Symmetric Topological Phases

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Plan:

- Today:

The introduction of symmetry fractionalization:

(1) AKLT chain

(2) Generalized symmetry fractionalizations for:

topological defects (dislocations in topological insulators)

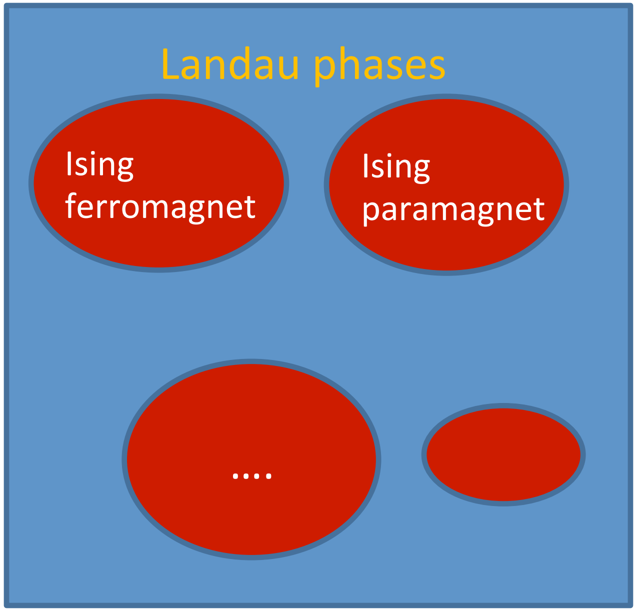
topological excitations in topologically ordered phases.

- Tomorrow:

(1) Quantum spin liquid phases in frustrated magnets, and related experiments in materials

(2) Parton constructions of quantum spin liquids, and symmetry fractionalization

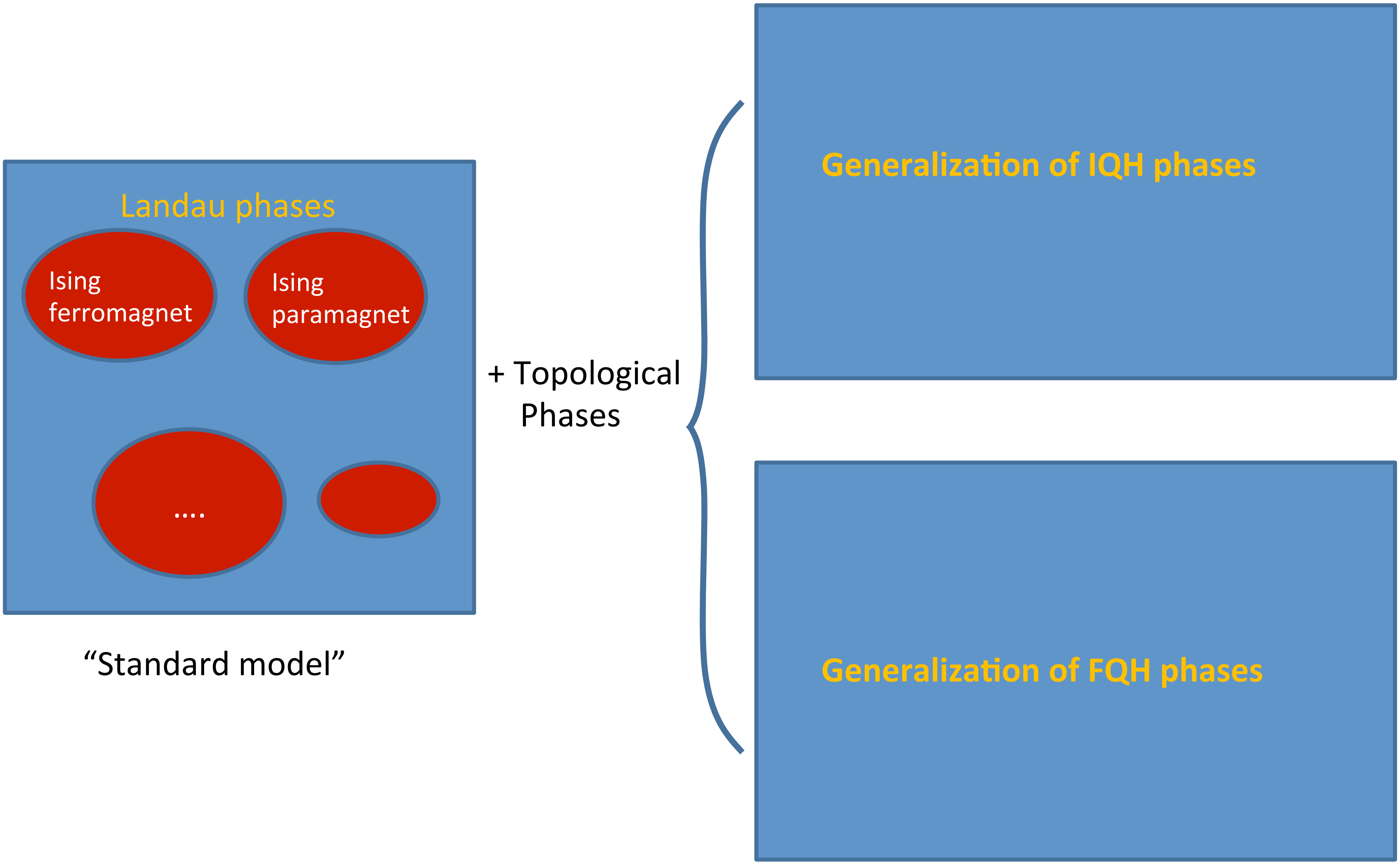
The modern view of gapped quantum phases



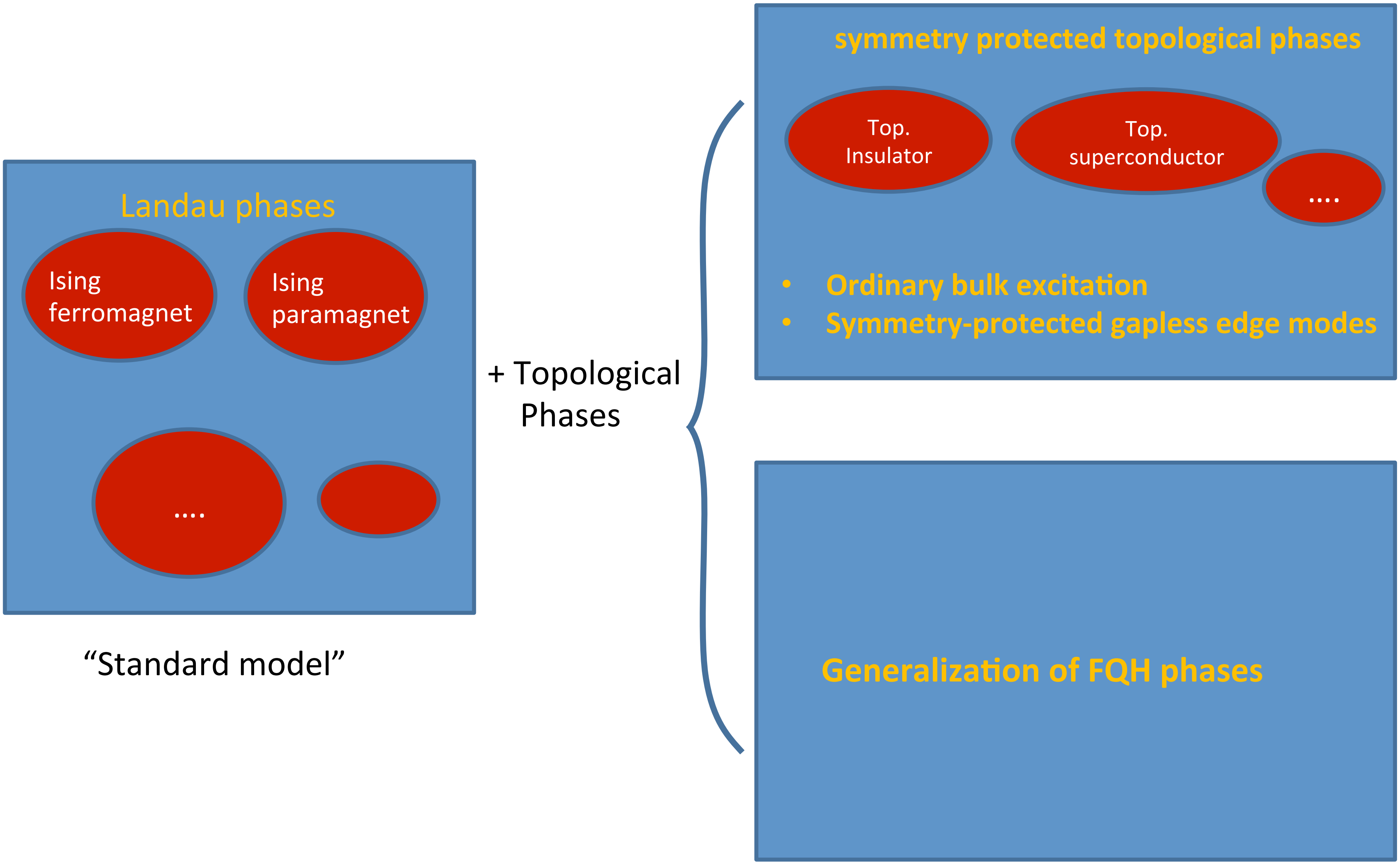
“Standard model”

+ Topological Phases

The modern view of gapped quantum phases



The modern view of gapped quantum phases



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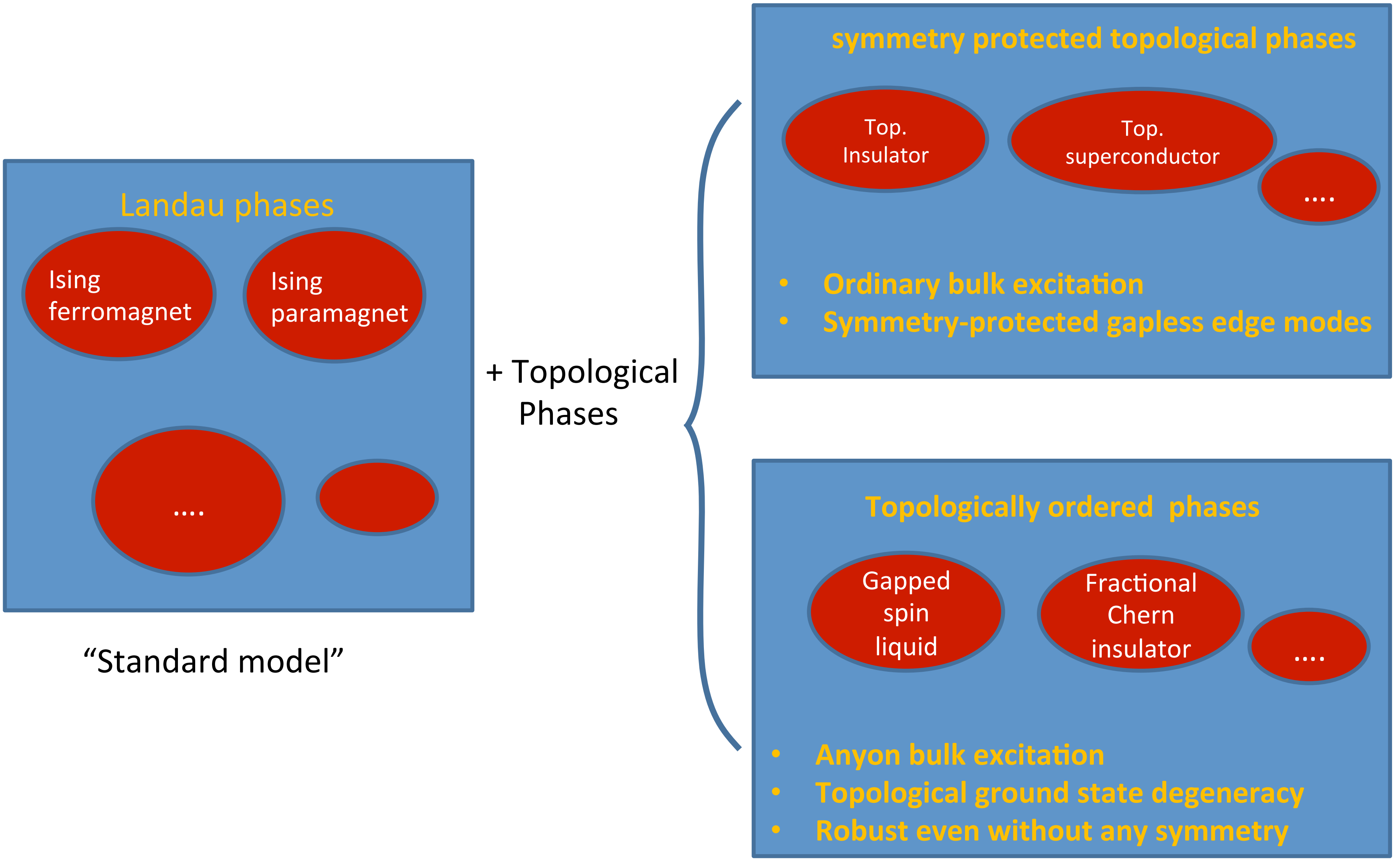
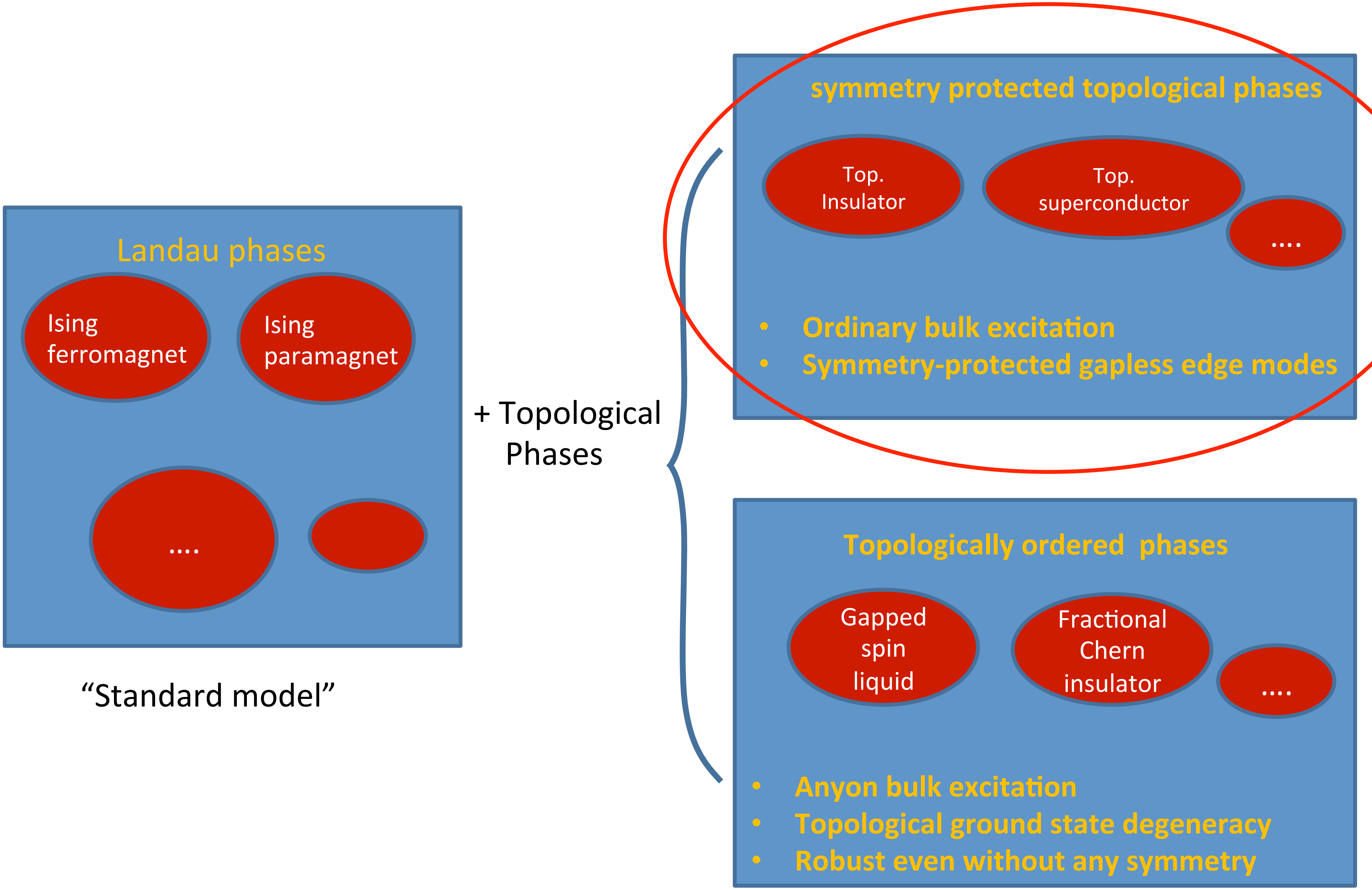
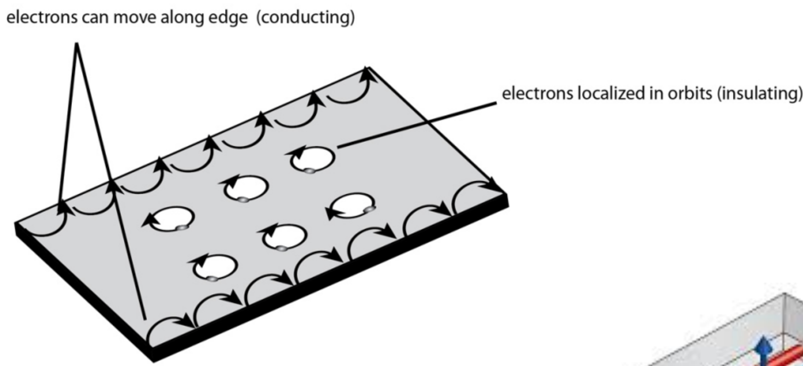


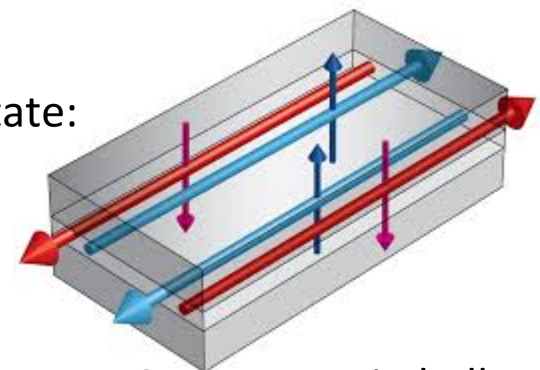
The modern view of gapped quantum phases



Symmetry protected topological phases



Integer quantum hall state:
Chiral edge state



Quantum spin hall state:
helical edge state

symmetry protected topological phases

Top. Insulator

Top. superconductor

....

- **Ordinary bulk excitation**
- **Symmetry-protected gapless edge modes**

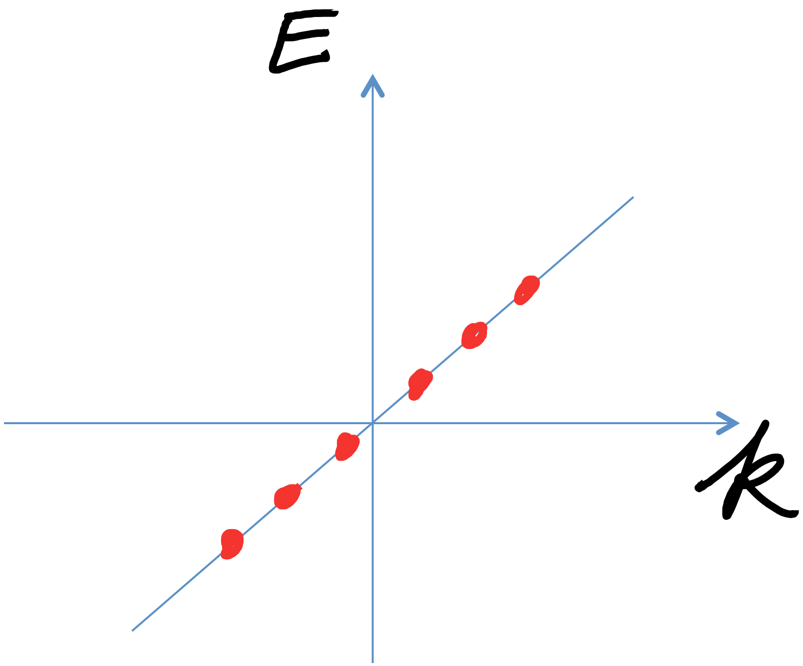
- These characteristic gapless edge states are "anomalous". Namely, they can never be realized in a (d-1)-dimension system, assuming certain global symmetry is respected.

Example: Anomalous edge states

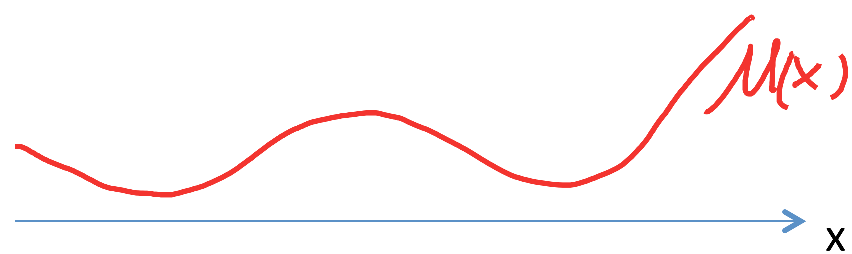
- Chiral edge state of integer quantum hall liquid cannot be realized in 1-spatial dimension:

Easy to show: $j = \frac{e}{h} \cdot \mu$

current Chemical potential



Namely, a position dependent chemical potential will break current conservation in 1D.



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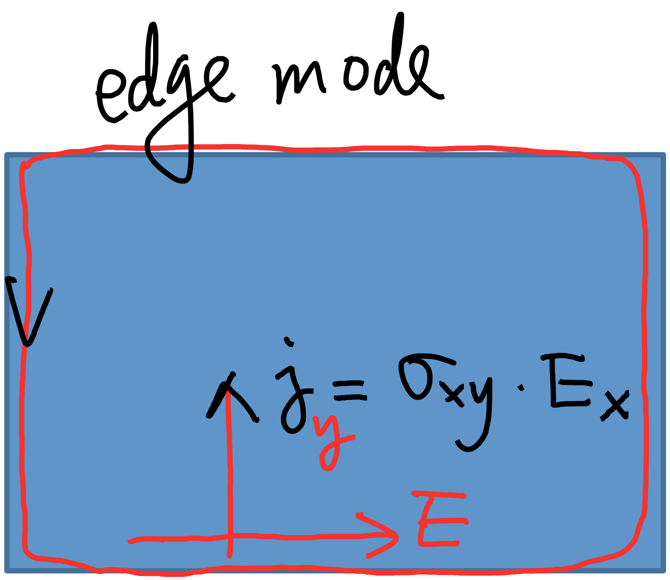
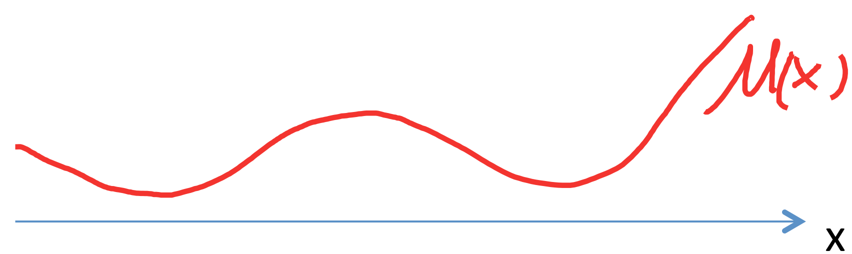
current　　　Chemical potential



This is just the hall effect at 2D boundary.

Namely, a position dependent chemical potential will break current conservation in 1D.



SPT phases --- Key feature:

- Gapped bulk. Conventional bulk excitations.
- Anomalous edge states that cannot be realized in local (d-1)-dimensional quantum systems (assuming certain global symmetries).
- In some sense, this is precisely why these d-dimenional topological phases are robust: One CANNOT think about the system as a trivial bulk glued with a (d-1)-dimensional gapless system.

Symmetry fractionalization in 1D SPT phases

- Next, I will present the first example of symmetry protected topological phases beyond quantum hall liquids

  --- the AKLT spin-1 chain model. Affleck, Kennedy, Lieb, Tasaki (PRL 1987)

- AKLT model has a gapped bulk, but gapless spin-1/2 edge states.

  --- Edge states are also anomalous. No way to realize in 0-d.

  symmetry become “fractionalized”.

- Confirmed in experiments: (e.g., in NENP spin-1 chain)

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Observation of $S=\frac{1}{2}$ Degrees of Freedom in an $S=1$ Linear-Chain Heisenberg Antiferromagnet

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Ian Affleck
Canadian Institute for Advanced Research and Physics Department, University of British Columbia, Vancouver, British Columbia, Canada V6T 2A6

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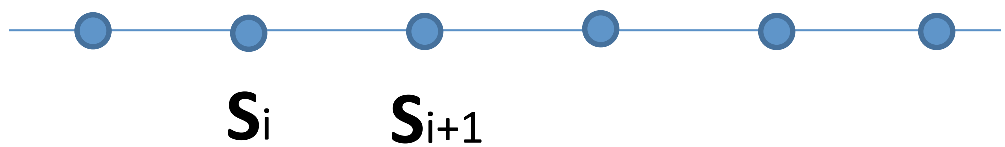
J. P. Renard
Institut d'Electronique Fondamentale, Bâtiment 220, Université Paris-Sud, 91405 Orsay CEDEX, France

(Received 31 July 1990)

Spin-1 chain: the AKLT model

- Consider an antiferromagetic spin-1 chain:



- Let's modify the usual Heisenberg model a little bit:

$$H = K\sum_i [\mathbf{S}_i \cdot \mathbf{S}_{i+1} + \beta(\mathbf{S}_i \cdot \mathbf{S}_{i+1})^2] \ (K > 0)$$

I will show:

when ¯=1/3, the model is (quasi-)exactly solvable,

with interesting ground state. Affleck, Kennedy, Lieb, Tasaki (PRL 1987)

Here "(quasi-)" means that one can solve the ground state(s) exactly, but not the excited states.

Spin-1 chain: the AKLT model

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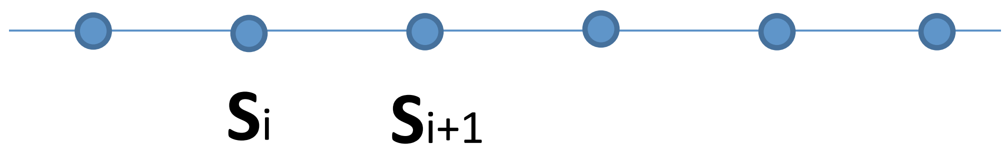

- Let's modify the usual Heisenberg model a little bit:

$$H = K\sum_i [\mathbf{S}_i \cdot \mathbf{S}_{i+1} + \beta(\mathbf{S}_i \cdot \mathbf{S}_{i+1})^2] \;\; (K > 0)$$

Note that unlike the spin-1/2 case , for spin-1, $(\mathbf{S}_i \cdot \mathbf{S}_j)^2$ is an independent operator.

Let $\vec{J} \equiv \vec{S}_i + \vec{S}_j$, $\vec{J}^2 = j(j+1) = \begin{cases} 0 \text{ or } 2 & \text{if } S=\frac{1}{2} \\ 0,2,6 & \text{if } S=1 \end{cases}$

$$\vec{S}_i \vec{S}_j = \frac{1}{2}(\vec{J}^2 - \vec{S}_i^2 - \vec{S}_j^2)$$

$$= \begin{cases} \frac{1}{2}(\vec{J}^2 - \frac{3}{2}) & \text{if } S=\frac{1}{2} \\ \frac{1}{2}(\vec{J}^2 - 4) & \text{if } S=1 \end{cases}$$

Spin-1 chain: the AKLT model

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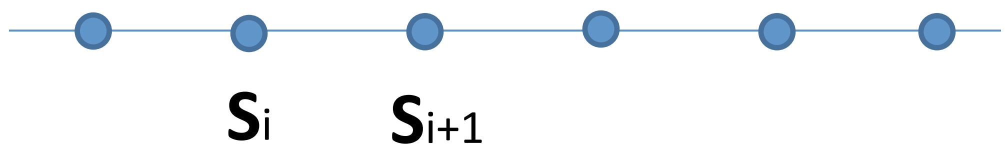

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$$H = K \sum_i [\mathbf{S}_i \cdot \mathbf{S}_{i+1} + \beta(\mathbf{S}_i \cdot \mathbf{S}_{i+1})^2]$$

for $s=1$: $\vec{J} = \vec{S}_i + \vec{S}_j$

	$j=0$	$j=1$	$j=2$	
$\vec{J}^2$	0	2	6	$\leftarrow j\cdot(j+1)$
$\vec{S}_i \cdot \vec{S}_j$	-2	-1	1	$\leftarrow \frac{1}{2}(\vec{J}^2 - 4)$
$(\vec{S}_i \cdot \vec{S}_j)^2$	4	1	1	
$\frac{1}{2}[\vec{S}_i \cdot \vec{S}_j + \frac{1}{3}(\vec{S}_i \cdot \vec{S}_j)^2 + \frac{2}{3}]$	0	0	1	

this is why $\beta = \frac{1}{3}$ is special!

projector into $j=2$

Spin-1 chain: the AKLT model

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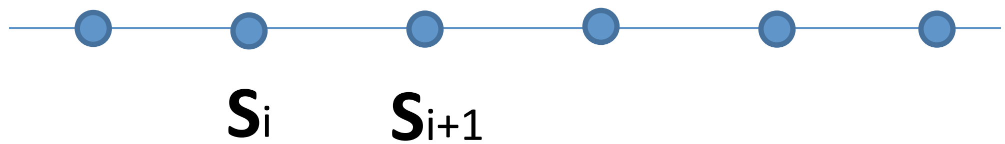

- Let's modify the usual Heisenberg model a little bit:

$$H = \frac{K}{2}\sum_i \delta[(\mathbf{s}_i + \mathbf{s}_{i+1}) = 2] + const$$

for $s=1$: $\vec{J} = \vec{S}_i + \vec{S}_j$

	$j=0$	$j=1$	$j=2$	
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Spin-1 chain: the AKLT model

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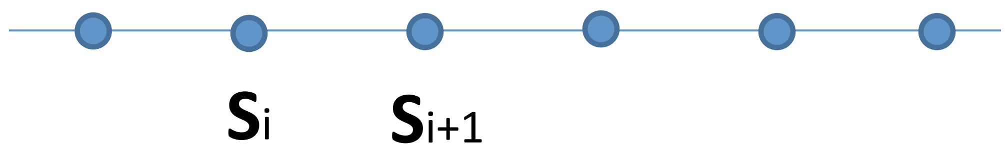


- Let's modify the usual Heisenberg model a little bit:

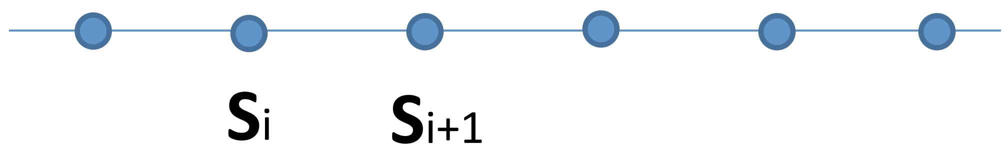
$$H = \frac{K}{2}\sum_i \delta[(\mathbf{s}_i + \mathbf{s}_{i+1}) = 2] + const$$

If we can find a quantum state |ª>, such that the combined spin of nearest neighbors can only be 0 or 1, then |ª> will certainly be one ground state.

Surprisingly, it is quite easy to write down such a state |ª> .

Spin-1 chain: the AKLT model

- Consider an antiferromagetic spin-1 chain:

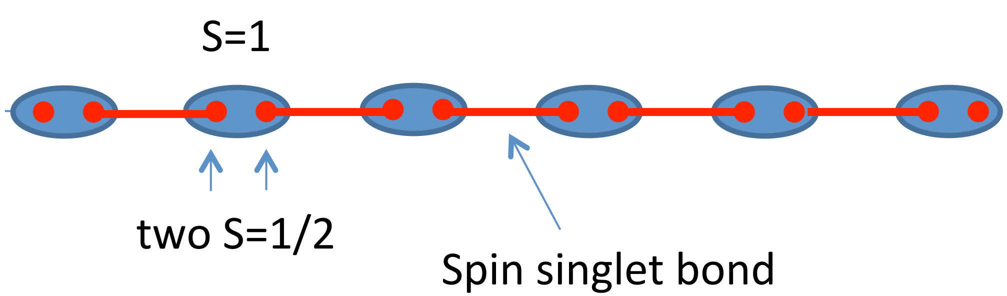


- Let's modify the usual Heisenberg model a little bit:

$$H = \frac{K}{2}\sum_i \delta[(\mathbf{s}_i + \mathbf{s}_{i+1}) = 2] + const$$

If we can find a quantum state $|^{a}>$, such that the combined spin of nearest neighbors can only be 0 or 1, then $|^{a}>$ will certainly be one ground state.

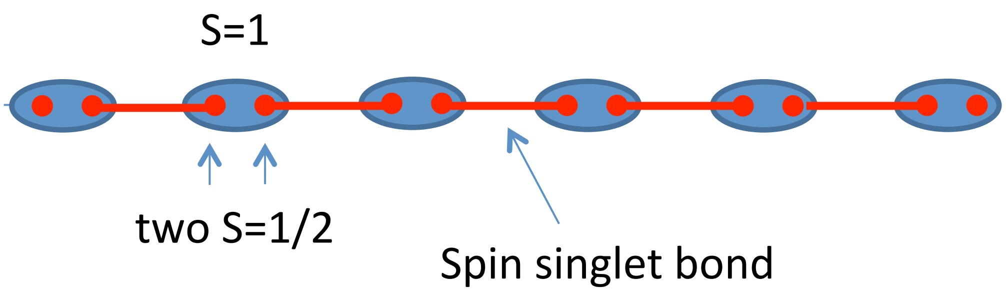
The idea to write down $|^{a}>$ is to split each spin-1 into two auxillary spin-1/2's:



Argument:
for every two n.n. sites,
2 of the 4 spin-1/2's form
singlet, the other two can
only form J=0 or 1.

AKLT model: the exact ground state(s)

$$H = \frac{K}{2}\sum_i \delta[(\mathbf{s}_i + \mathbf{s}_{i+1}) = 2] + const$$



Let's construct the ground state $|^{a}\rangle$ explicitly:

represent 3 states @ site-i as: using 2 spin-$\frac{1}{2}$

$$|\phi_{\uparrow\uparrow}\rangle = |\uparrow\rangle\otimes|\uparrow\rangle, \quad |\phi_{\downarrow\downarrow}\rangle = |\downarrow\rangle\otimes|\downarrow\rangle$$

$$|\phi_{\uparrow\downarrow}\rangle = |\phi_{\downarrow\uparrow}\rangle = \frac{1}{2}\left(|\uparrow\rangle\otimes|\downarrow\rangle + |\downarrow\rangle\otimes|\uparrow\rangle\right)$$

namely:

$$|\phi_{\uparrow\uparrow}\rangle = |m=+1\rangle$$

$$|\phi_{\downarrow\downarrow}\rangle = |m=-1\rangle$$

$$|\phi_{\uparrow\downarrow}\rangle = 1/\sqrt{2}\cdot|m=0\rangle.$$

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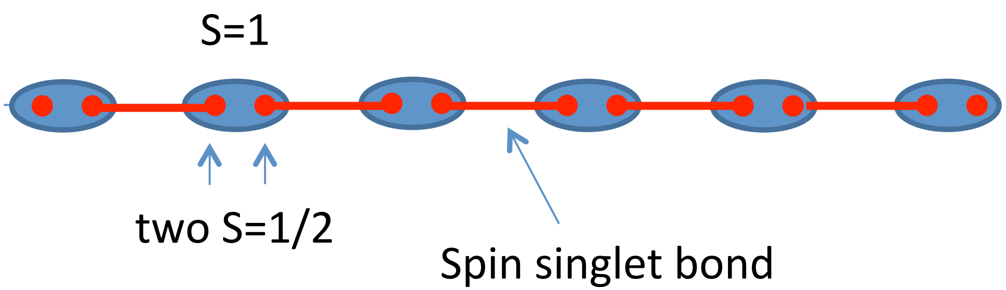

Let's construct the ground state $|{}^{a}\rangle$ explicitly:

represent 3 states @ site-i as: $|\phi_{\uparrow\uparrow}\rangle = |\uparrow\rangle \otimes |\uparrow\rangle$, $|\phi_{\downarrow\downarrow}\rangle = |\downarrow\rangle \otimes |\downarrow\rangle$
using 2 spin-$\frac{1}{2}$

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under spin-1 rotation:

$$\vec{S}|\phi_{\alpha\beta}\rangle = \left(\frac{\vec{\sigma}}{2}\right)_{\alpha'\alpha} |\phi_{\alpha'\beta}\rangle + \left(\frac{\vec{\sigma}}{2}\right)_{\beta'\beta} |\phi_{\alpha\beta'}\rangle$$

<u>exactly like 2 spin-$\frac{1}{2}$.</u>

AKLT model: the exact ground state(s)

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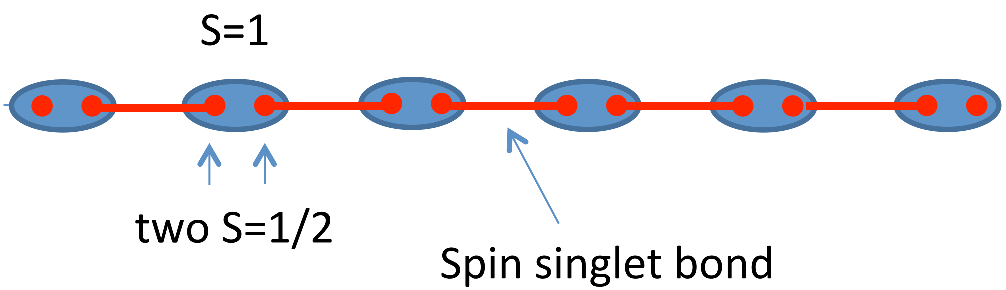

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$|\phi_{\uparrow\downarrow}\rangle = |\phi_{\downarrow\uparrow}\rangle = \frac{1}{2}(|\uparrow\rangle \otimes |\downarrow\rangle + |\downarrow\rangle \otimes |\uparrow\rangle)$

e.g. ① $S^z |\phi_{\alpha\beta}\rangle = (S^z_\alpha + S^z_\beta)|\phi_{\alpha\beta}\rangle$

② $S^+ |\phi_{\downarrow\downarrow}\rangle = S^+ |m=-1\rangle$

$= \sqrt{2}\,|m=0\rangle = 2\,|\phi_{\uparrow\downarrow}\rangle$

$= \left(\frac{\sigma^+}{2}\right)_{\alpha'\downarrow} |\phi_{\alpha'\downarrow}\rangle + \left(\frac{\sigma^+}{2}\right)_{\beta'\downarrow} |\phi_{\downarrow\beta'}\rangle$

under spin-1 rotation.

$\vec{S}|\phi_{\alpha\beta}\rangle = \left(\frac{\vec{\sigma}}{2}\right)_{\alpha'\alpha} |\phi_{\alpha'\beta}\rangle + \left(\frac{\vec{\sigma}}{2}\right)_{\beta'\beta} |\phi_{\alpha\beta'}\rangle$

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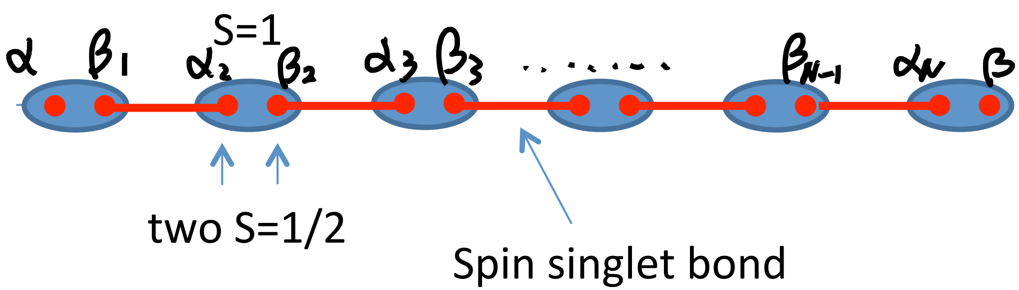


Let’s construct the ground state |a> explicitly:

consider two-site chain:

enforcing singlet bond

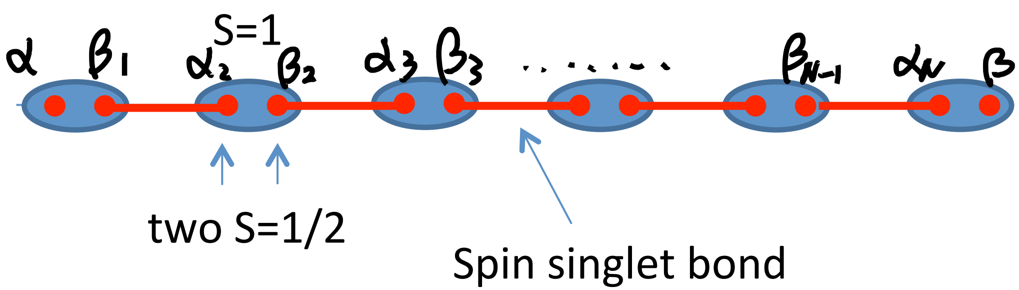
G.S.: $|\psi_{\alpha\beta}\rangle \equiv \sum_{\beta_1\alpha_2} |\phi^1_{\alpha\beta_1}\rangle \otimes |\phi^2_{\alpha_2\beta}\rangle \cdot \varepsilon_{\beta_1\alpha_2}$ $(\alpha,\beta = \uparrow,\downarrow)$

only form J=0 or 1

$\varepsilon_{\uparrow\downarrow} = -\varepsilon_{\downarrow\uparrow} = 1.$

AKLT model: the exact ground state(s)

$$H = \frac{K}{2}\sum_i \delta[(\mathbf{s}_i + \mathbf{s}_{i+1}) = 2] + const$$



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$\varepsilon_{\uparrow\downarrow} = -\varepsilon_{\downarrow\uparrow} = 1.$

proof:

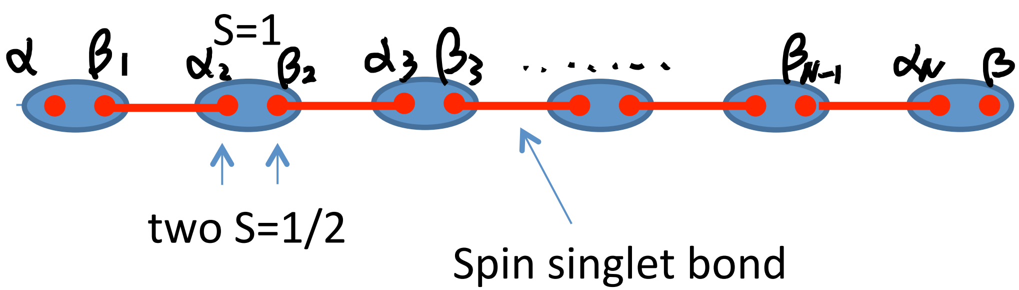
$$(\vec{S}_1+\vec{S}_2)|\psi_{\alpha\beta}\rangle = \left(\frac{\vec{\sigma}}{2}\right)_{\alpha'\alpha}|\psi_{\alpha'\beta}\rangle + \left(\frac{\vec{\sigma}}{2}\right)_{\beta'\beta}|\psi_{\alpha\beta'}\rangle$$

$$+ \left(\frac{\vec{\sigma}}{2}\right)_{\beta_1'\beta_1}|\phi^1_{\alpha\beta_1'}\rangle|\phi^2_{\alpha_2\beta}\rangle\varepsilon_{\beta_1\alpha_2} + \left(\frac{\vec{\sigma}}{2}\right)_{\alpha_2'\alpha_2}|\phi^1_{\alpha\beta_1}\rangle|\phi^2_{\alpha_2'\beta}\rangle\varepsilon_{\beta_1\alpha_2}$$

cancels.

AKLT model: the exact ground state(s)

$$H = \frac{K}{2}\sum_i \delta[(\mathbf{s}_i + \mathbf{s}_{i+1}) = 2] + const$$



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proof:

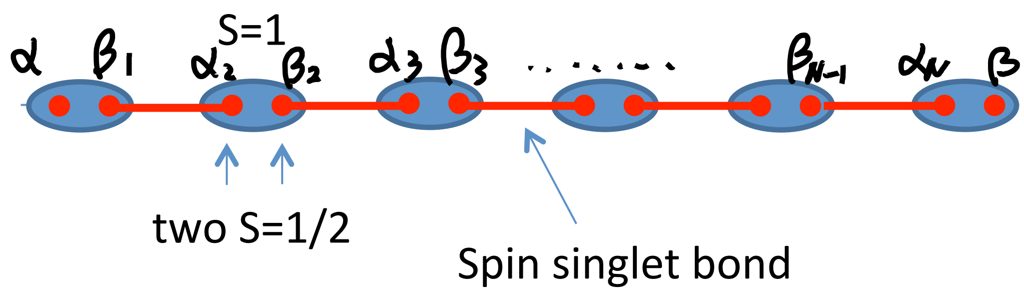
$$(\vec{S}_1+\vec{S}_2)|\psi_{\alpha\beta}\rangle = \left(\frac{\vec{\sigma}}{2}\right)_{\alpha'\alpha} |\psi_{\alpha'\beta}\rangle + \left(\frac{\vec{\sigma}}{2}\right)_{\beta'\beta} |\psi_{\alpha\beta'}\rangle$$

Be cautious: $|\psi_{\alpha\beta}\rangle$ are NOT orthonormal.

But enough to prove $\delta(S_1+S_2=2)=0$.

AKLT model: the exact ground state(s)

$$H = \frac{K}{2}\sum_i \delta[(\mathbf{s}_i + \mathbf{s}_{i+1}) = 2] + const$$



Let's construct the ground state $|^a>$ explicitly:

general G.S. (totally 4 of them)

$$|\psi_{\alpha\beta}\rangle \equiv \sum_{\beta_1,\alpha_2,\beta_2,\cdots,\alpha_N} \Big[|\phi^1_{\alpha\beta_1}\rangle \otimes |\phi^2_{\alpha_2\beta_2}\rangle \otimes |\phi^3_{\alpha_3\beta_3}\rangle \otimes \cdots \otimes |\phi^N_{\alpha_N\beta}\rangle \cdot \varepsilon_{\beta_1\alpha_2} \cdot \varepsilon_{\beta_2\alpha_3} \cdot \varepsilon_{\beta_3\alpha_4} \cdots \varepsilon_{\beta_{N-1}\alpha_N} \Big]$$

total spin rotation:

$$\Big(\sum_i \vec{S}_i\Big)|\psi_{\alpha\beta}\rangle = \Big(\frac{\vec{\sigma}}{2}\Big)_{\alpha'\alpha}|\psi_{\alpha'\beta}\rangle + \Big(\frac{\vec{\sigma}}{2}\Big)_{\beta'\beta}|\psi_{\alpha\beta'}\rangle$$

exactly like two spin-$\frac{1}{2}$ at ends.

Is the spin-1/2 fake or real?

$$\left(\sum_i \vec{S}_i\right)|\psi_{\alpha\beta}\rangle = \left(\frac{\vec{\sigma}}{2}\right)_{\alpha'\alpha}|\psi_{\alpha'\beta}\rangle + \left(\frac{\vec{\sigma}}{2}\right)_{\beta'\beta}|\psi_{\alpha\beta'}\rangle$$

Although formally looks like two spin-1/2 at the edges:

- Two things to worry about:

(1) $|\psi_{\alpha\beta}\rangle$ are not orthonormal.

(2) I have not constructed a local operator acting only on one edge that implements the spin-1/2 rotation.

Local spin rotations

- Two things to worry about:

(1) $|\psi_{\alpha\beta}\rangle$ are not orthonormal.

--- orthonormal up to exponentially small error as L increases

$$\langle\psi_{\uparrow\uparrow}|\psi_{\uparrow\uparrow}\rangle = \langle\psi_{\downarrow\downarrow}|\psi_{\downarrow\downarrow}\rangle$$
$$\simeq \langle\psi_{\uparrow\downarrow}|\psi_{\uparrow\downarrow}\rangle = \langle\psi_{\downarrow\uparrow}|\psi_{\downarrow\uparrow}\rangle \sim \left(\frac{3}{2}\right)^L$$

and $\langle\psi_{\uparrow\downarrow}|\psi_{\downarrow\uparrow}\rangle \sim \left(\frac{1}{2}\right)^L$. other overlaps are zero.

Local spin rotations

- Two things to worry about:

(1) $|\psi_{\alpha\beta}\rangle$ are not orthonormal.
--- orthonormal up to exponentially small error as L increases

(2) I have not constructed a local operator acting only on one edge that implements the spin-1/2 rotation.

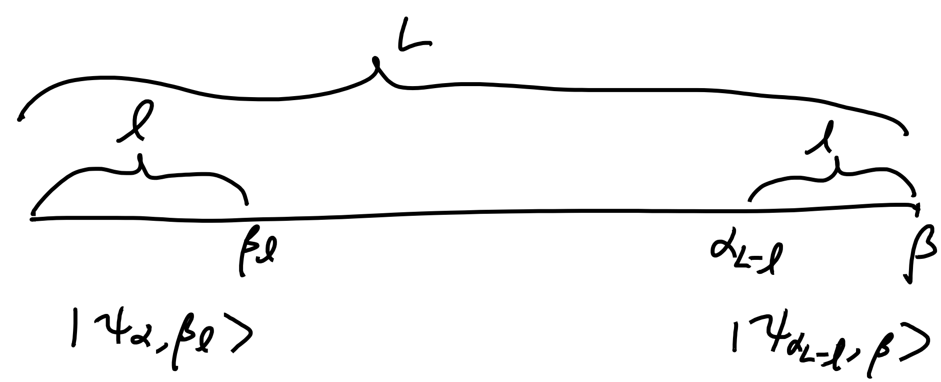
--- result in (1) allows us to construct such an operator:

Consider a long chain:

Define local unitary (almost) operators for the two edge segments:

$$\vec{S}_L |\psi_{\alpha,\beta_\ell}\rangle = \left(\frac{\vec{\sigma}}{2}\right)_{\alpha'\alpha} |\psi_{\alpha',\beta_\ell}\rangle$$

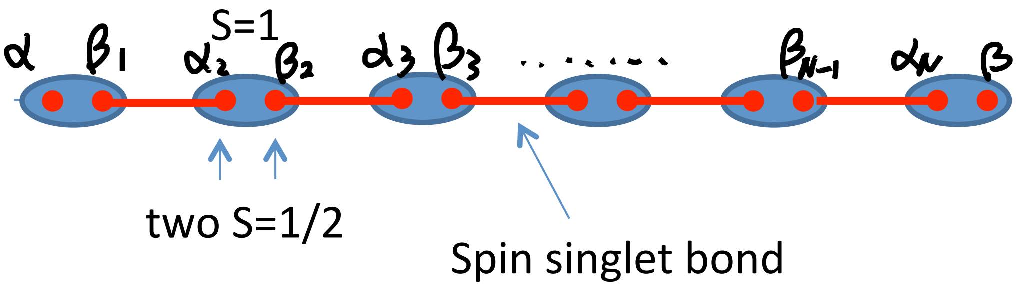
$$\vec{S}_R |\psi_{\alpha_{L-\ell},\beta}\rangle = \left(\frac{\vec{\sigma}}{2}\right)_{\beta'\beta} |\psi_{\alpha_{L-\ell},\beta'}\rangle$$



Key feature of symmetry fractionalization

We find:

$$H = \frac{K}{2}\sum_i \delta[(\mathbf{s}_i + \mathbf{s}_{i+1}) = 2] + const$$



total spin rotation:

$$\left(\sum_i \vec{S}_i\right)|\psi_{\alpha\beta}\rangle = \left(\frac{\vec{\sigma}}{2}\right)_{\alpha'\alpha}|\psi_{\alpha'\beta}\rangle + \left(\frac{\vec{\sigma}}{2}\right)_{\beta'\beta}|\psi_{\alpha\beta'}\rangle$$

$$= \vec{S}_L\,|\psi_{\alpha\beta}\rangle + \vec{S}_R\,|\psi_{\alpha\beta}\rangle.$$

local operator.

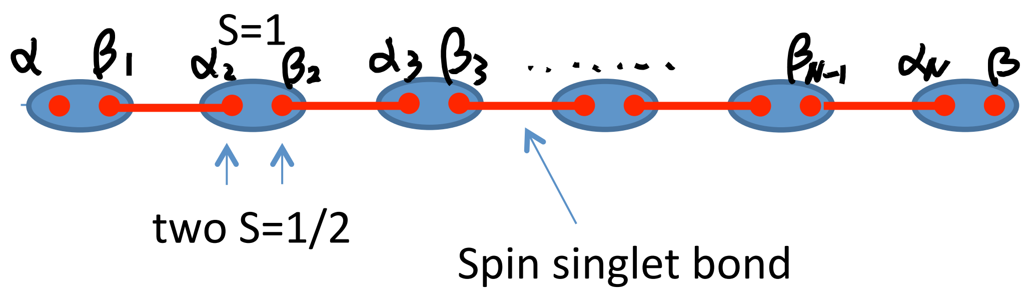
$$\hat{U}(\hat{n},\phi)\,|\psi_{\alpha\beta}\rangle = \hat{U}_L(\hat{n},\phi)\circ\hat{U}_R(\hat{n},\phi)\,|\psi_{\alpha\beta}\rangle$$

spin rotation.

Key feature of symmetry fractionalization

We find:

$$H = \frac{K}{2} \sum_i \delta[(\mathbf{s}_i + \mathbf{s}_{i+1}) = 2] + const$$



total spin rotation:

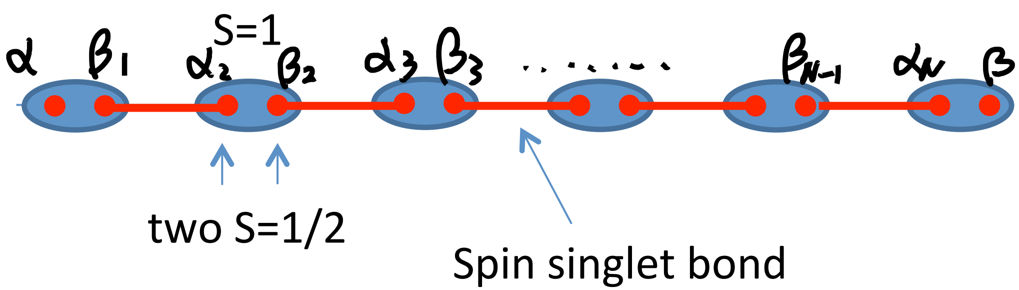
$$\left(\sum_i \vec{S}_i\right) |\psi_{\alpha\beta}\rangle = \left(\frac{\vec{\sigma}}{2}\right)_{\alpha'\alpha} |\psi_{\alpha'\beta}\rangle + \left(\frac{\vec{\sigma}}{2}\right)_{\beta'\beta} |\psi_{\alpha\beta'}\rangle$$

$$= \vec{S}_L |\psi_{\alpha\beta}\rangle + \vec{S}_R |\psi_{\alpha\beta}\rangle.$$

This is striking: we started from SO(3) spin-1 model, but we got spin-1/2 on edges.

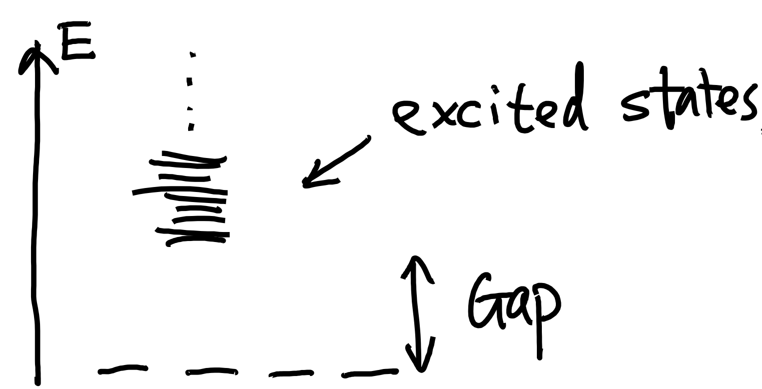
AKLT model: the exact ground state(s)

$$H = \frac{K}{2}\sum_i \delta[(\mathbf{s}_i + \mathbf{s}_{i+1}) = 2] + const$$



We find 4 ground states for open chain. The ground state degeneracy comes from the unpaired spin-1/2 on each ends of the chain.

One can further show that these are the only four ground states, and the many-body excitation spectrum of the chain has a FINITE energy gap:



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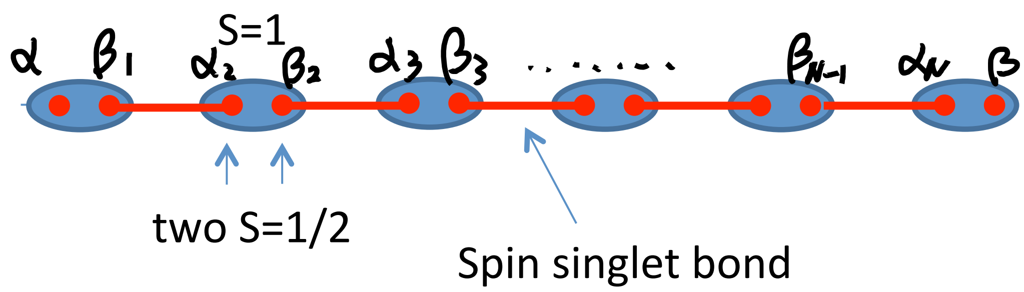
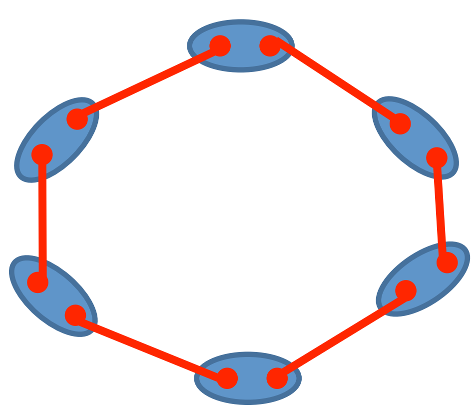
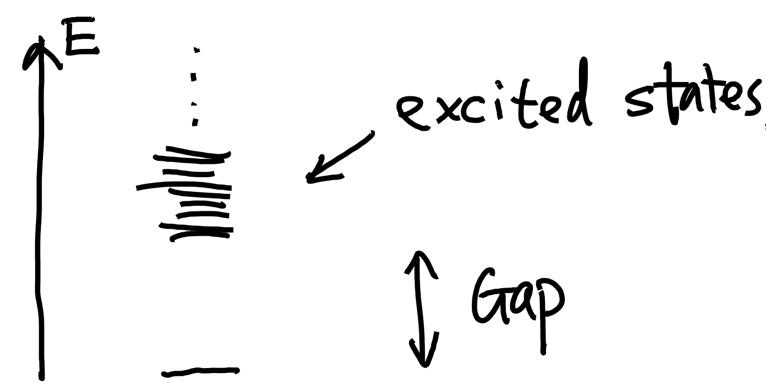


We find 4 ground states for open chain. The ground state degeneracy comes from the unpaired spin-1/2 on each ends of the chain!

However if the chain is a closed loop (periodic boundary condition), there is only a UNIQUE ground state:





The crucial feature of symmetry fractionalization

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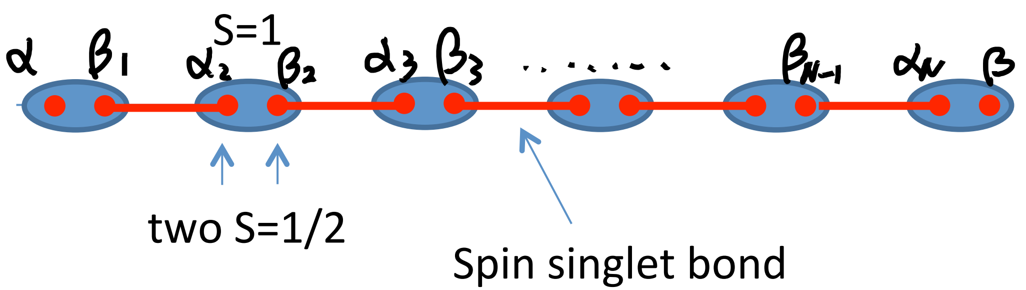

Symmetry fractionalization at chain edges --- Crucial feature

In a degenerate sector of energy eigenstates, global symmetry is implemented by product of two local operators spatially far away from each other:

$$\hat{U}(g)|\psi_a\rangle = \hat{U}_L(g)\cdot\hat{U}_R(g)|\psi_a\rangle$$

$g \in SG$ (symmetry group)

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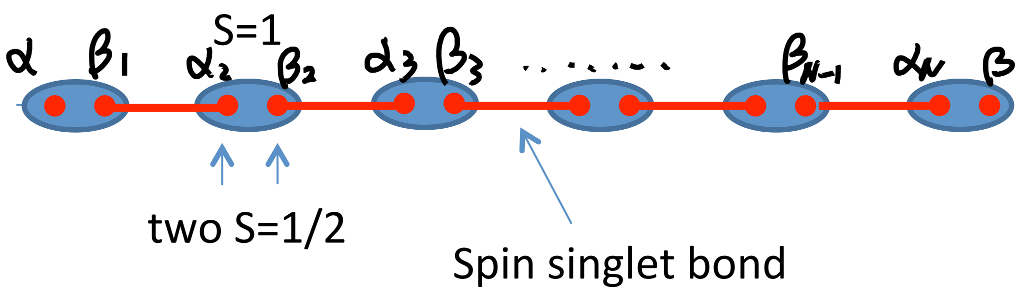

This is why it is possible to have spin-1/2 edge states in a spin-1 model --- although a single spin-1/2 is NOT a representation of SO(3) symmetry, the product of two spin-1/2's is a representation: ½ x ½ = 0 + 1

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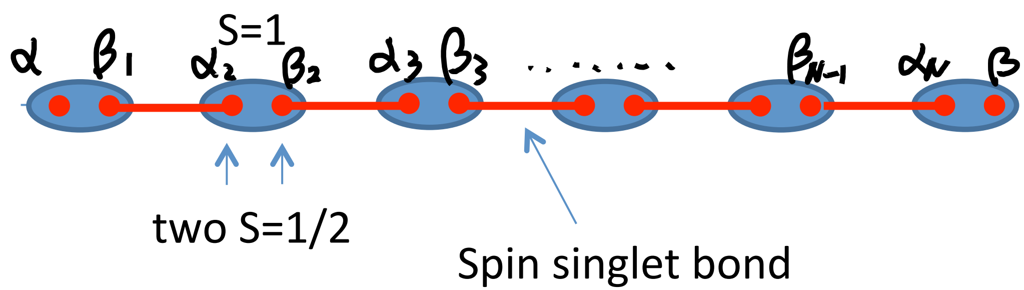

This is also why the 4-fold degeneracy is robust: consider a local SO(3) symmetric perturbation V:

$$[\hat{V}, \hat{U}(g)] = 0 \Rightarrow [\hat{V}, \hat{U}_L(g)] = 0 \text{ AND } [\hat{V}, \hat{U}_R(g)] = 0$$

The two SU(2) symmetries are individually respected!

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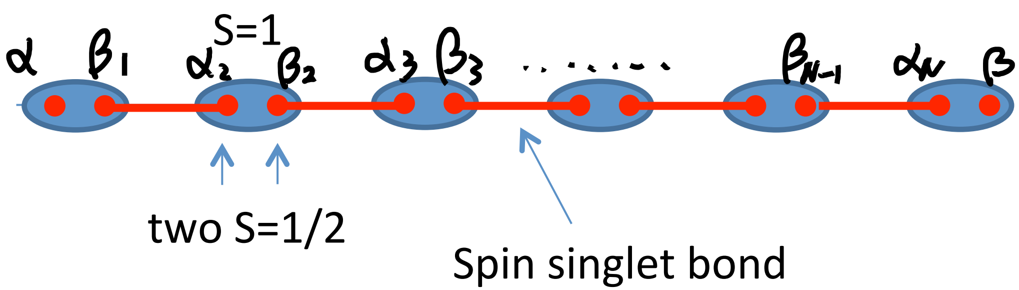

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However if perturbation is not SO(3) symmetric, the 4-fold degeneracy is gone.

Such spin-1/2 edge states are characteristic feature in the whole AKLT phase. You can kill the spin-1/2 edge states only by (1) a phase transition OR (2) removing the symmetry. ---- called Symmetry Protected Topological Phase

How to generally understand 1D SPT phases?

Turner, Pollmann, Berg, Oshikawa, Chen, Gu, Wen….

- From AKLT chain, we know, under internal symmetry g:

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There is a potential phase ambiguity in the definition of $\hat{U}_{L/R}(g)$.

They do not have to form reps of SG. Instead they can be

"projective representation" of SG. (e.g. spin-1/2 is projective rep of SO(3))

$$\hat{U}_L(g_1) \circ \hat{U}_L(g_2) = e^{i\theta(g_1, g_2)} \hat{U}_L(g_1 \circ g_2)$$

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- Mathematically, the phase factor here is called a factor system. It is a function of group elements and have to satisfy consistency condition:

$$[\hat{U}_L(g_1)\circ\hat{U}_L(g_2)]\circ\hat{U}_L(g_3) = \hat{U}_L(g_1)\circ[\hat{U}_L(g_2)\circ\hat{U}_L(g_3)]$$

$$\Rightarrow e^{i\theta(g_1,g_2)}\, e^{i\theta(g_1g_2,g_3)} = e^{i\theta(g_1,g_2g_3)}\, e^{i\theta(g_2,g_3)}$$

Mathematically this is 2-cocycle condition.

Inequivalent proj. reps are classified by 2nd cohomology group: $H^2[SG, U(1)]$

A simple example:

Turner, Pollmann, Berg, Oshikawa, Chen, Gu, Wen….

$$\hat{U}_L(g_1) \circ \hat{U}_L(g_2) = e^{i\theta(g_1,g_2)} \hat{U}_L(g_1 \circ g_2)$$

$$[\hat{U}_L(g_1) \circ \hat{U}_L(g_2)] \circ \hat{U}_L(g_3) = \hat{U}_L(g_1) \circ [\hat{U}_L(g_2) \circ \hat{U}_L(g_3)]$$

$$\Rightarrow e^{i\theta(g_1,g_2)} \cdot e^{i\theta(g_1 g_2, g_3)} = e^{i\theta(g_1, g_2 g_3)} e^{i\theta(g_2,g_3)}$$

Consider SG=Z2 x Z2

$$\uparrow$$

$$\{\mathbb{1}, \sigma\} \times \{\mathbb{1}, \tau\}.$$

$$H^2(Z_2 \times Z_2, U(1)) = Z_2.$$

$Z_2 = 0$: trivial.

$Z_2 = 1$: $\Rightarrow$ $U_L(\sigma) \cdot U_L(\tau)$

$$= - U_L(\tau) \cdot U_L(\sigma)$$

"Spin-$\frac{1}{2}$"

How to generally understand 1D SPT phases?

Turner, Pollmann, Berg, Oshikawa, Chen, Gu, Wen….

- 2^{nd} cohomology group can be used to classify 1D (bosonic) SPT phases.

$$H^2[SG, U(1)]$$

Symmetry of Hamiltonian	Number of Different Phases
None	1
$SO(3)$	2
D_2	2
T	2
$SO(3)+T$	4
D_2+T	16

Chen, Gu, Wen (2010)

Summary of discussion so far, and outlook

- SPT phase protected by local symmetries:

  trivial bulk + gapless anomalous edge states.

  higher dimensions:

→ Fermion: Integer quantum hall states, topological insulators

(can be realized even with weak interaction)

Boson: Bosonic integer quantum hall states, bosonic topological insulators

(require strong interaction to realize)

- In 1D, we find SPT phases host symmetry fractionalized edge states.

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- In 1D, we find SPT phases host symmetry fractionalized edge states.

Next, let's attempt to generalize these concepts.

Some imaginations

- The essence of SPT phases are:

anomalous lower dimensional gapless states are realized at the edge of the system.

What is so special about edges? Can one realize these anomalous lower dimensional states in other situations?

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What is so special about edges? Can one realize these anomalous lower dimensional states in other situations?

Edges are special because:

(1) In 1d, edges must be created in pairs. (This is why fractional quantum number can be realized.)

(2) In 2d, edge must form a closed loop. (This is why non-stoppable helical/chiral modes can be realized.)

(3) In 3d, edge must form a closed surface.(This is why single Dirac-cone can be realized in TI with time-reversal symmetry.)

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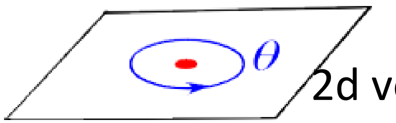
replacing by point-like topological objects:



2d vortex

domain walls in 1d, vortices in 2d, … and gauge charges >=2 dimensions…

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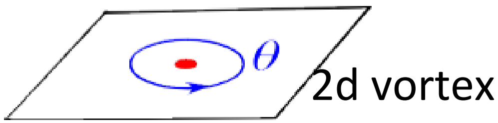

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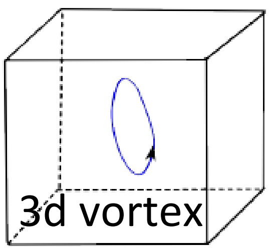

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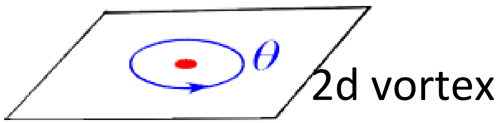

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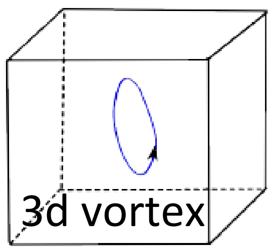

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replacing by surface-like topological objects: domain walls in 3d ….

Some imaginations

- **These objects in principle also could host anomalous lower dimensional states!** **Indeed, there are already lots of examples.**

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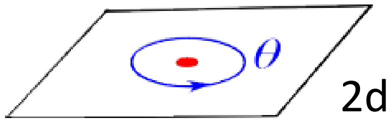

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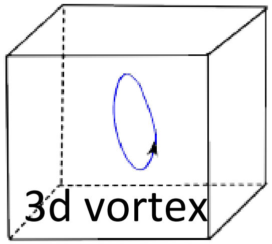

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Topological defects:

- Vortex hosted majorana modes in p+ip 2d superconductor.

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Topological dynamical excitations: --- "symmetry enriched" phenomena

- Symmetry fractionalizations for gauge charges

  Laughlin state e*=e/3

  Spin-charge separation in gapped quantum spin liquids.

- A large class of exactly solvable models: (Mesaros&YR, 2012)

Showing: gauge charge hosted symmetry fractionalization in 2d

gauge flux loop hosted anomalous line states in 3d…

Plan:

- Tomorrow I will talk about symmetry fractionalization for gauge charge excitations in quantum spin liquids.

- Today, if time is allowed, let's have a simple derivation of the so-called "worm-hole" effect in 3D TI. (G. Rosenberg, H.-M. Guo, M. Franz, 2010)

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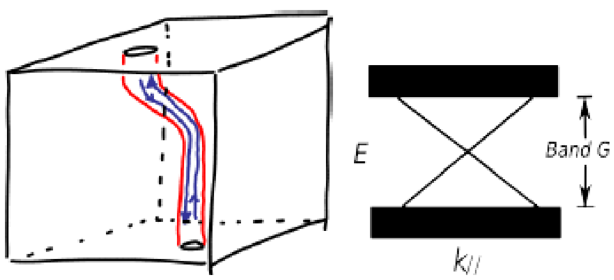
---If a pi-flux loop (TR sym.)is threaded through the bulk 3D strong TI, the loop is topological bound with helical modes:
(same as the anomalous edge state of 2D TI).



---If one interpret the pi-flux as a dynamical Z_2 gauge flux excitation, namely if the TI are not formed by electrons, but by fermions carrying Z_2 gauge charge, this effect is an example of symmetry-enriched phenomena.

The wormhole effect

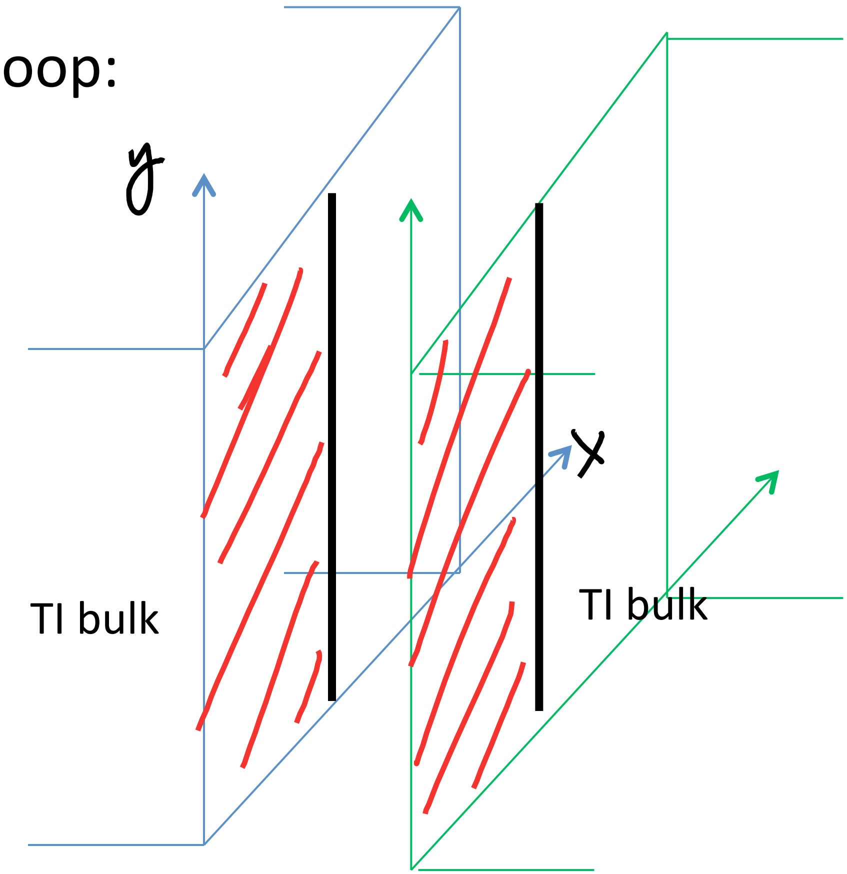
- A simple model of the pi-flux loop:

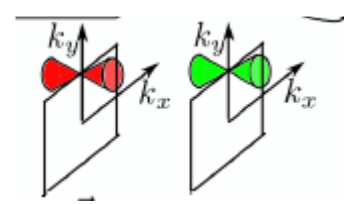
Step (1): Cut the 3D TI

Two surfaces-- two sets of Dirac nodes:

$$H = \vec{P} \cdot \vec{\sigma} \mu_z$$

$(\mu_z = \pm 1 : \text{ L/R surfaces})$





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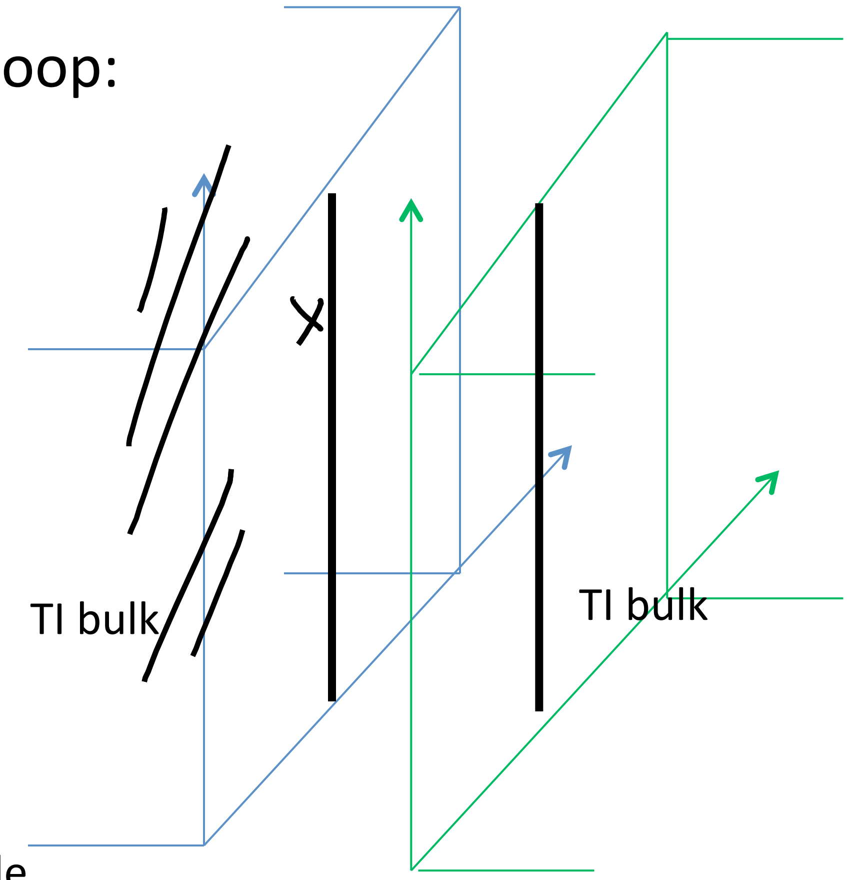

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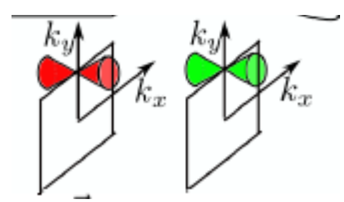
Step (2): Gluing back,
but trapping a pi flux (black) line in the middle

Hopping from left to right: $m \cdot \mu_x$

m<0 in red region
m>0 in uncolored region



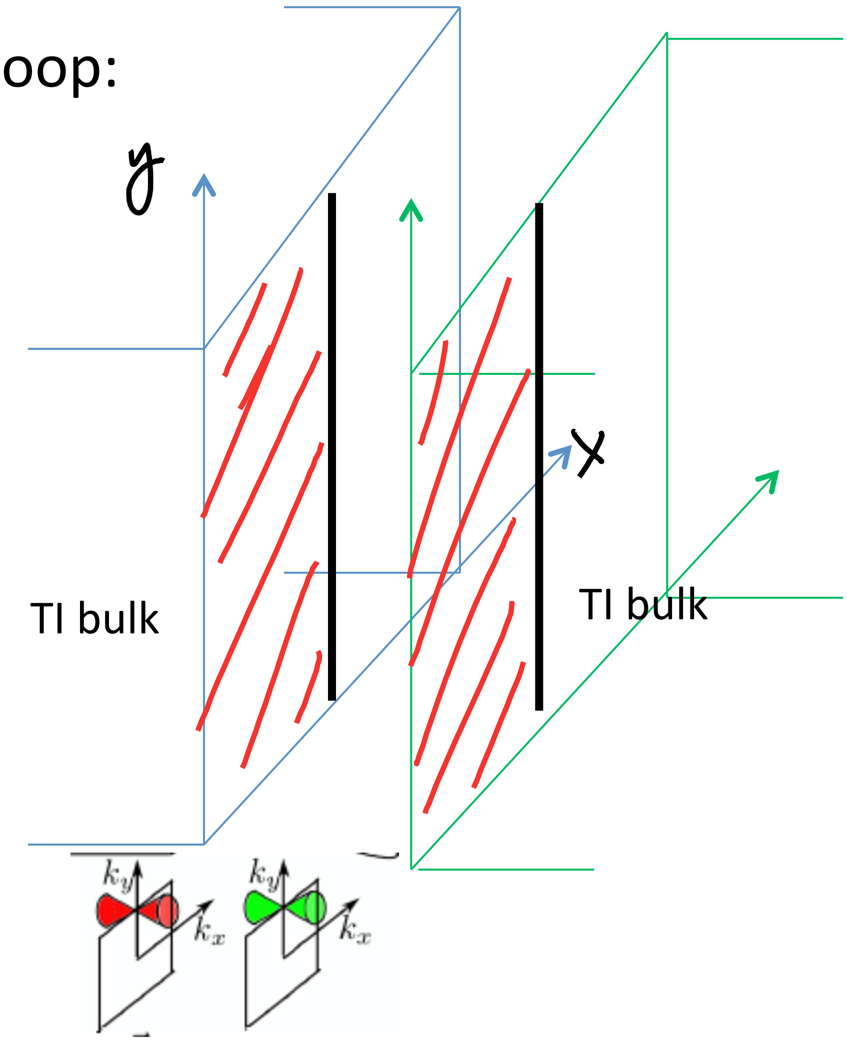
The wormhole effect

- A simple model of the pi-flux loop:

Dirac equation mass changing sign:

$$H = (k_x\sigma_x + k_y\sigma_y)\mu_z + m(x)\mu_x$$

$$m(x) = \begin{cases} +m & \text{if } x > 0 \\ -m & \text{if } x < 0 \end{cases}$$



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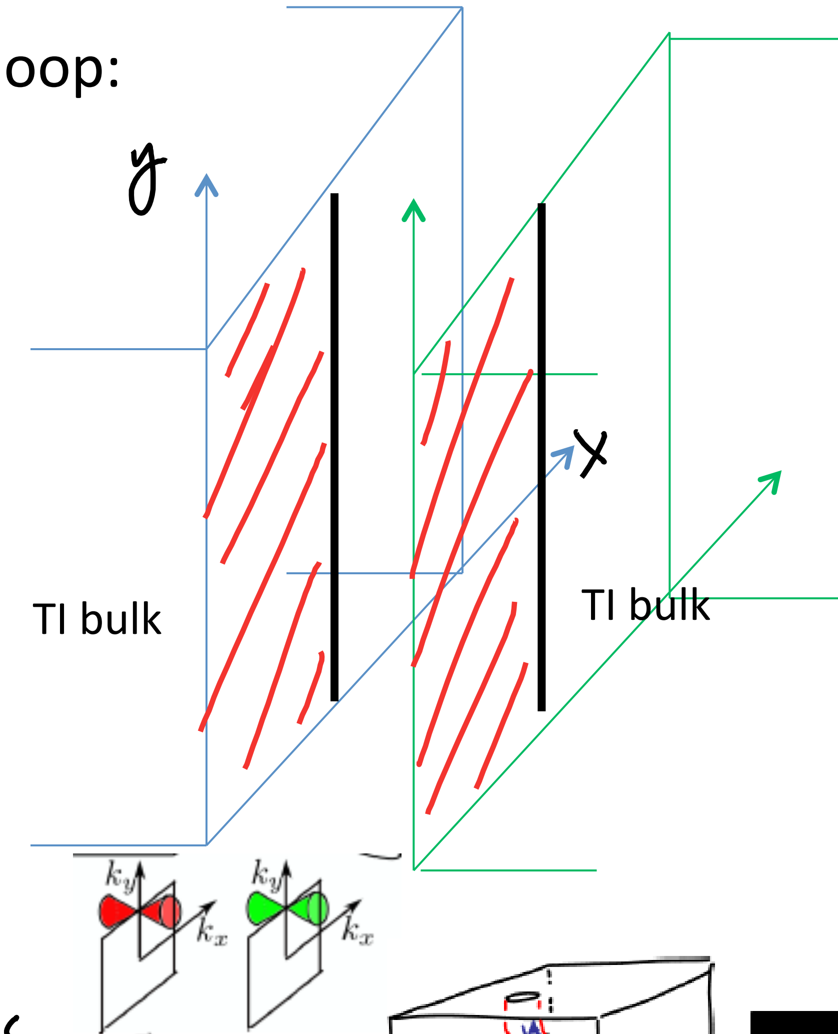
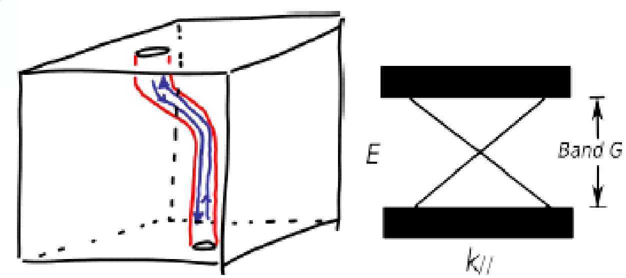
@ $k_y = 0$.

midgap modes: $\psi_\pm(x) = e^{-\int_0^x m(x')dx'} \cdot \psi_{0,\pm}$

where $\sigma_x \mu_y \cdot \psi_{0,\pm} = \psi_{0,\pm}$.

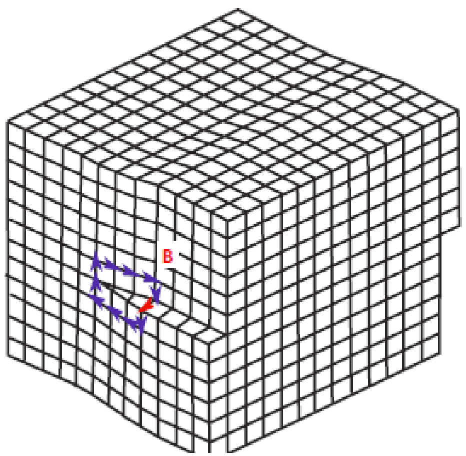
adding $k_y \neq 0 \Rightarrow$ helical modes.

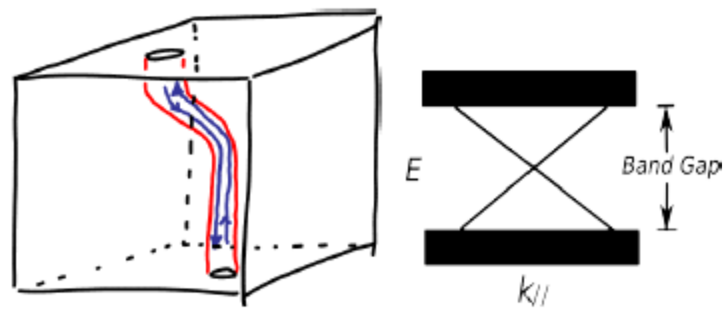




Dislocations in 3D TI

- Although magnetic pi-flux is difficult to realize in TI, even here in magnetic lab, the crystalline topological defects --- dislocations can have similar effect. (YR, Zhang, Vishwanath 2009)





Condition for existence of helical modes: $\vec{B} \cdot \vec{M}_\nu = \pi (\mathrm{mod} 2\pi)$

$\vec{B}$: Burger's vector in real space. $\vec{M}_\nu$: Weak index vector in momentum space

Realized in TI with nonzero weak index: e.g., SmB6….

Plan:

I was mentioning Z_2 gauge excitations, like flux loops, and their symmetry enriched phenomena.

But can these be realized in materials?

- Tomorrow:

(1) Quantum spin liquid phases in frustrated magnets, and related experiments in materials

(2) Parton constructions of quantum spin liquids, and symmetry fractionalization.