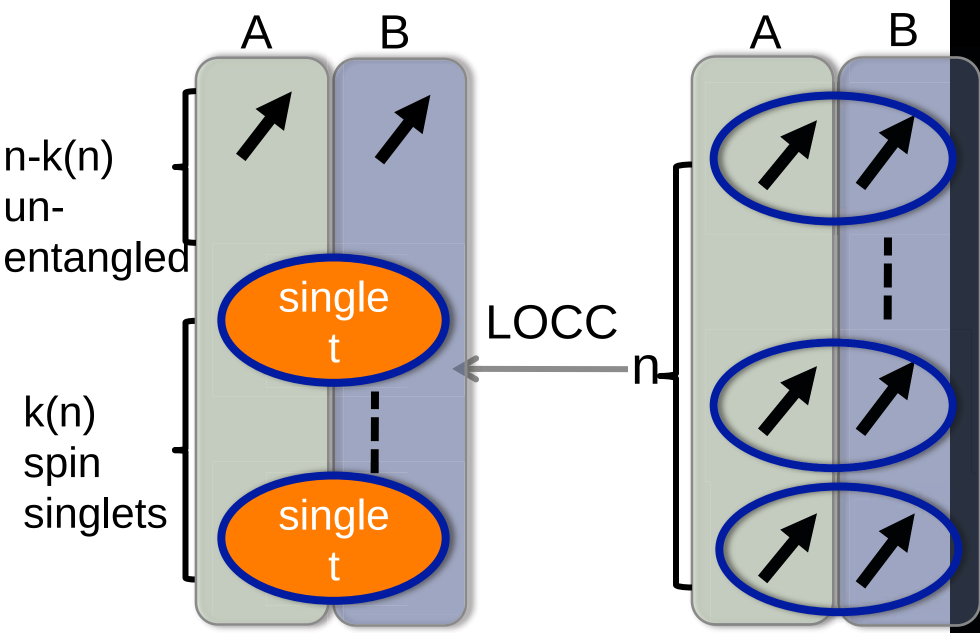
QUANTUM INFORMATION INTERPRETATION: (BENNET ET AL.96)

Given $\psi_{AB} \in H_A \otimes H_B$

k(n) = # of maximally entangled pairs that can be extracted by local operations from $\psi_{AB}^{\otimes n}$

Then :

$$\lim_{n\to\infty} \frac{k(n)}{n} = S_A$$

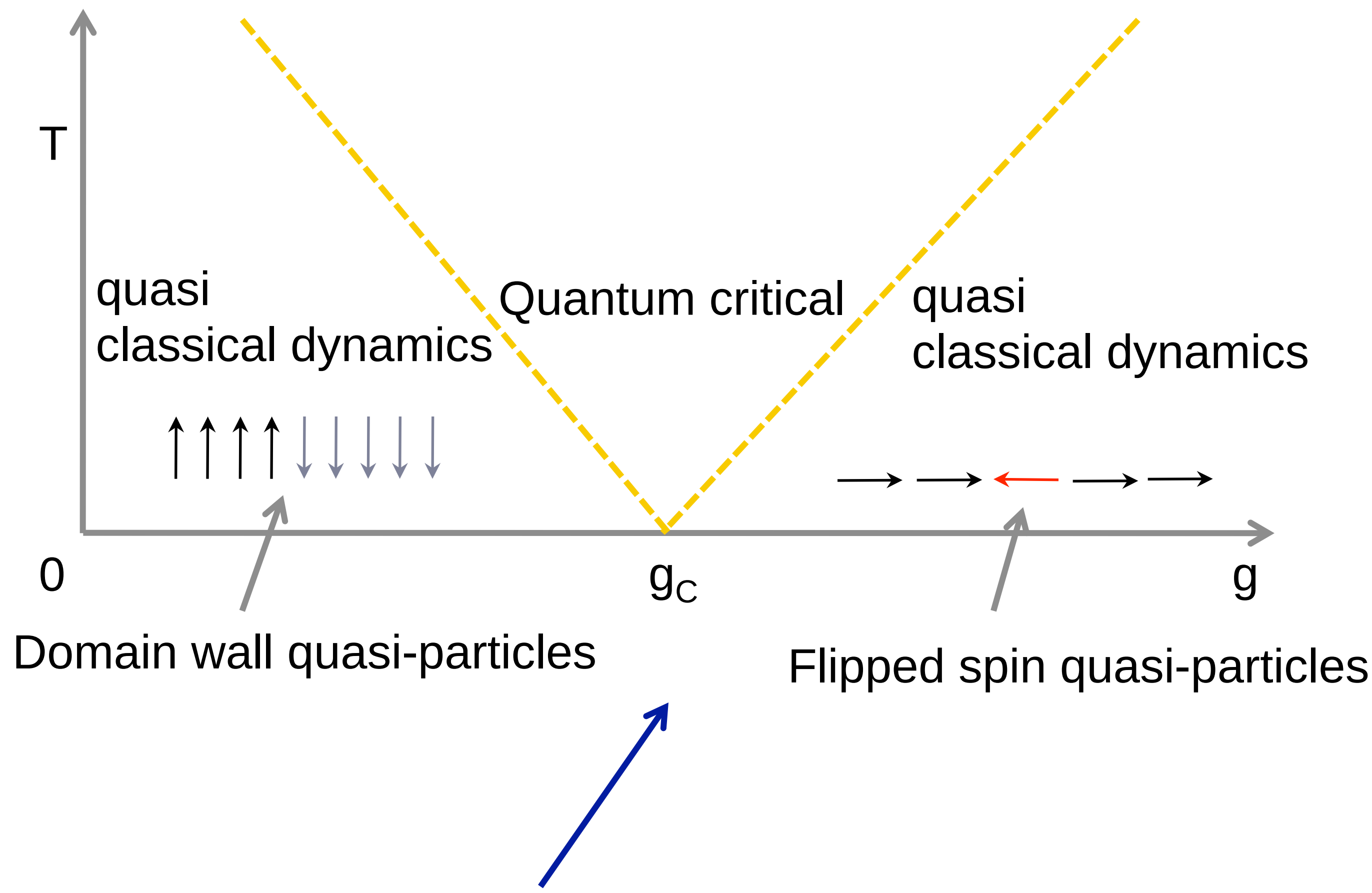


Generalization:

How many maximally entangled pairs can be extracted between two parts of a spin chain?

Quantum Criticality

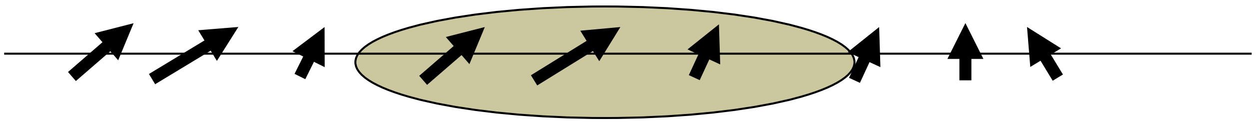
$$H_I = J\sum_i \left(g\sigma_i^x + \sigma_i^z\sigma_{i+1}^z\right)$$



gap closes, system described critical theory

Features 1d:

SCALING OF ENTANGLEMENT ENTROPY:



General feature at Qauntum phase transition point:

To physicists all objects are point like froma far, characterised by their s-wave scattering.

From afar, many quantum critical 1d systems are conformal field theories, characterized by their conformal charge

For CFTs:

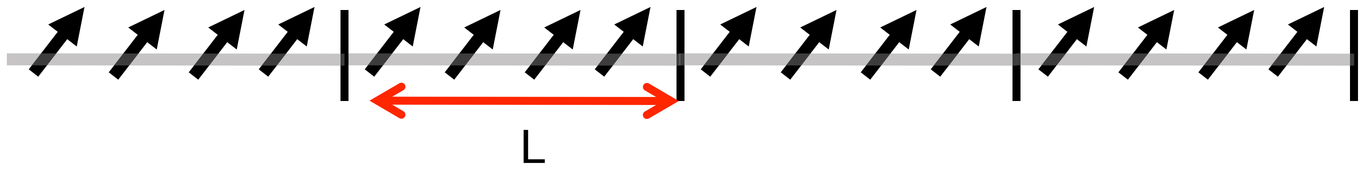
$$S_L \xrightarrow[L\to\infty]{} \frac{c+\bar{c}}{6}\log L$$

c is the central charge of the critical conformal field theory (Holzhey, Larsen & Wilczek 94, Calabrese & Cardy 04)

Universality

ENTROPY SCALING AND SIMULATIONS

A gapped system in 1D allows for classical simulation:



DMRG (white 92) , MPS Fannes Nachtengale Werner (92): chop into bigger and bigger blocks. size of effective block depends on correlation length. # of effective degrees of freedom ~ exp(entanglement entropy of the block)

1d gapped exp(S)~ saturates => do-able

1d Critical exp(S)~ L => still doable, but not very good

d>1 => area laws => much harder. Progress in 2D, MERA, PEPS etc..

Fermions much harder than bosons due to worse area law

d>1:

Boson fields:

Entropy of a scalar field restricted to a subsystem was first studied by Bombelli et al (87) as a quantum contribution to Bekenstein-Hawking entropy. Area law for bosonic systems (Srednicki93,Rigorized Plenio et al. (05))

Topological states (2+1) dimensions (Kitaev&Preskill, Levin&Wen).

$$S_A \sim \alpha L_A - \gamma_{top} + \ldots \qquad ; \qquad \gamma_{top} = \log D \quad ;$$

$$D = \text{"total quantum dimension"}$$

Expect an area law for gapped systems at D>1. Not proven (yet).

In 1D area law for gapped states proved by Hastings.
With an exponentially improved bound: Arad et al (2013)

MODELS WITH EXACTLY COMPUTABLE ENTROPY ARE SCARCE

MAIN BENCHMARK SYSTEMS: SYSTEMS WITH QUADRATIC HAMILTONIANS

NEXT: FERMIONS

LEONID LEVITOV (MIT)
DIMITRY GIOEV (ROCHESTER)
GIL REFAEL (CALTECH)
ALESSANDRO SILVA (ICTP, TRIESTE)
CHRISTIAN FLINDT (GENEVA),
STEPHAN RACHEL, H. FRANCIS SONG,
KARYN LE HUR (YALE)

Reduced density matrices for Gaussian states

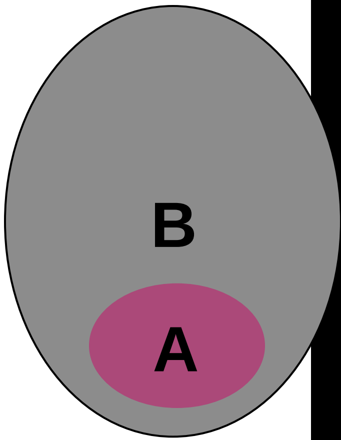
A quasi free state, or gaussian state, by **Wicks theorem** is characterized
In terms of it's two-point correlations.

$$\langle A_1 .. A_{2n} \rangle = \sum (\pm 1)^P \langle A_{P_1} A_{P_2} \rangle .. \langle A_{P_{2n-1}} A_{P_{2n}} \rangle$$

In particular Wick's theorem holds for two point
correlations functions restricted inside A

→ Given the restriction of the two point function to A

The state of fermions in A is can be read from it as:



Explicitly:

Recall Fermi-Dirac:

$$n_k = \langle a_k^+ a_k \rangle = \frac{1}{1+e^{\beta E(k)}}$$

Excersise : True in a general basis

$$M_{ij} = \langle a_i^+ a_j \rangle = \left(\frac{1}{1+e^{\beta H}} \right)_{ij}$$

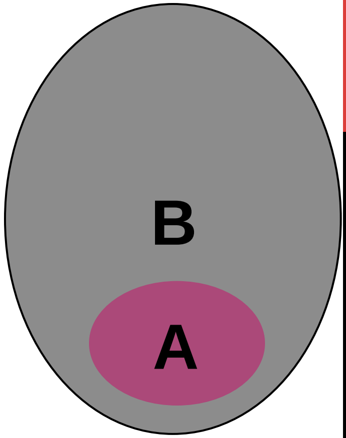
$$\Rightarrow H_{eff} = Log\left(\frac{M_A - 1_A}{M_A} \right)$$

$M_A = M$ restricted to A

"entanglement Hamiltonian"

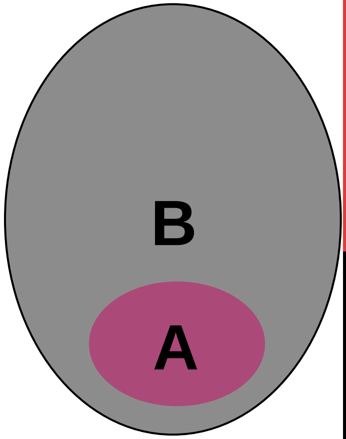
Reduced density matrix, explicitly:

$$\rho_A = \frac{e^{-(H_{eff})_{ij} a_i^+ a_j}}{Z_A}$$



Projection on A
P(x)=1 if x in A

PROPERTIES OF M



Projection on A
P(x)=1 if x in A

$$M_{ij} = P_A(i)\langle a_i^+ a_j \rangle P_A(j) = \left(\frac{1}{1+e^{H_{eff}}}\right)_{ij}$$

M positive, and M<1

Spectrum of M corresponds to occupation probability of the eigenmodes of the effective H. Related to entanglement spectrum (Haldane's talk)

$$\rho_A = \frac{e^{-(H_{eff})_{ij}\, a_i^+ a_j}}{Z_A} \quad ; \quad H_{eff} = Log\left(\frac{M-1}{M}\right)$$

$$diagonalize \Rightarrow$$

$$\rho_A = \frac{e^{-Log\left(\frac{M_k-1}{M_k}\right) c_k^+ c_k}}{Z_A}$$

Note **number fluctuations**:

$$\langle \Delta N_A \rangle^2 = Tr\ M(1-M)$$

$$S_A = -Tr\ (M \log M + (1-M)\log(1-M))$$

FERMIONS IN A FERMI SEA

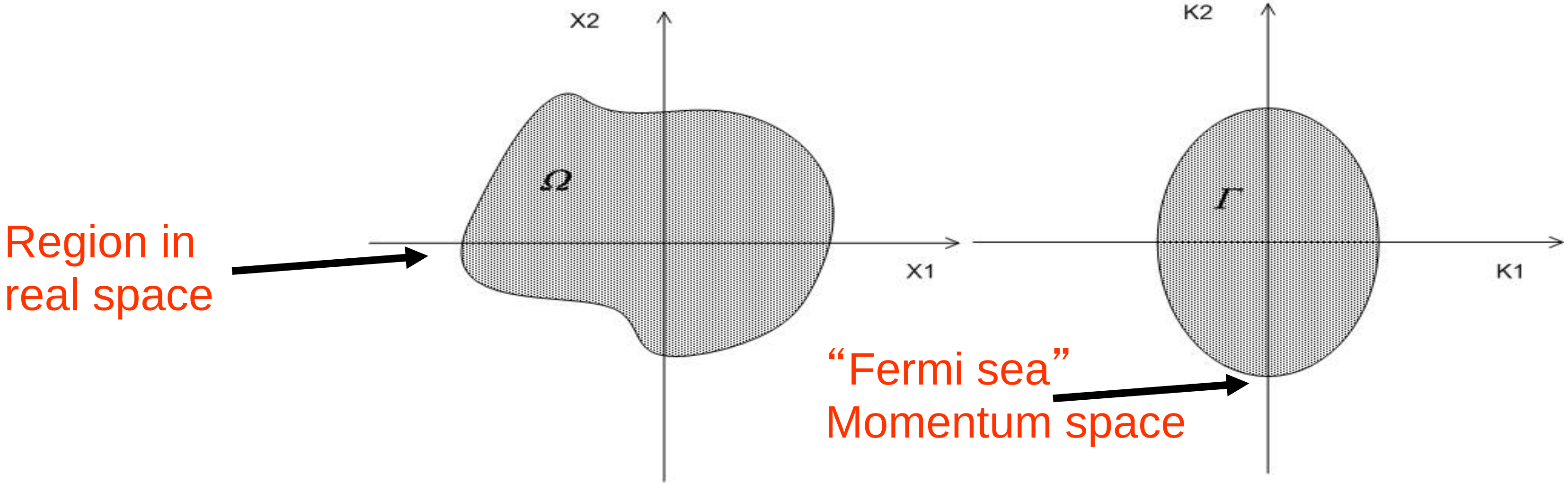
Hamiltonian

$$H = \int (E(k) - E_F) a^+(k) a(k) d^k k$$

Ground state (lowest eigenvector of H)

$$\psi = \prod_{k \in \Gamma} a^+(k) \,|\, 0 >$$

Fermi surface $\partial\Gamma = \{\mathrm{k} \mid \mathrm{E(k)} = \mathrm{E_F}\}$



M FOR A FERMI SEA:

$$\text{At T} = 0 \text{ the two point function is:}$$

$$\langle a_x{}^+ a_{x'} \rangle = \langle x | \theta(H - E_f) | x' \rangle$$

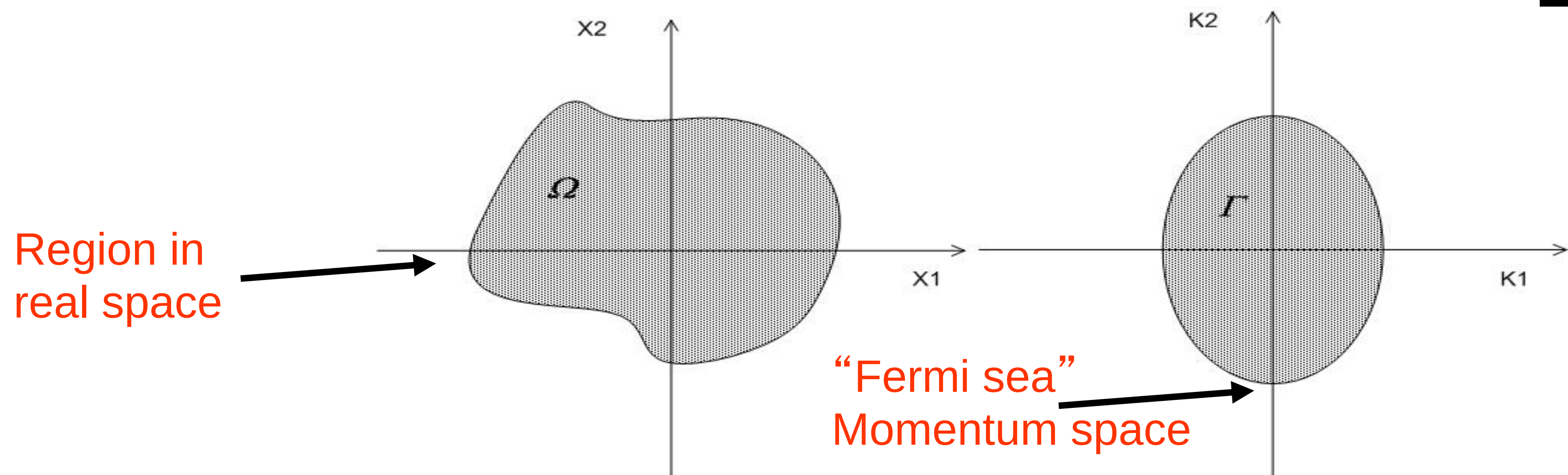
Fermi step in a general basis

$$So:$$

$$M = P_A Q P_A$$

$$Explicitly: \qquad \langle x | M | x' \rangle = \chi_\Omega(x) \chi_\Omega(x') \int_\Gamma \frac{e^{ik(x-x')}}{(2\pi)^d} dk$$

M tries to simultaneously "localize" in space and momentum



TRANSLATIONALLY INVATRIANT SYSTEMS

$$\langle x|M|x'\rangle = \chi_\Omega(x)\chi_\Omega(x')\int_\Gamma \frac{e^{ik(x-x')}}{(2\pi)^d}dk = \chi_\Omega(x)g(x-x')\chi_\Omega(x')$$

In 1D M is a block of a Toeplitz matrix (Operator)

$$\begin{pmatrix} m_1 & m_2 & m_3 & m_4 & m_5 \\ m_{-1} & m_1 & m_2 & m_3 & m_4 \\ m_{-2} & m_{-1} & m_1 & m_2 & m_3 \\ m_{-3} & m_{-2} & m_{-1} & m_1 & m_2 \\ m_{-4} & m_{-3} & m_{-2} & m_{-1} & m_1 \end{pmatrix}$$

Def: A is a **Toeplitz matrix** if A_{ij} =g(i-j)

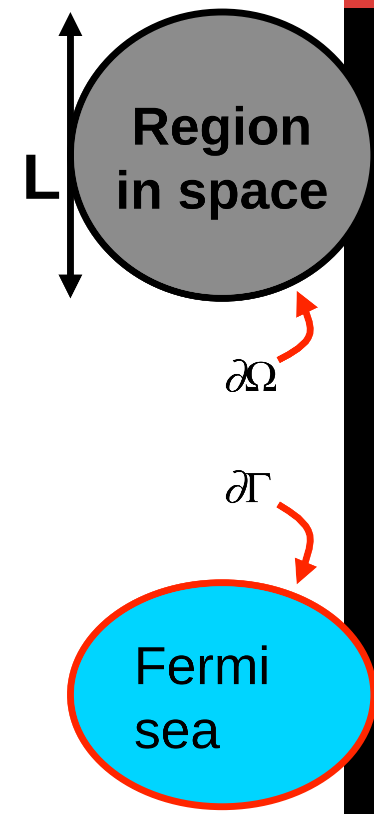
Spectrum can be studied using Szego theorems/Hartwig-Fisher asymptotics

$$Det_{1,..N}\, A \xrightarrow{N\to\infty} C\, \exp\oint g(k)\frac{dk}{2\pi}$$

Fourier trans of g

FORMULA BASED ON WIDOM'S CONJECTURE

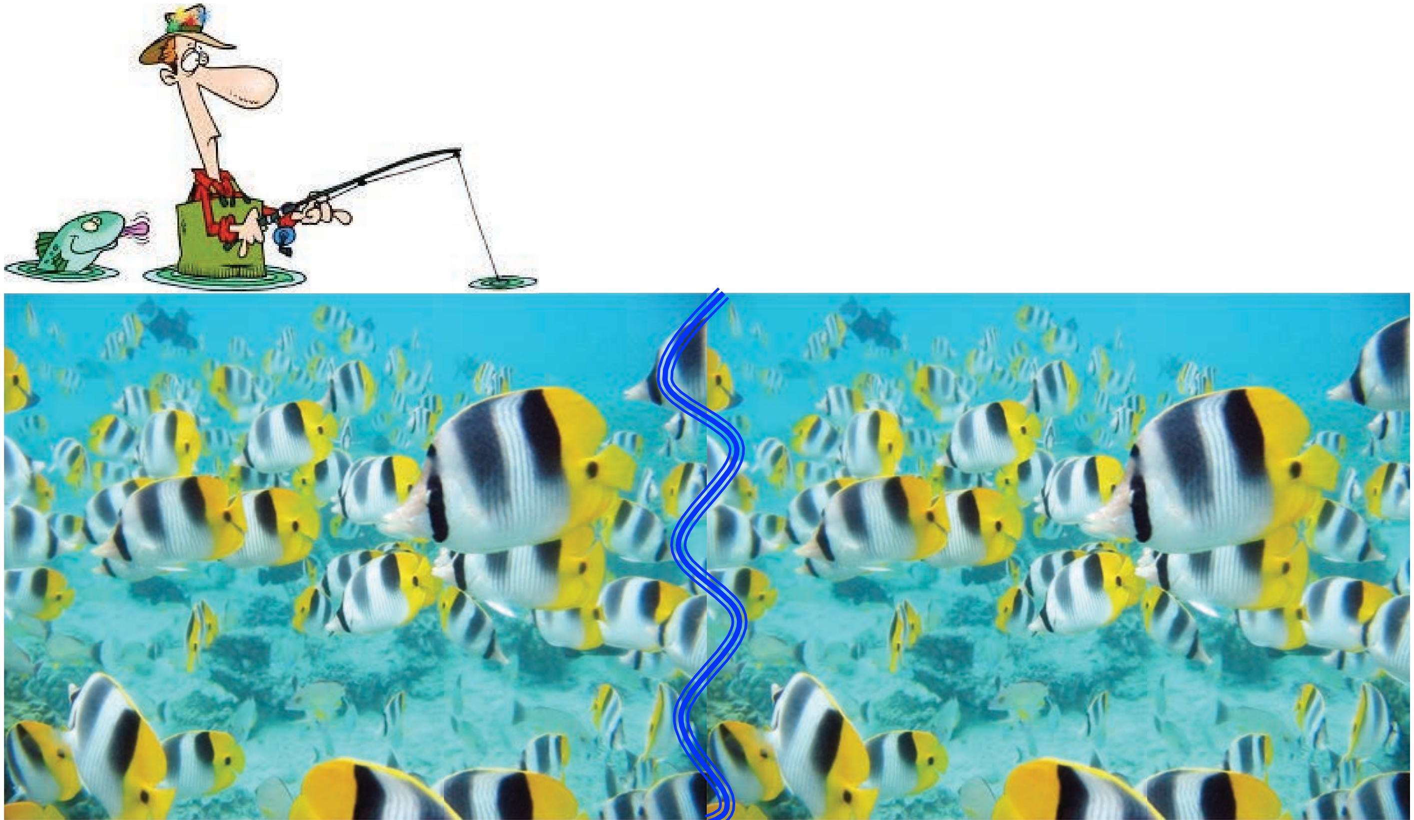
$$S_\Omega \sim \frac{L^{d-1}\log L}{12(2\pi)^{d-1}} \iint_{\partial\Omega,\partial\Gamma} |n_x \cdot n_p| \, dS_x \, dS_p + o(L^{d-1}\log L)$$



➔ **Logarithmic violation of area law**

Seidel et al. prb2012: extension to interacting systems using higher dimensional Bosonization. Widom proved by Sobolev 2013

Measuring entanglement entropy for fermions



FERMION NUMBER FLUCTUATIONS:

Particle number fluctuations share with entropy two traits:

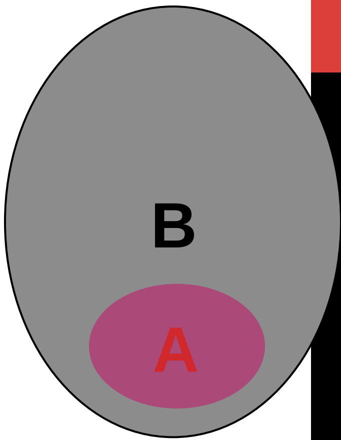
1) Subadditivity

2) Symmetry

Can be used as an indication for entanglement!

Can we do more?

Main Tool: Relation to "Full Counting Statistics" (FCS)



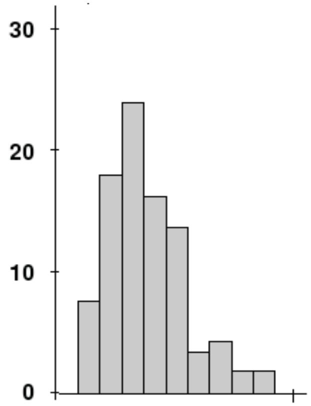
p_n = Probability of having n fermions in A

$$\chi(\lambda) = \sum p_n e^{i\lambda n}$$

$$\log \chi(\lambda) = \sum \frac{(i\lambda)^n}{n!} C_n$$ ⟷ **"Cumulants"**

signal　　　　Noise　　　　Skewness

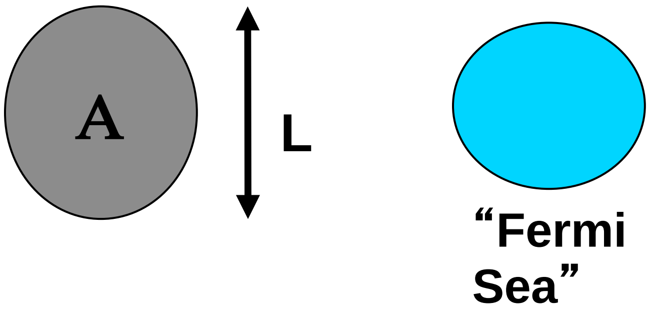
$$C_1 = \langle n \rangle \quad ; \quad C_2 = \langle \delta n^2 \rangle \quad ; \quad C_3 = \langle\langle \delta n^3 \rangle\rangle \ldots$$



COUNTING STATISTICS FOR PARTICLE NUMBERS:

$$M = P_A P_E P_A$$

Counting of particles in



$$\chi(\lambda) = \sum_n p_n e^{i\lambda n} = \det(1 - M + Me^{i\lambda})$$

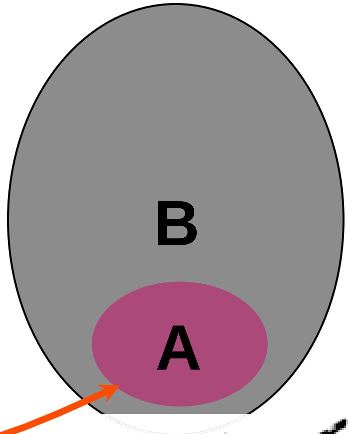
Recall:

$$S = -Tr[M \log M + (1 - M)\log(1 - M)]$$

We can find the spectral density of M from the counting statistics generating function and use it to express S

MAIN RESULT:

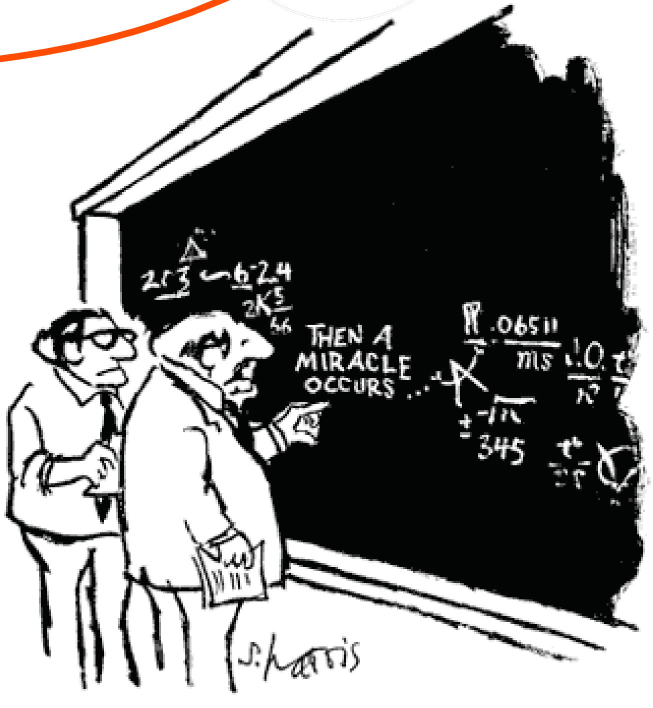
$$S_A = \sum_{m\geq 2, Even} \frac{|B_m|(2\pi)^m}{m!} C_m$$



B_m are Bernoulli numbers

$$S_A = \frac{\pi^2}{3} C_2 + \frac{\pi^4}{15} C_4 + \frac{2\pi^6}{945} C_6 + \ldots$$

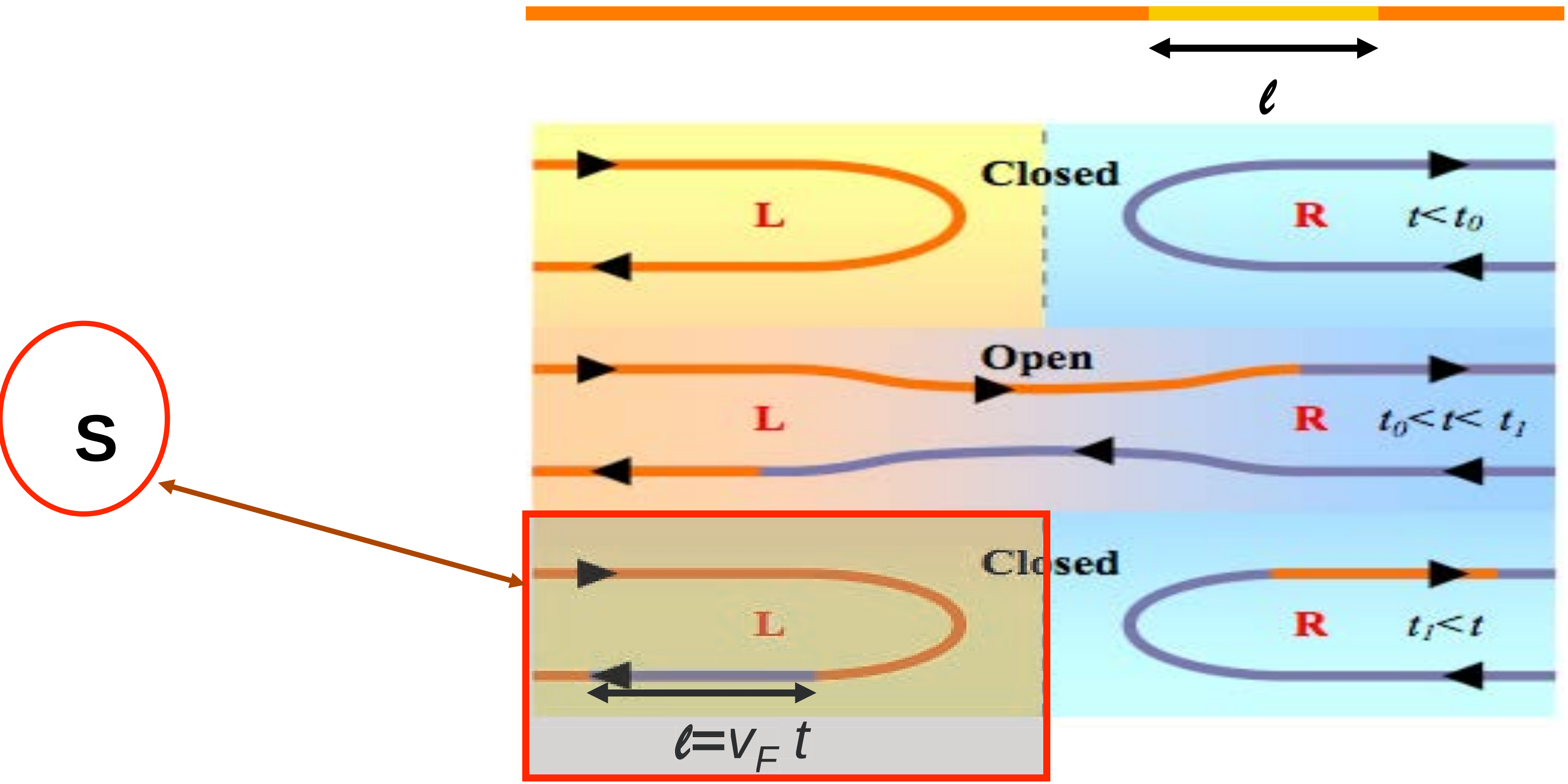
Coefficients are universal!



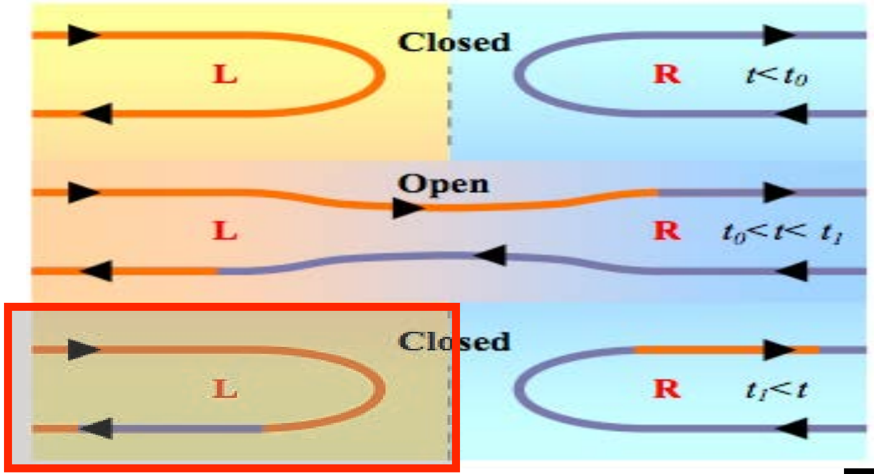
"I think you should be more explicit here in step two."

IDEA: ENTANGLEMENT ENTROPY ➔ TRANSPORT PROBLEM

Linear dispersion means excitations travel without changing shape, typical situation in 1d systems



ABRUPT AND PERFECT CONNECTION:



Gaussian FCS

$$\chi(\lambda) = e^{-\frac{\lambda^2}{2\pi^2}\log(t/\tau)}$$

$$C_m = 0 \text{ for m} > 2$$

$$S = \frac{1}{3}\log(t/\tau)$$

"t" is duration of connected state

short time cutoff (switching time)

➔ **Recovered the result of Holzhey Larsen& Wilczek!**

RELATION (GENERICALLY) DOESN'T CONVERGE!

H. Francis-Song, C. Flindt, S. Rachel, IK and K. Le-Hur2011

Re-summation:

The convergent expression is:

$$S_A = \lim_{K\to\infty} \sum_{m\geq 2, Even}^{K} a_m(K) C_m$$

$$a_m(K) = 2\sum_{j=m-1}^{K} \frac{s_1(j,m-1)}{j!\,j}$$

s_1 - unsigned Stirling numbers of first kind

BOSONIC GAUSSIAN STATES

Crash course on Entanglement in Gaussian states:

$$\text{Let } (x_1, p_1, \ldots x_n, p_n) = (O_1, ..O_{2n}) \text{ where x,p conjugate}$$

$$[O_j, O_k] = i\hbar\sigma_{jk}$$

$$\sigma = \oplus_{j=1}^{n} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Let γ be the covariance matrix:

$$\gamma_{ij} = 2\,\mathrm{Re}\left\langle (O_j - \langle O_j \rangle)(O_k - \langle O_k \rangle) \right\rangle$$

Entropy depends on simplectic eigenvalues of γ

Bombelli et al. 87. Modern outlook and
reviews J. Eisert, M.B. Plenio JQI(2003),
A. Botero and B. Reznik PRA (2003) etc..

Transformations

$$\begin{pmatrix} O_1 \\ \dots \\ O_{2n} \end{pmatrix} \longrightarrow S \begin{pmatrix} O_1 \\ \dots \\ O_{2n} \end{pmatrix}$$

preserving commutations $[O_j, O_k] = i\hbar\sigma_{jk}$

are the symplecitc matrices Sp(2n) $\equiv$ all S s.t. S $\sigma S^T = \sigma$

Under a symplectic transformation

$\gamma \rightarrow S\gamma S^T$

Williamson / Darboux Thm: there is an S s.t.

$S\gamma S^T = diag(\lambda_1, \dots, \lambda_n, \lambda_1, \dots, \lambda_n)$

Use the symplectic transformation to get to normal form, then:

$$\gamma \rightarrow S\gamma S^T = diag(\lambda_1,\ldots,\lambda_n,\lambda_1,\ldots,\lambda_n)$$

and:

$$\rho_{field} - \frac{1}{Z} e^{-\sum_m \log\left(\frac{\lambda_m + 1/2}{\lambda_m - 1/2}\right) a_m^+ a_m}$$

Finding the λ_i can be tricky, but assuming $\langle p_k x_l \rangle = 0$

Then one can check that $|\lambda_i|^2$ are eigenvalues of

$$\Gamma = 4\sum_k G_{jk} H_{kl} \qquad G = \langle x_j x_k \rangle \quad H = \langle p_k p_l \rangle$$

Entropy:

$$S - \sum_i h(\lambda_i)$$

$$h(\lambda) - \frac{\lambda+1}{2}\log\frac{\lambda+1}{2} - \frac{\lambda-1}{2}\log\frac{\lambda-1}{2}$$

λ_i are simplectic eigenvalues of the covarianc matrix

Note h(1)=0. All eigenvalues are larger than 1 due to uncertainty relation:

$$\lambda = 2\Delta x \Delta p \geq 1$$

APPLICATION: RADIATION MATTER ENTANGLEMENT

IK Radiation matter entanglement, arXiv:1208.2474

IK On the entanglement of a quantum field with a dispersive medium, Phys. Rev. Lett. 109, 061601 (2012)

Entanglement cuts do not have to be spatial!

Consider an EM modes in a dielectric. What is the entanglement between the modes and the matter?

To summarize: effective action

$$S_{eff} = \frac{1}{4\pi}\int d^3x d\omega \varphi^*(x,\omega)[\omega^2\varepsilon(x,\omega) - \nabla^2]\varphi(x,\omega)$$

Model dielectric function

Allows to compue $\left\langle \varphi(x,t)\varphi(x',t') \right\rangle$ correlators.

$$\left\langle \varphi(x,t)\varphi(x',t') \right\rangle = \frac{1}{4\pi}\int_0^\infty d\omega \frac{e^{i\omega(t-t')}}{-\nabla^2 + \omega^2\varepsilon(x,i\omega)}$$

$need:$

$$\left\langle \varphi(x,0)\varphi(x',0) \right\rangle = \frac{1}{4\pi}\int_0^\infty d\omega \frac{1}{-\nabla^2 + \omega^2\varepsilon(x,i\omega)}$$

And assume conjugate momentum obeys $\pi_\varphi = \dot{\varphi}$

$$S = \frac{1}{4\pi}\int d^3x d\omega \varphi^*(x,\omega)[\omega^2\varepsilon(x,\omega) - \nabla^2]\varphi(x,\omega)$$

Trivial example :

$\varepsilon(x,\omega) \equiv \varepsilon(x)$ *Independent of* ω

No Entropy : Described by a hamiltonian

$$H = \frac{1}{4\pi}\int d^3x \left(\frac{\pi^2}{\varepsilon(x)} + (\nabla\varphi)^2\right)$$

EXAMPLE:

Translationally invariant system, entropy per unit volume

Check free space:

$$S=\frac{1}{2}\int d^3x\,\frac{dw}{2\pi}\varphi_w^*(x)(w^2+\nabla^2)\varphi_w(x)$$

$$\langle\varphi(k)\varphi(k')\rangle_{Free}=\frac{\delta_{kk'}}{\pi}\int_0^\infty dw\;\frac{\hbar}{w^2+k^2}=\frac{\hbar}{2k}\delta_{kk'}$$

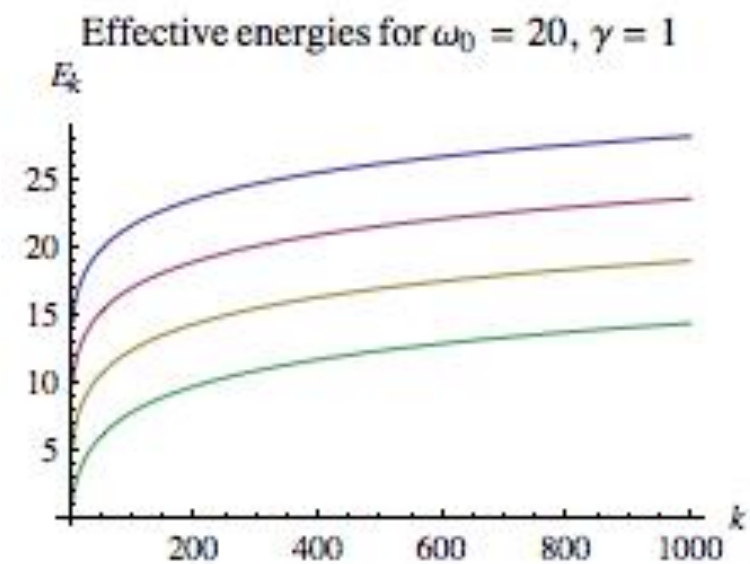
$$\langle\pi(k)\pi(k')\rangle_{Free}=\frac{\delta_{kk'}}{\pi}\int_0^\infty dw\;\frac{\hbar k^2}{w^2+k^2}=\frac{k\hbar}{2}\delta_{kk'}$$

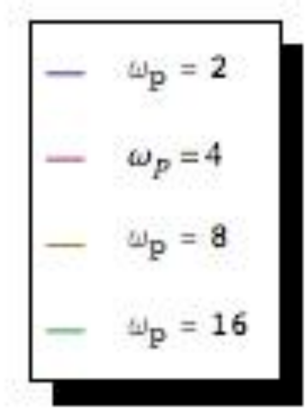
$$\Rightarrow\boxed{GH(k)-\frac{4}{\hbar}\langle\varphi^2\rangle_k\langle\pi^2\rangle_k-1}\quad\Rightarrow\quad No\;\;entropy\;\;\surd$$

MAIN RESULTS:

$$Typical\ dielectric: \varepsilon(\omega) = 1 + 4\pi \frac{{\omega_p}^2}{{\omega_0}^2 - \omega^2 - i\gamma\omega}$$

$$\frac{1}{Z} e^{-\sum_k E(k) c_k^+ c_k} = \frac{1}{Z} e^{-\sum_k (Log[\lambda_k + 1/2] - Log[\lambda_k - 1/2]) c_k^+ c_k}$$





Not linear → not a simple thermal state!
Too many conserved quantities
(see Huse's talk)

HOW MUCH ENTANGLEMENT CAN A LOCAL HAMILTONIAN SUPPORT?

Supported by:

A Ahmadain, Z Zhang (Uva)
R Alexander (UNM)
H Katsura, T Udagawa (Tokyo)
V Korepin, O Salberger (Stony Broo

Refs:
Z. Zhang and IK, J. Phys. A, 50, 42 (2017);
Z. Zhang, A. Ahmadain and IK, PNAS, 114, 20 (2017);
O. Salberger, T. Udagawa, Z. Zhang, H. Katsura, IK and V. Korepin, J. Stat. Mech (2017): 063103

ENTANGLEMENT SCALING IN TYPICAL SYSTEMS:

Entanglement entropy:

$$S_A = -Tr\rho_A \log \rho_A \text{ where } \rho_A = Tr_B \rho$$

Generic states in Hilbert space have extensive entanglement

(page prl 93,foong prl 94,sen prl 96)

$$S_A \approx \begin{cases} L^d & \textit{generic state} \\ L^{d-1} & \textit{gapped, "area law"} \\ L^{d-1}\log L & \textit{free fermions} \\ \frac{c}{3}\log L & \textit{conformal} \end{cases}$$

- L^d — generic state (Page prl 93)
- L^{d-1} — gapped, "area law" (Hastings 07,1d)
- $L^{d-1}\log L$ — free fermions (Gioev IK 06,M Wolf 06,…)
- $\frac{c}{3}\log L$ — conformal (Holzhey Larsen Wilczek 96, many many more)

EXTENSIVELY ENTANGLED STATES

First local Hamiltonian with volume scaling: Irani 2010.

local Hilbert space dimension is 21

Simpler models but without translational invariance, and with exponentially varying couplings:

Gottesman Hastings 2010 (not frustration free)

Rainbow ground states:Vitagliano Riera Latorre 2010, Ramirez Rodriguez-Laguna Sierra 2014

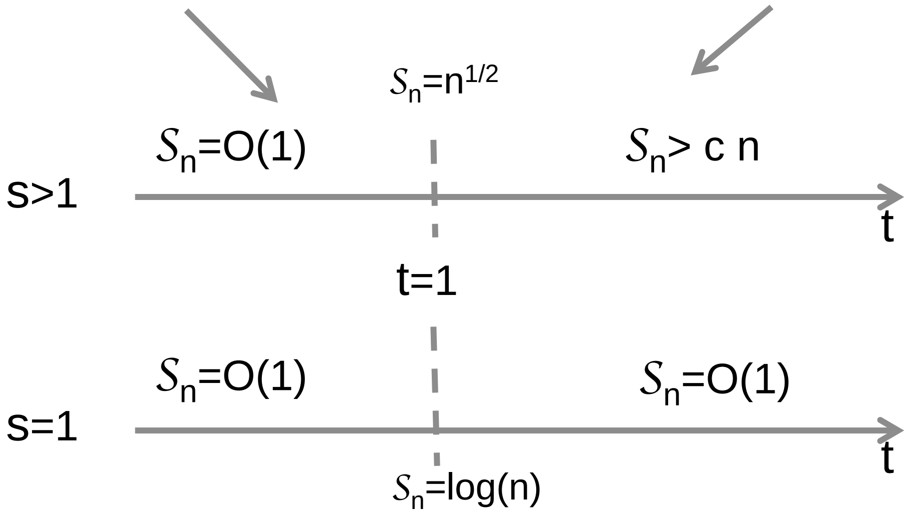
Translationally invariant but with a square root scaling:

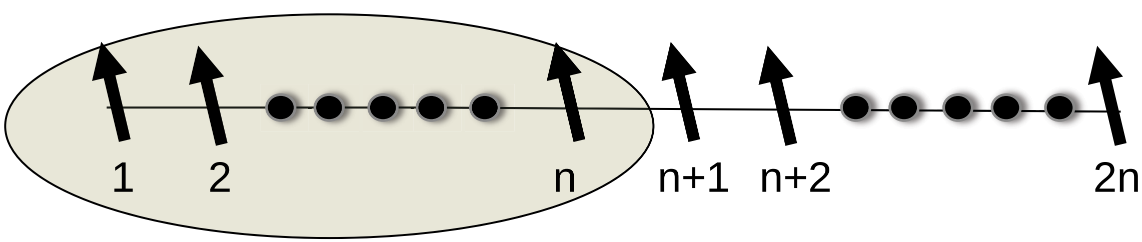
Movassagh Shor (2014), Salberger Korepin (2016)

Here: a simple spin chain with remarkable phase transition:

Product state

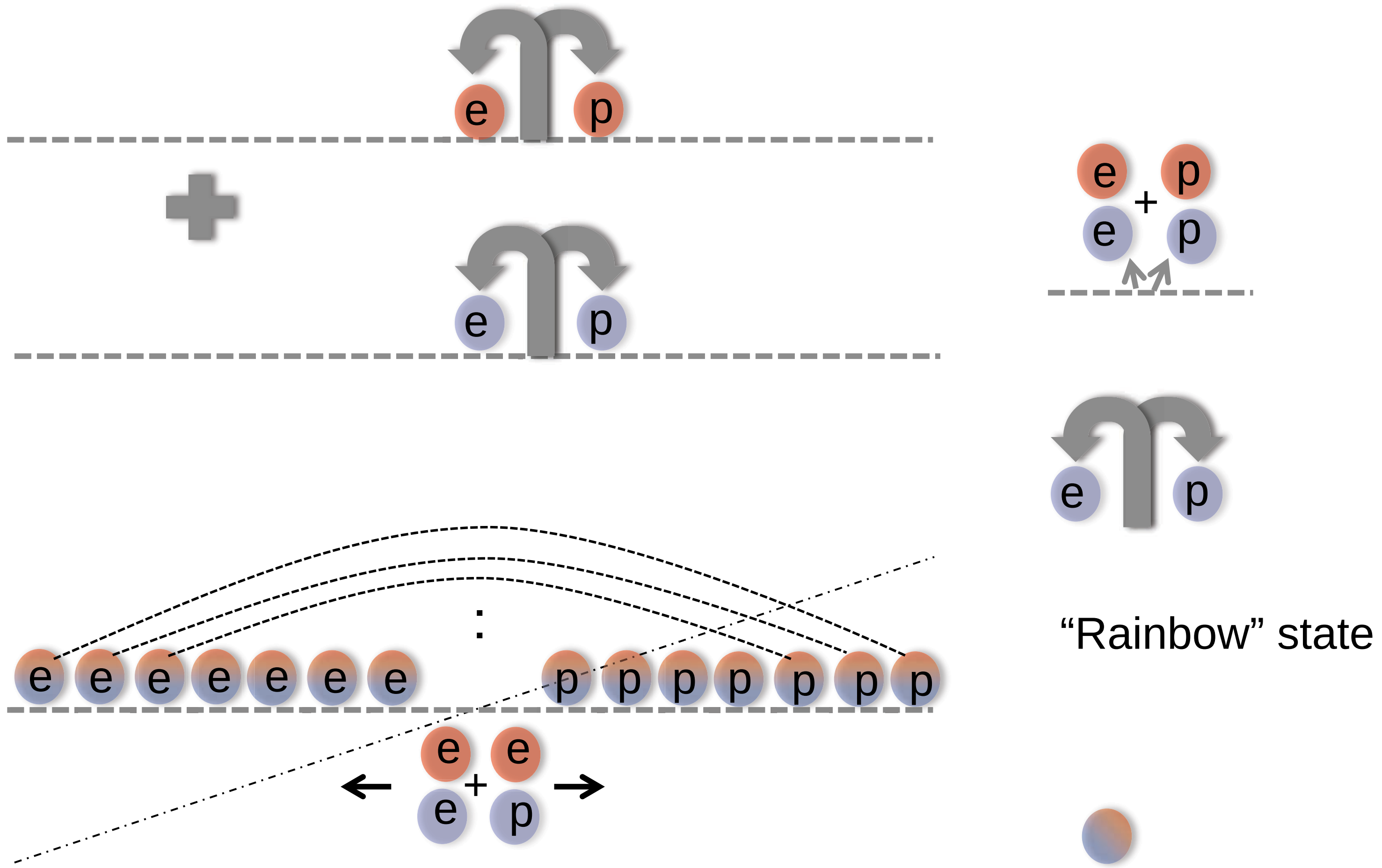
“Rainbow” state





Basic intuition: How to create a highly entangled state?

EPR: electron-positron pair generation in an electric field as a source of entanglement

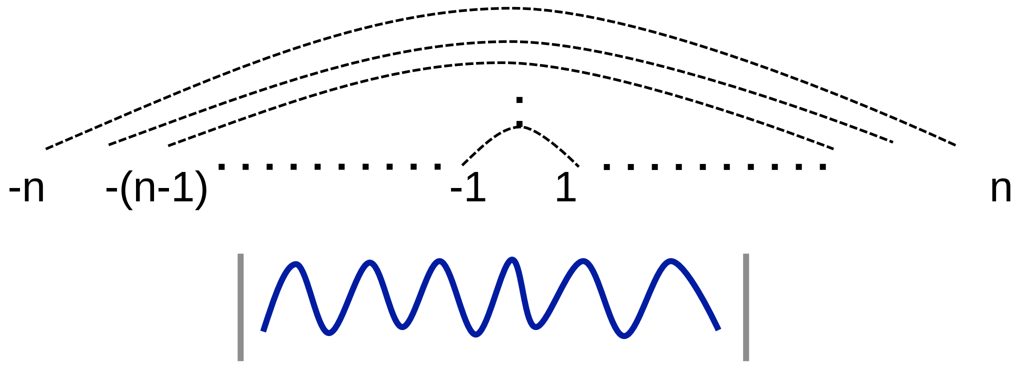


ANOTHER TYPE OF RAINBOW STATE IN THE LAB!

Pfister et al, 2004
Chen Meniccuci Pfister PRL2014, 60 mode cluster state

Optical frequency comb

Cavity eigenmodes



Nonlinear cavity

$$\omega_{in} \rightarrow \omega_n + \omega_{-n} = \omega_{in}$$

Incoming laser

CRITICALITY WITHOUT FRUSTRATION

Frustration free Hamiltonians:

$H = \sum H_i$, H_i are local non negative (i.e. $\langle f | H_i | f \rangle \geq 0$ for all f)

$H |\Psi\rangle = 0$ and $H_i |\Psi\rangle = 0$ are local

Examples:

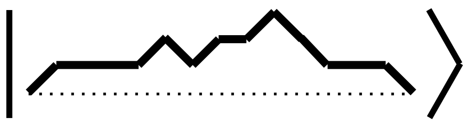
- Classical Hamiltonians such as Ising
- Toric Code
- AKLT model

Typically commuting and gapped.

MOTZKIN WALK HAMILTONIANS

Bravyi et al. 2012 “Criticality without frustration”

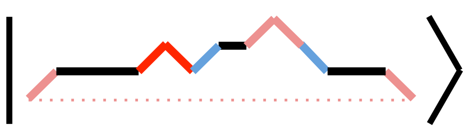
$$|\Psi\rangle = \sum_{\substack{\text{Motzkin}\\ \text{paths}}}$$



$$S_n \propto \frac{1}{2}\log(n)$$

Movassagh and Shor 2014 “Power law violation of the area law in quantum spin chains”

$$|\Psi\rangle = \sum_{\substack{\text{colored}\\ \text{Motzkin}\\ \text{paths}}}$$



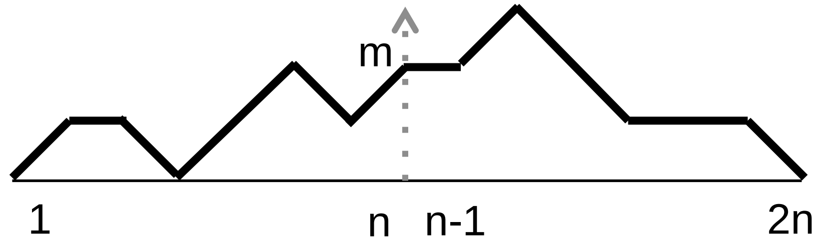
$$S_n \propto \sqrt{n}$$

REPRESENTING SPIN STATES AS MOTZKIN WALKS

Motzkin paths:

|1 , 0 , -1 , 1 , 1 , -1 , 1 , 0 , 1 , -1 , -1 , 0 , 0 , -1 >

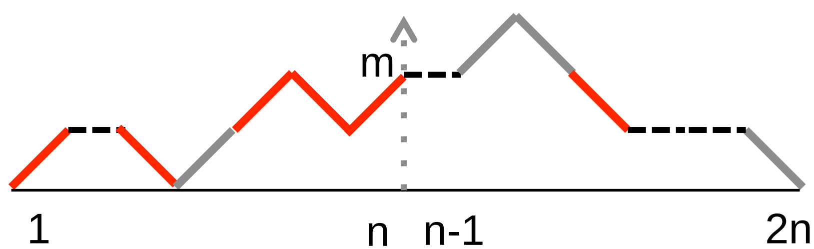
(-) (( ) (- ( )) - -)



Colored Motzkin paths:

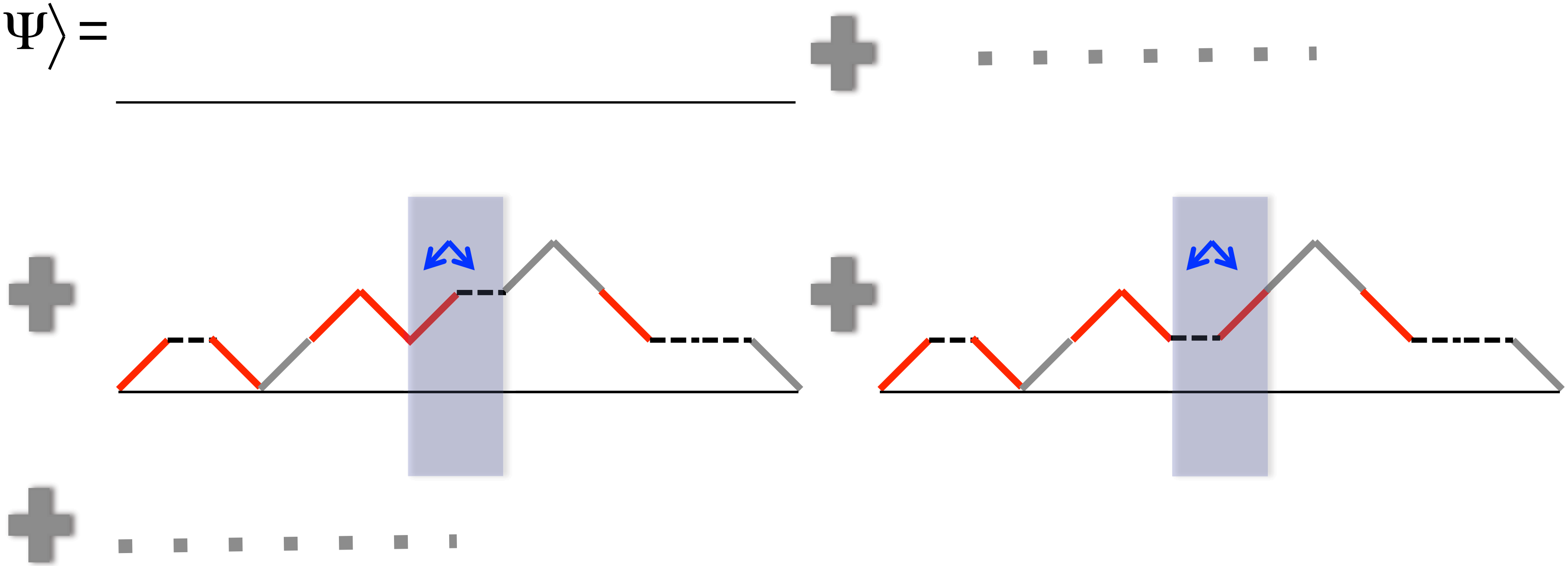
|1 , 0, -1 , 2 , 1 ,-1 , 1 , 0, 2 , -2, -1 , 0, 0, -1>

(-) [( ) (- [ ]) - -]

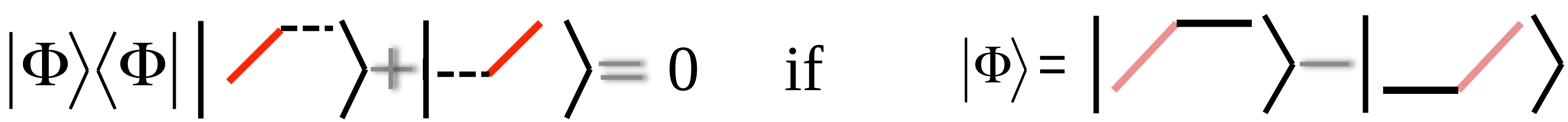


MOTZKIN HAMILTONIANS

$|\Psi\rangle =$

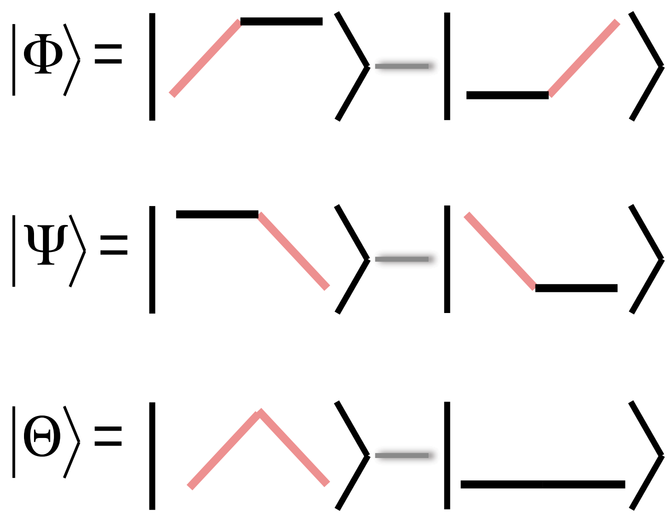


Basic idea - locally:



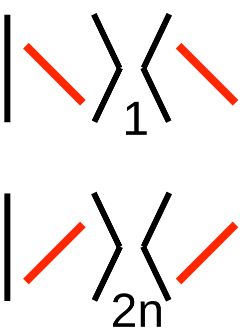
MOTZKIN HAMILTONIANS

Enforce a ground state superposition made of Motzkin paths by using projectors like:



$$H = \sum |\Theta\rangle\langle\Theta| + |\Psi\rangle\langle\Psi| + |\Phi\rangle\langle\Phi| + h_1 + h_{2n} + (\textit{penalty unmatched colors})$$

Boundary terms:



HOW COLOR ENHANCES ENTROPY

$\text{H}eight\ \ after\ \ n\ \ steps = \#\ of\ \ unmatched\ up\ steps$

$For\ \ n >> 1,\ typical\ Motzkin\ walk\ is\ like\ a\ Brownian\ walk.$

$\Rightarrow$

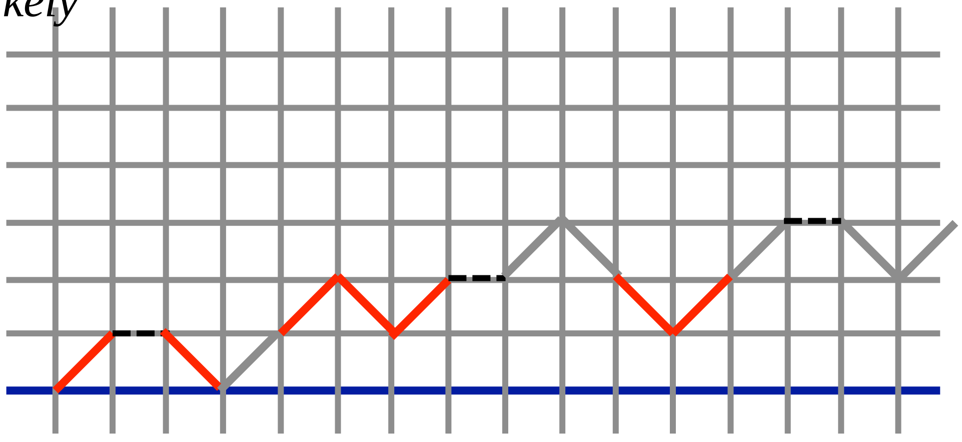
$Typical\ \ height\ \ after\ \ n\ \ steps \propto \sqrt{n}$

$\Rightarrow$

$\#\ of\ \ \text{colorings of}\ unmatched\ up\ steps \propto s^{\sqrt{n}}$

$all\ \text{coloring schemes of}\ unmatched\ equally\ \ likely$

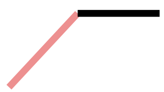
$\Rightarrow\ \ S_n \propto \sqrt{n}$

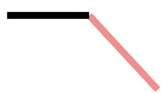
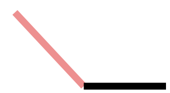


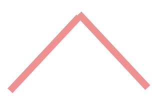
CAN WE SKEW THE MODEL TO PREFER RAINBOW STATES?

Main idea – up moves are like electrons and down moves are like positrons. They should go in different directions!

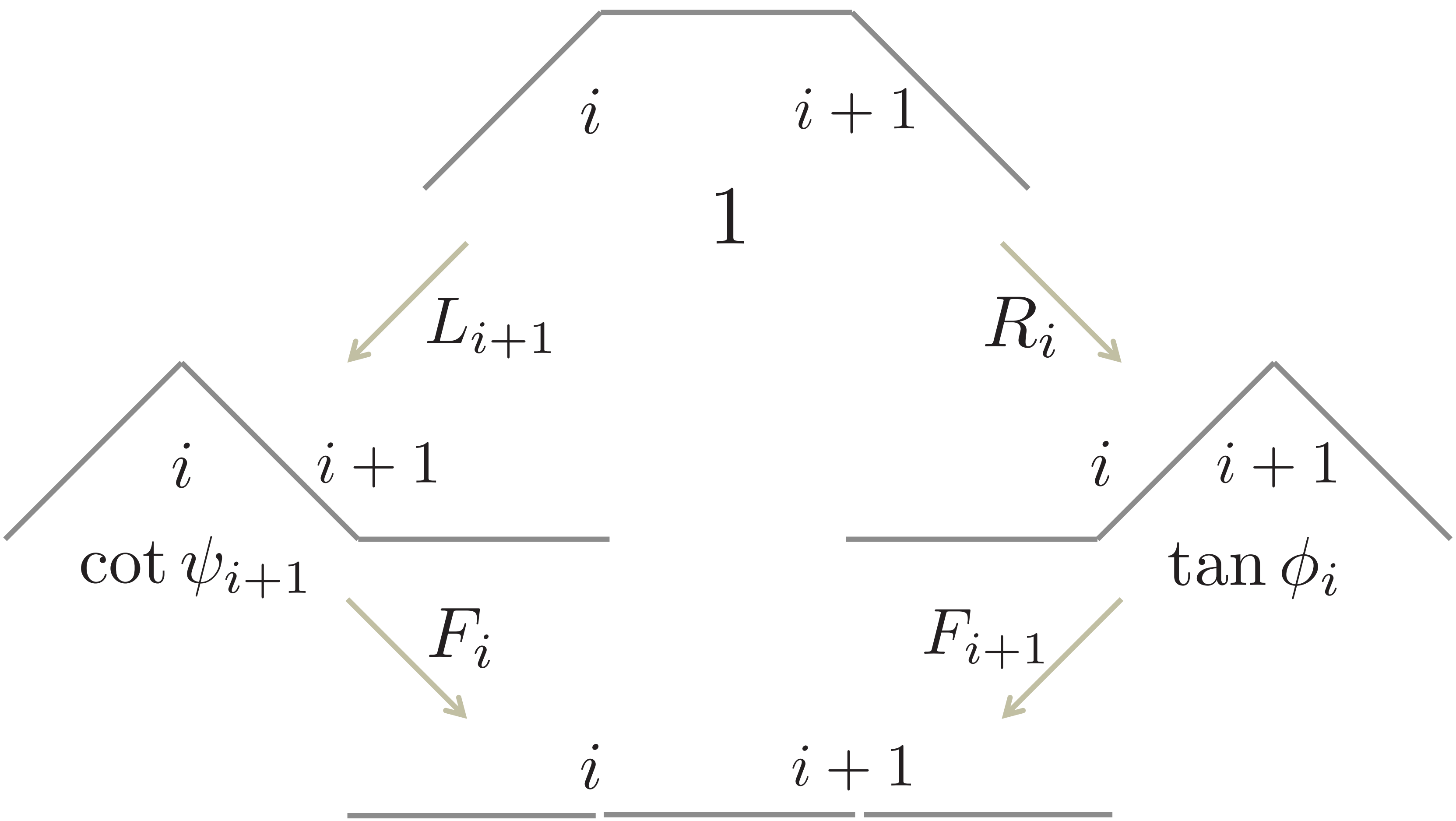
Can try:

$|\Phi\rangle = \cos\varphi_i$ |  $\rangle - \sin\varphi_i$ |  $\rangle$

$|\Psi\rangle = \cos\psi_i$ |  $\rangle - \sin\psi_i$ |  $\rangle$

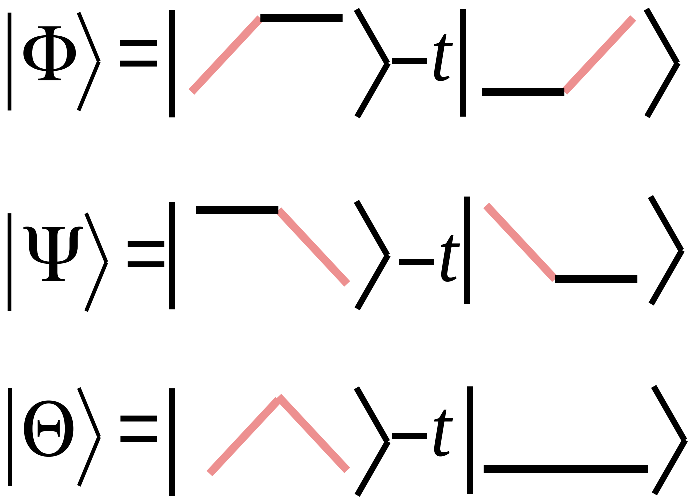
$|\Theta\rangle = \cos\theta_i$ |  $\rangle - \sin\theta_i$ | $\rangle$

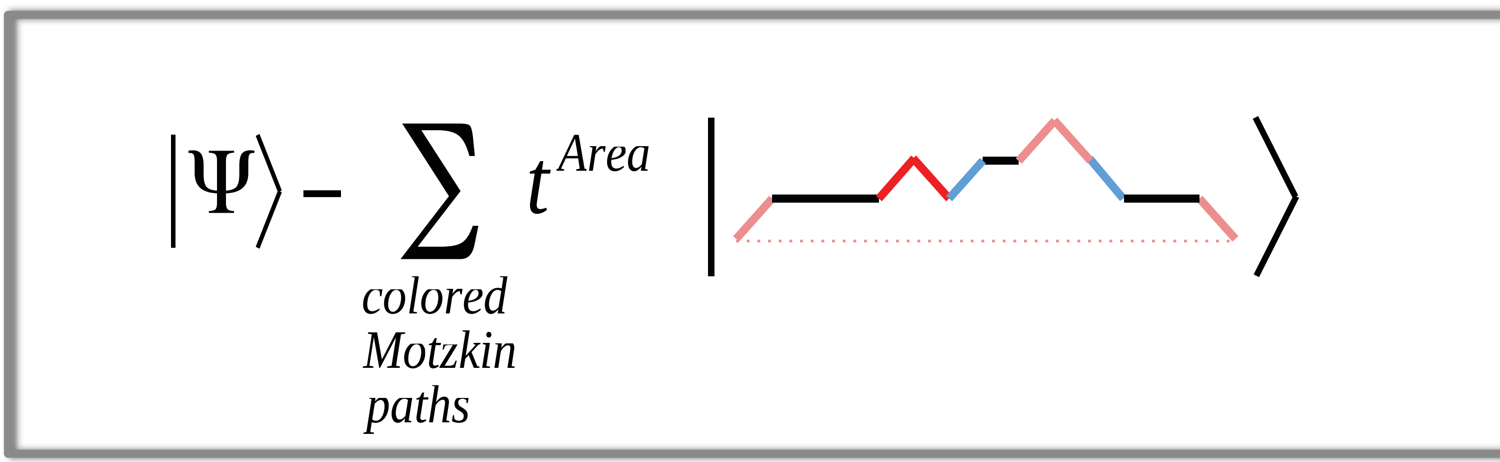
Choice of angles must satisfy a consistency condition:



$$\cot\psi_{i+1}\tan\theta_i \equiv \tan\phi_i \tan\theta_{i+1}$$

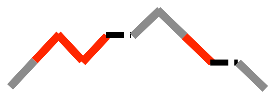
THE UNIFORM MODEL

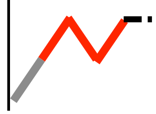
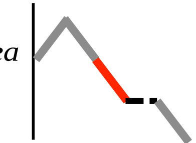




ENTANGLEMENT ENTROPY

Schmidt decomposition

$$|\Psi\rangle = \sum_{\substack{\text{colored}\\ \text{Motzkin}\\ \text{paths}}} t^{Area} \Big|$$  $$\Big\rangle$$

$$|\Psi\rangle \to \sum_{m=0}^{n} \sqrt{p_{n,m}} \sum_{\substack{\text{coloring}\\ \text{scheme}}} \left(\sum_{\substack{\text{paths from 0}\\ \text{to height } m}} t^{Area} \Big|$$  $$\Big\rangle \right) \otimes \left(\sum_{\substack{\text{paths from}\\ \text{height } m \text{ to } 0}} t^{Area} \Big|$$  $$\Big\rangle \right)$$

$$p_{n,m} = \frac{M_{n,m}^2}{N_n}$$

$$M_{n,m} = \sum_{i=0}^{(n-m)/2} s^i \sum_{\substack{\text{path from 0 to}\\ \text{height } m \text{ with}\\ i \text{ unpaired colors}}} t^{\text{Area under path}}$$

$$N_n = \sum_{m=0}^{n} s^m M_{n,m}^2$$

SCALING OF ENTROPY.

$$S = -\sum s^m p_{n,m} \log p_{n,m}$$

We need the asymptotics of $M_{n,m}$

$$M_{n,m} = \sum_{i=0}^{(n-m)/2} s^i \sum_{\substack{path\ from\ 0\ to \\ height\ m\ with \\ i\ unpaired\ colors}} t^{Area\ under\ path}$$

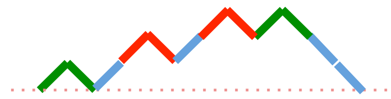
$$\sum_{\substack{path\ from\ 0\ to \\ height\ m\ with}} t^{Area\ under\ path} \approx \int_{X(0)=0}^{X(n)=m} dX[\tau]\ e^{-\int_0^n (\frac{dX}{ds})^2 - \log(t) X(s) ds}$$

Charged particle in a field,
Brownian particle with a drift

FREDKIN CHAIN

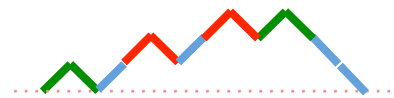
The Fredkin model of Salberger/Korepin 2016 has as ground state superposition of Dyck paths:

$$|\Psi\rangle = \sum_{\substack{colored\\ Dyck\\ paths}} \Big| \quad \Big\rangle$$



We deform it into:

$$|\Psi\rangle = \sum_{\substack{colored\\ Dyck\\ paths}} t^{Area\ \ under} \Big| \quad \Big\rangle$$



Entropy scales linearly with n log(s)! Same phase diagram.

Model has 3-nearest neighbor interactions.

O. Salberger, T. Udagawa, Z. Zhang, H. Katsura, **IK** and V. Korepin, JSTAT (2017)

EXCITATION GAP

$$uncolored\ \ Motzkin \quad S \sim \frac{1}{2}\log(n) \quad \Delta \leq n^{-c},\ \ c \sim 2+$$

$$t = 1,\ \ Motzkin, \quad S \sim \sqrt{n}\log(s) \qquad n^{-c} \leq \Delta \leq n^{-2}\,(c >> 1)$$

$$t = 1,\ \ Fredkin$$

$$Here:$$

$$t > 1,\ \ Motzkin, \quad S \sim n\,l\,og(s) \qquad \Delta \leq 8nst^{-n^2/3}$$

Levine and Movassagh, JphysA 2017

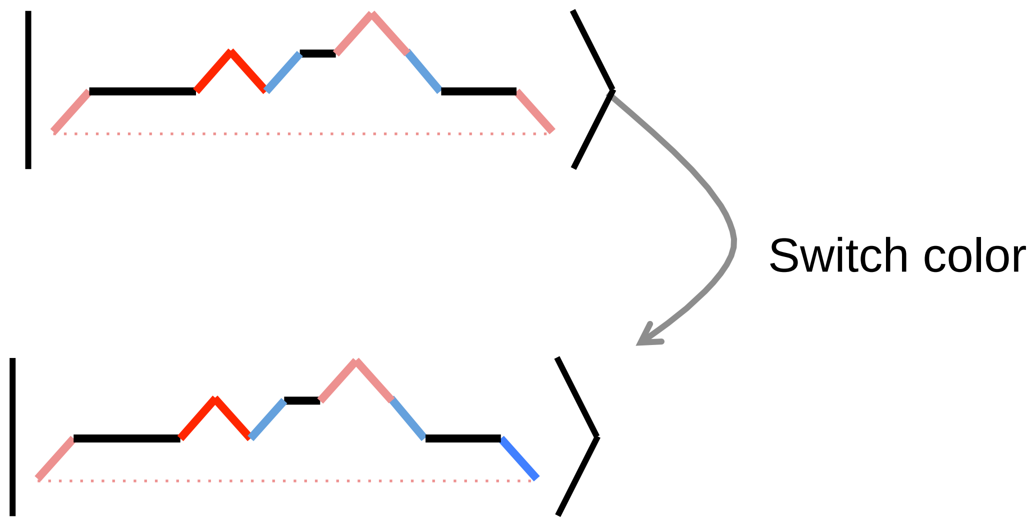
Beautiful proof uses mapping to Markov Chains and Cheeger Inequality

Alternative approach →

VARIATIONAL PROOF FOR GAP SCALING:

$$|\Psi\rangle = \sum t^{Area}$$

$$\sigma|\Psi\rangle = \sum t^{Area}$$



Result is orthogonal to g.s. Energy exponentially small with t.

More sophisticated: flip the color of the last down making the larges interval gives t^-n^2/2 gap.

EXCITATION GAP IN COLORLESS MODEL

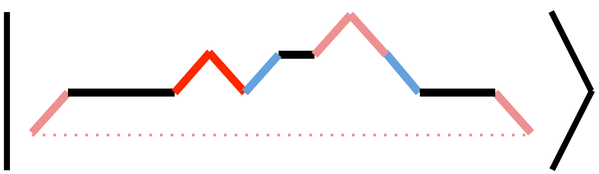
Colorless model:

Gap for t<1

Gapless for t>1 (although entropy obeys area law!)

Variational approach:

$$|\Psi\rangle = \sum t^{Area}$$



Z Zhang and IK, JPA2017

TENSOR NETWORK FOR AREA-LAW STATES

Matrix Product States are a useful description for chains with area law.
Take D matrices A:

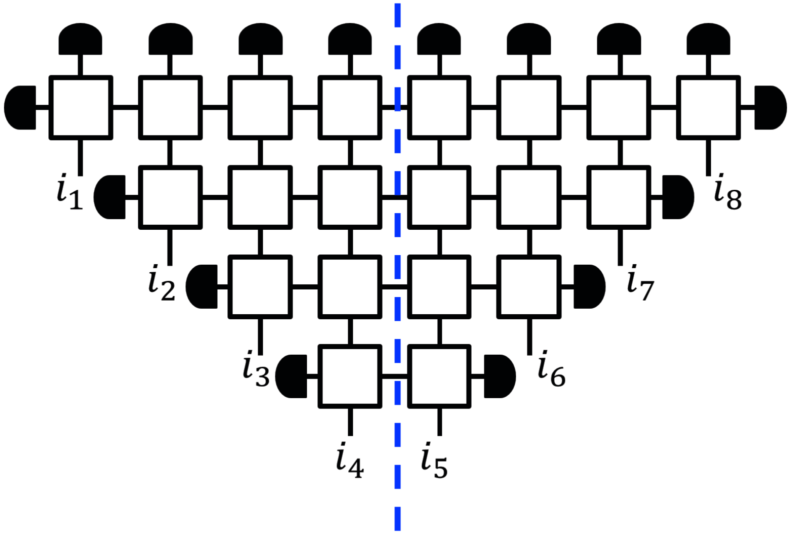
$$\left|\Psi\right\rangle = \sum C(\sigma_1 ... \sigma_N)\left|\sigma_1 ... \sigma_N\right\rangle \qquad \sigma_1 \in \{1, 2, ..D\}$$

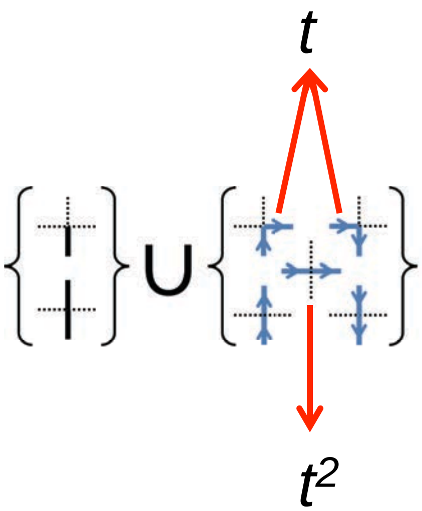
$$C(\sigma_1\sigma_2...\sigma_N) = A^{\sigma_1}_{s_1 s_2} A^{\sigma_1}_{s_2 s_3} A^{\sigma_1}_{s_3 s_4} ... A^{\sigma_1}_{s_N s_{N+1}} = \left\langle s_1 \right| A_{\sigma_1} A_{\sigma_2} ... A_{\sigma_N} \left| s_{N+1} \right\rangle$$

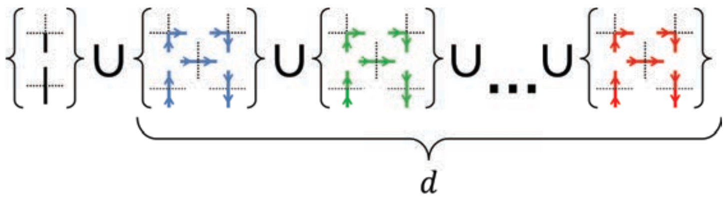
Tensor network description:

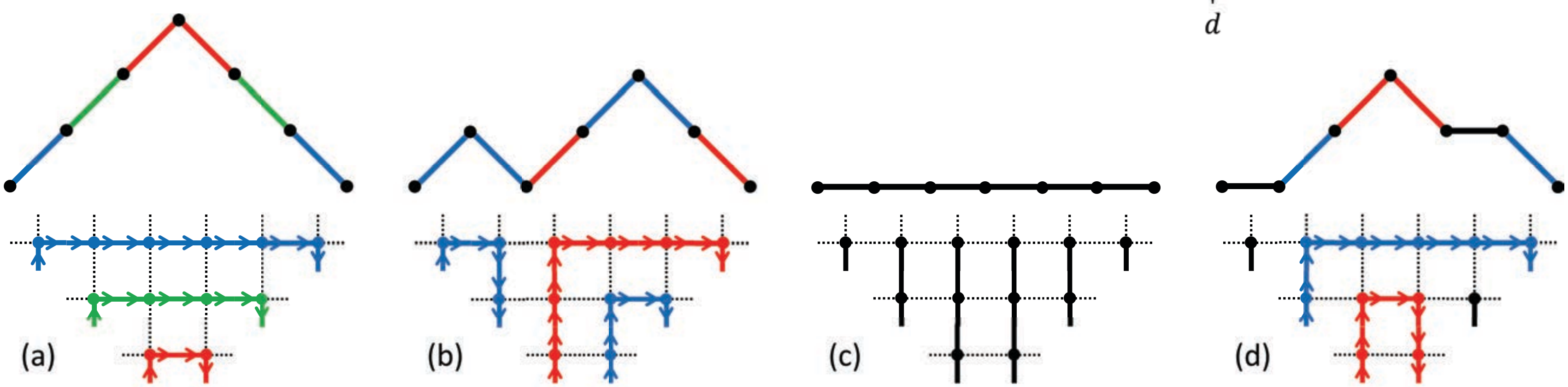
Entanglement obtained by cutting a bond. It is bounded by log (dimension A).

EXACT HOLOGRAPHIC TENSOR NETWORK



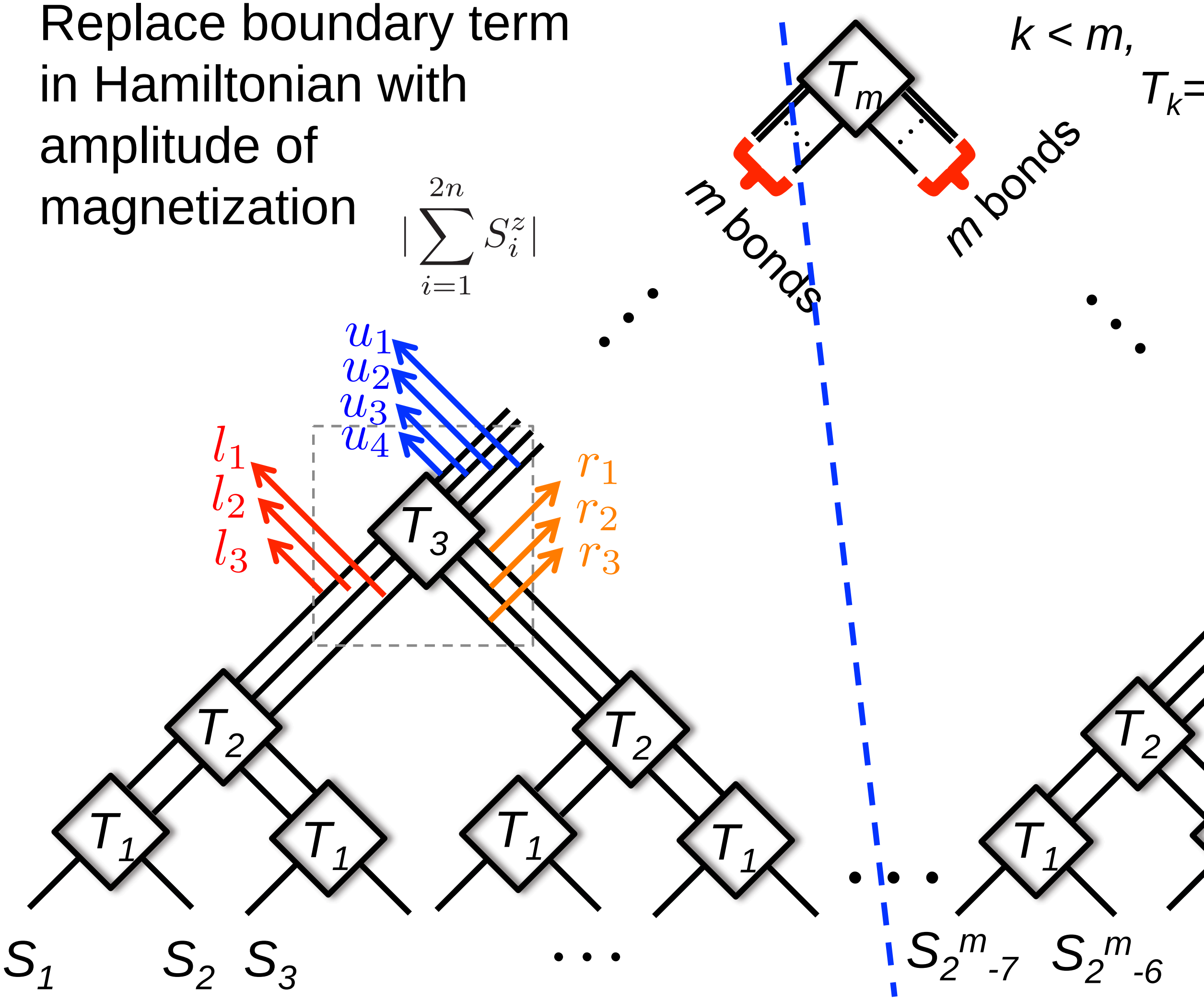






THE TN IS NOT OPTIMAL FOR T=1. CAN WE DO BETTER?

Replace boundary term in Hamiltonian with amplitude of magnetization $|\sum_{i=1}^{2n} S_i^z|$

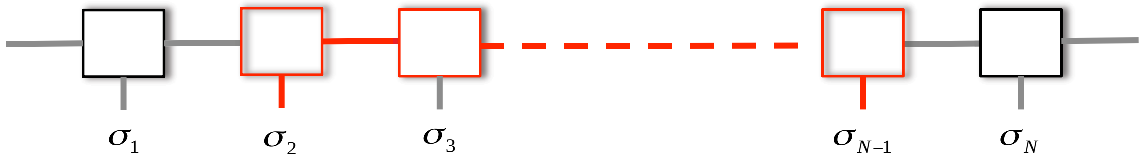


$k < m$, $T_k = \begin{cases} 1, \text{ if } \sum_{d=1}^{k} 2^{d-1}(l_d + r_d) = \sum_{d=1}^{k+1} 2^{d-1} u_d \\ 0, \text{ else} \end{cases}$

$$T_m = \prod_{d=1}^{m} \delta(l_d, r_d)$$

$$S_i = \begin{cases} 0, \text{ for } \downarrow \\ 1, \text{ for } \uparrow \end{cases}$$

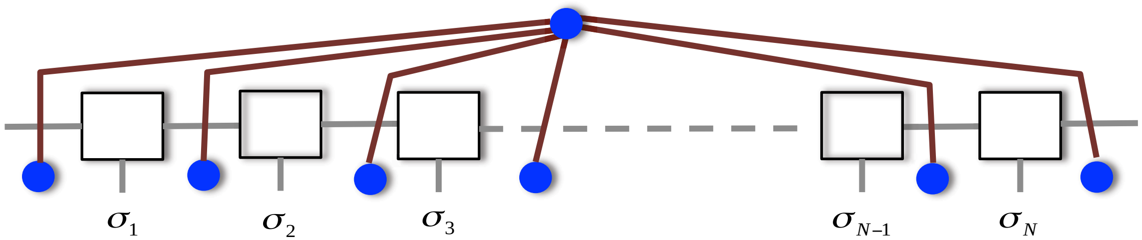
Remark about holographic metric

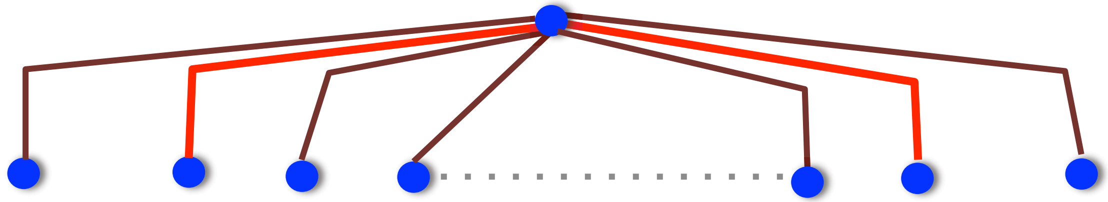


Exponential decay of correlations
$<\sigma_x\sigma_y>\sim\exp(-a|y-x|)$

Consistent with graph distance $D(x,y)=|y-x|$

But entanglement is bound:





$D_{dual}(x,y)=2$

S=const