Introduction to the Quantum Hall Effects

Lecture by B. I. Halperin, Harvard University

at the Winter School *on*

“Correlations in flat bands: From the FQHE to Moiré”

NHFML, Tallahassee, January 9-13, 2023

The Quantum Hall Effects

Large set of peculiar phenomena in two-dimensional electron systems, at low temperatures in strong magnetic fields.

Usually: electrons in semiconductor structures: e.g. electrons trapped in a thin layer of GaAs, surrounded by AlGaAs, “quantum well structure”.

More recently: QHE seen in monolayer and bilayer graphene.

Samples can differ widely in electron densities and freedom from defects. Magnetic fields range from 0.1 to 45 Tesla.

Variety of Quantum Hall States

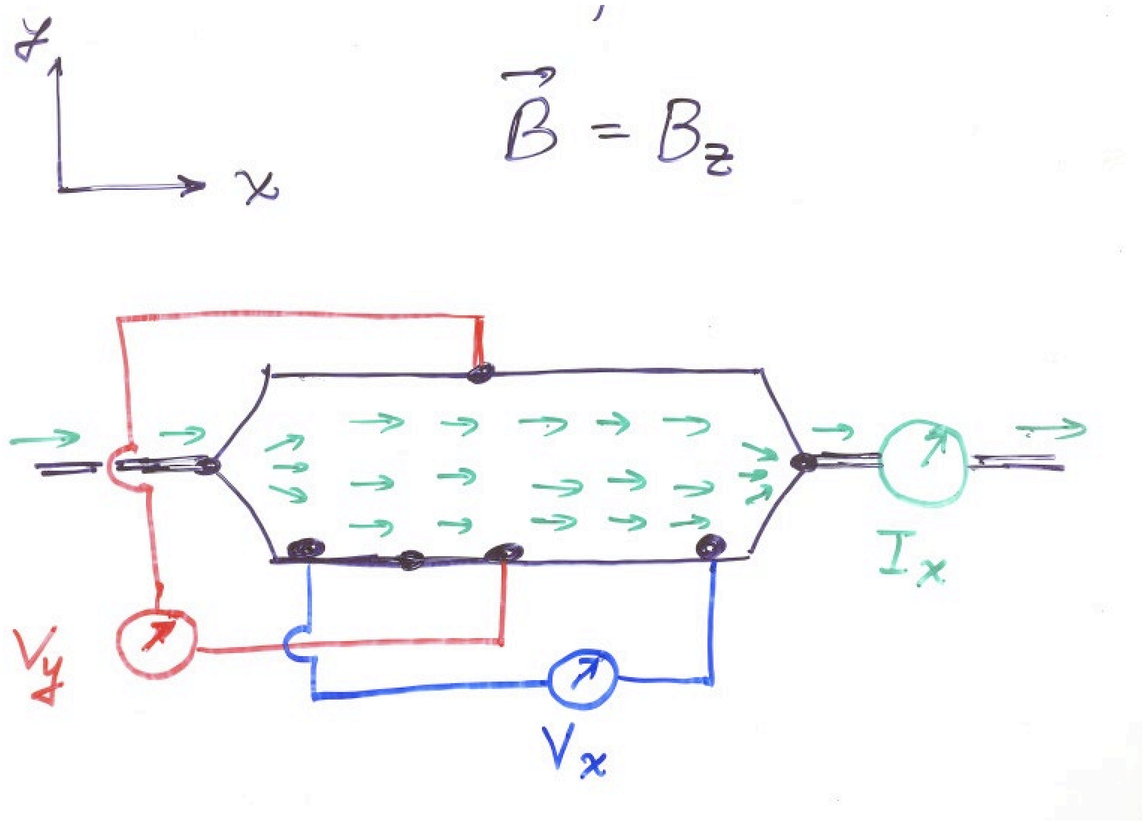
- Integer and Fractional Quantized Hall States
- Wigner Crystals and Striped Phases
- Unquantized Quantum Hall States (Fermi liquid of composite fermions)

- Note: Quantized Hall Effects can also occur in zero magnetic field in certain systems with spontaneously broken time-reversal symmetry.

Outline

- Current lecture will focus on general properties of Quantized Hall systems (Integer and Fractional)
- See how they arise in the simplest case: Integer QHE for non-interacting electrons.
- Excitations with fractional charge in Fractional QHE systems.

Hall Geometry



Hall resistance: $R_H = V_y / I_x$

Longitudinal Resistance: $R_{xx} = V_x / I_x$

Quantized Hall Effects

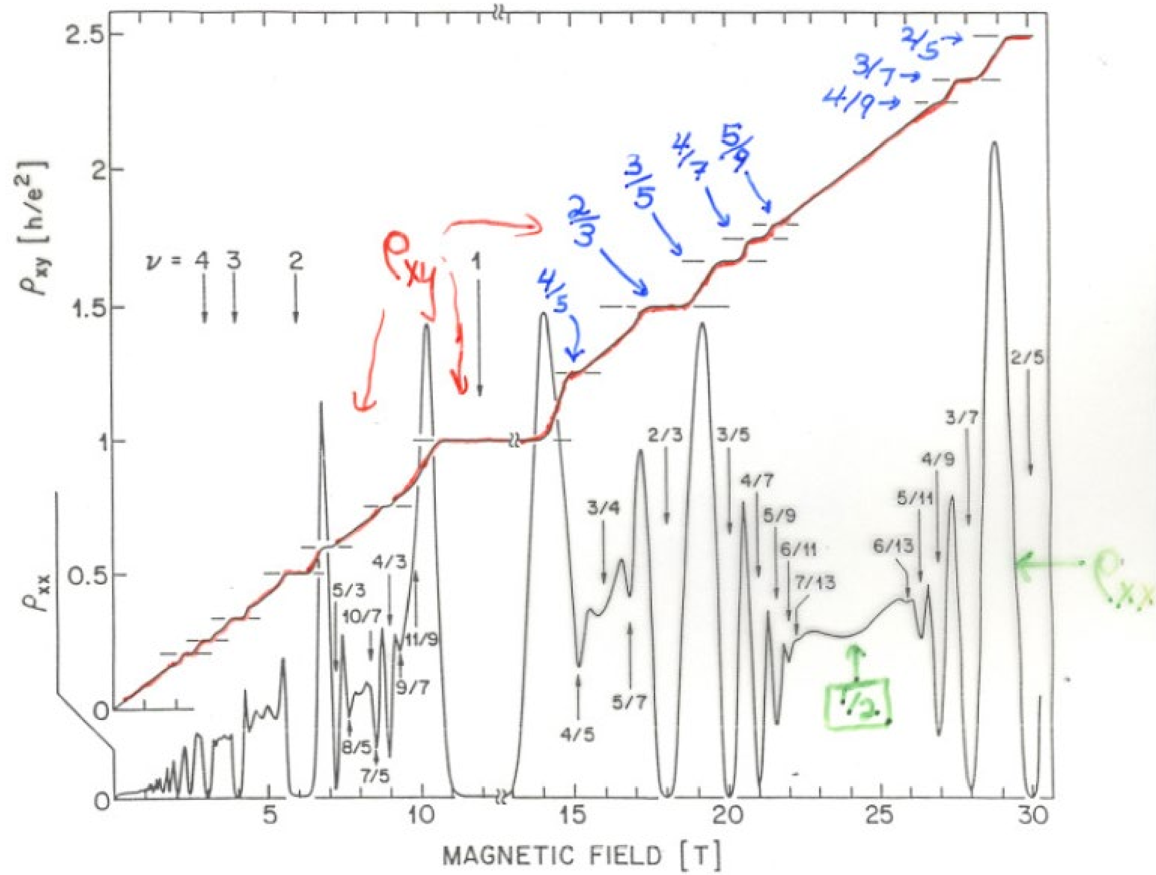
Under appropriate conditions, it is observed that the Hall resistance exhibits a series of plateaus, where it remain constant over a range of magnetic fields and carrier densities.

On the plateaus: $R_{xx}=0$, and $1/R_H = \nu\, e^2/h$,

where ν is a nonzero integer or simple rational fraction, and h/e^2 = 25, 812.02 ohms.

Independent of precise shape of sample, same in different materials, robust to small concentrations of impurities, etc.

GaAs / GaAlAs Heterostructure $n = 3.0 \times 10^{11}\ cm^{-2}$
$\mu = 1.3 \times 10^{6}\ cm^2/V\,sec$



R Willett, J.P. Eisenstein, H.L. Störmer, D.C. Tsui, A.C. Gossard, & J H English, Phys Rev. Lett. 59 1776 (1987)

Bulk Conductivity of a Quantized Hall State

For an infinite sample, uniform electric field:

$$J_i = \sigma_{ij} E_j$$

=> In quantized Hall state: Electrical conductivity tensor obeys

$$\sigma_{xx} = \sigma_{yy} = 0 , \qquad \sigma_{yx} = -\sigma_{xy} = \nu e^2/h .$$

Results are exact in limit of T -> 0, large sample.

Material requirements for the Quantized Hall Effect (Integer or Fractional)

In the ideal 2D bulk, far from the edges, no impurities, system should have an energy gap for creation of mobile charges.

In the presence of impurities, there can be localized states in the gap, but carriers must freeze out into localized states at low temperatures.

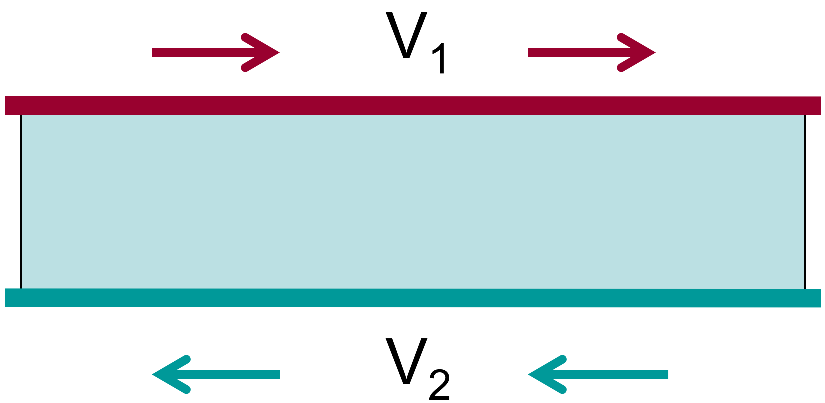
So the bulk is essentially an insulator: current cannot flow in the direction parallel to an applied electric field. $\sigma_{xx} = \sigma_{yy} = 0$

Current can flow in a direction perpendicular to the electric field, carried by electrons well below the Fermi level, giving rise to a Hall conductivity.
$\sigma_{xy} = -\sigma_{yx} = \nu e^2/h$.

Edge states

Theorem: If ν is nonzero, the energy gap must vanish along the sample edges. (Laughlin 1981, Halperin 1982). Edges are a peculiar type of one-dimensional metal: "Chiral metal": Over a large length scale, charge carriers travel in only one direction along the edge.

For a ribbon-shaped sample, if there is a chemical potential difference between the two edges, there will be a net edge current proportional to the chemical potential difference .



Total current

The voltage difference V, measured by a voltmeter connected to the two edges, will be the sum of the electrostatic potential difference and the chemical potential difference. Bulk current is produced by gradients in the electrostatic potential. The total current I along the ribbon will be the sum of the edge current and the current in the bulk.

Measured Hall conductance $G_H = I_x / V_y = \nu\, e^2/h$

regardless of how the current is divided between edge and bulk. Cannot change, as long as $\sigma_{xx} = \sigma_{yy} = 0$ in the bulk. (Consequence of charge conservation.)

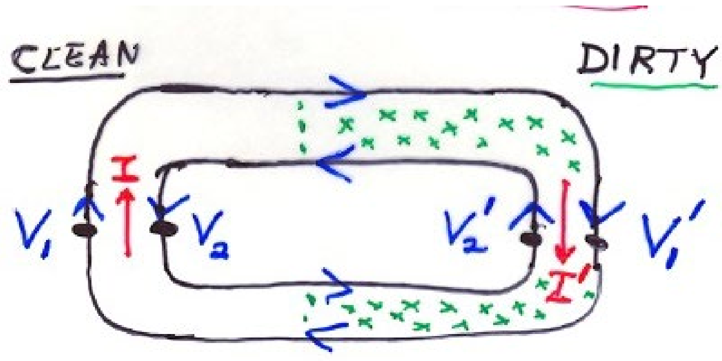
Exactness and robustness of

the quantized Hall conductance

(and neccessity of edge states)

and where does it break down.

Gedanken experiment: Annular (Corbino) geometry



Assume: Clean region is an ideal QHE state

$$I = \nu\,(e^2/h)\,(V_1 - V_2)$$

Assume: No extended states at Fermi level in dirty region or interface between clean and dirty regions. => No flow from inner to outer edge of sample (at T=0).

Can reach steady state with $V_1' = V_1 \neq V_2' = V_2$.
By current conservation: $I' = I = \nu\,(e^2/h)\,(V_1 - V_2)$

Integer QHE for non-interacting spinless electrons.

1. Ideal infinite system.

2. Effects of edges.

Landau Levels in 2-Dimensional Systems

Consider non-interacting electrons in uniform magnetic field B in 2D.

In quantum mechanics, energy levels are quantized into "Landau levels", with

$E_n = (h/2\pi)\, \omega_C\, (n+1/2)$, $n = 0, 1, 2, 3, ...$

The number of independent orbits, in each Landau level is equal to the number of flux quanta: $N_B \equiv B\, e\, \mathrm{Area} / h$

Define Landau level filling factor $f = N_e / N_B = (n_e/B)(h/e)$.

If the Fermi level is in an energy gap, between two Landau levels, then f will be an integer.

Relation of Hall conductance to Landau-Level Filling Factor $f=(n_e/B)(h/e)$

For electrons in a uniform positive background (no impurities) Hall conductivity is given by

$\sigma_{xy} = e\, n_e/B = f\,(e^2/h)$ (exact result)

System with Boundaries: An Infinite Strip

Confining potential $V(y)$

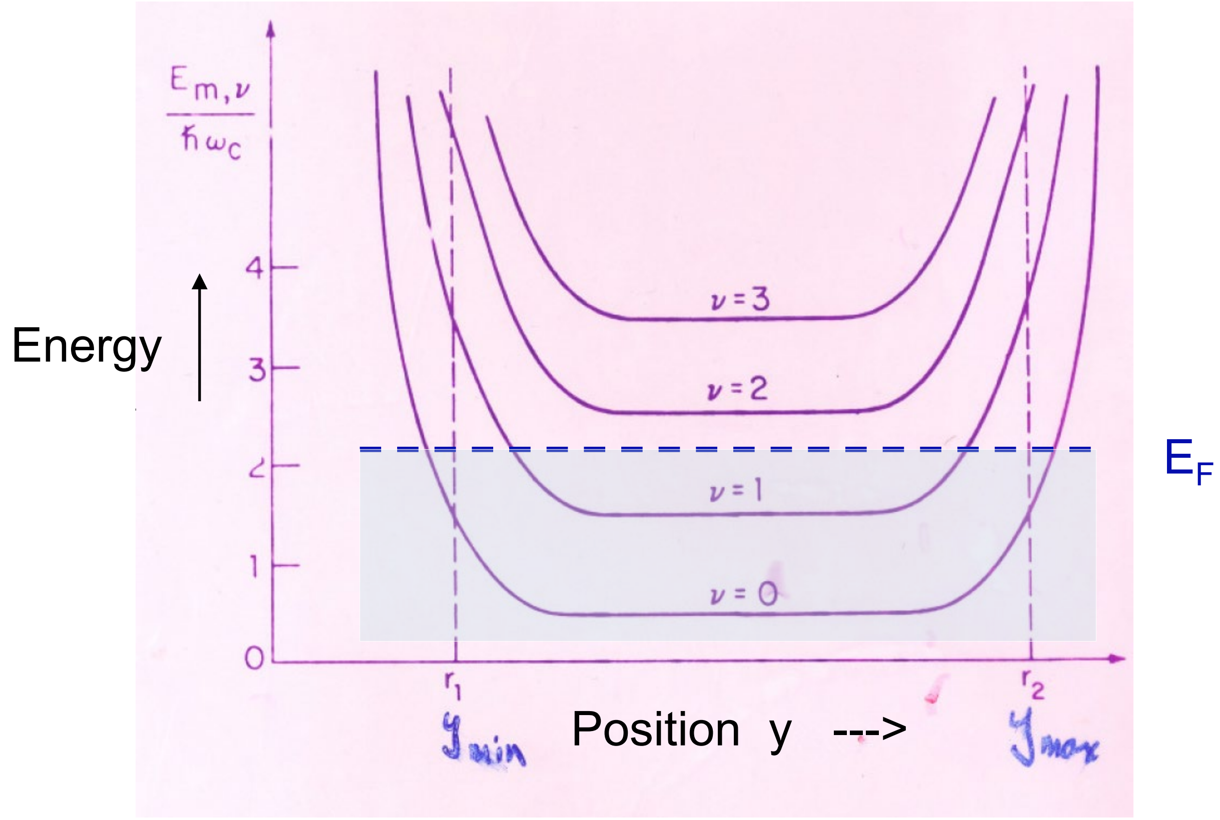
Use Landau Gauge: $A_x = B\,y,\; A_y = A_z = 0$

Eigenstates have form: $\psi(x,y) = e^{ikx}\varphi_{kn}(\mathrm{y})$

where n is a Landau level index and $\varphi_{kn}(y)$ is localized near $y = y_k = k\,l_B^2$, $l_B^2 = \frac{\hbar}{eB}$.

Landau levels in a strip of finite width

Energy levels E_{kn} are pushed up at edges of the strip.



Energy levels are filled up to Fermi Level E_F. Here f=2, with an energy gap in bulk. At each edge, there are two conducting states at the Fermi level. Hall conductance G = 2 e^2/h . (Spin is ignored, here).

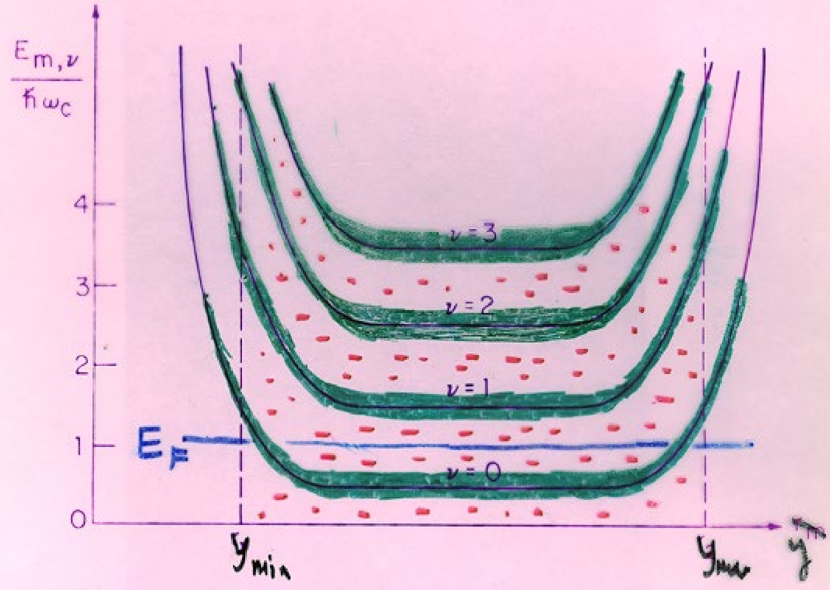
Eigenstates can carry current in the *x*-direction

- Specifically, for a filled eigenstate:

$$< I >_{kn} = \frac{e}{\hbar L} \frac{\partial}{\partial k} E_{kn}$$

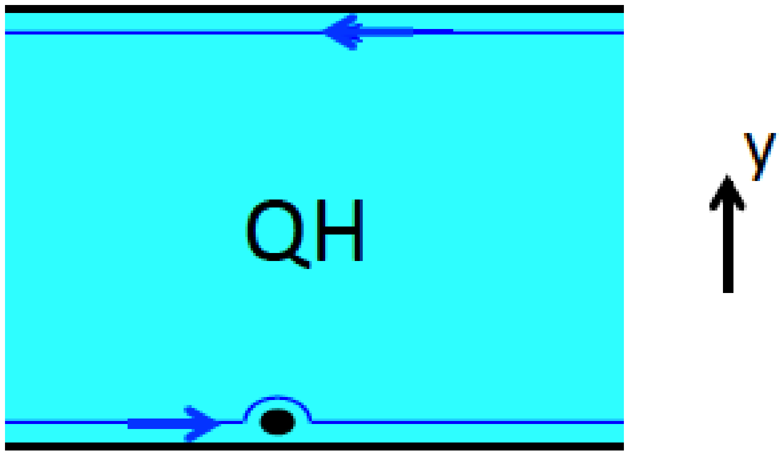
- Current has opposite sign at two edges
- Also get nonzero current in the bulk, given by the same formula, if there is an electric field in the *y*-direction.
- Total current carried by given Landau level is determined by the difference in Fermi levels at the two edges.

FINITE SYSTEM WITH IMPURITIES



A chiral edge state cannot be localized by disorder at the edge

Edge state can simply go around disordered region.



Effect of electron spin

- For non-interacting electrons, in the absence of Zeeman interaction, states are doubly degenerate, only even integer ν would be observed.
- In presence of Zeeman coupling, odd integers seen, but energy gap may be small.
- Electron-electron interaction enhances energy gap at odd integer fillings. ("Exchange energy".) Gives finite energy gap and spontaneous spin polarization (at T=0) even in absence of Zeeman field.

Fractional quantized Hall states

Need some way to produce an energy gap in a system with Hall conductivity different from an integer times e^2/h.

Electron-electron interactions are essential.

Models to explain the existence of fractional quantized Hall states will be discussed by Ady Stern and Bernd Rosenow in subsequent lectures.

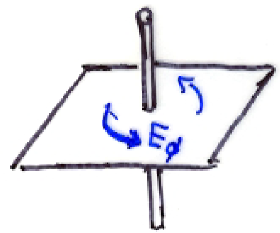
Regardless of specific models, FQH states have some peculiar features, including quasiparticles with fractional charge and fractional statistics.

I will address fractional statistics in my lecture on Wednesday

Here I address the necessity for fractional charge.

Existence of Fractional Charges

Laughlin Argument at $\nu = 1/3$



1. Insert SOLENOID of zero diameter at origin
2. Turn on current adiabatically until flux $\phi = \phi_0 \; (= \frac{1}{2\pi})$

- Azimuthal EMF $= 2\pi r E_\phi = \dfrac{d\phi}{dt}$

Far from origin:

- Radial current $j_r = \sigma_{xy} E_\phi = \dfrac{1}{3} \dfrac{E_\phi}{2\pi}$

- Cumulative charge near origin

$$q = -\int dt \; 2\pi r \, j_r = -\frac{e}{3}$$

But Hamiltonian with one extra flux at the origin is physically identical to original Hamiltonian H_0 (Differs only by a gauge transformation)

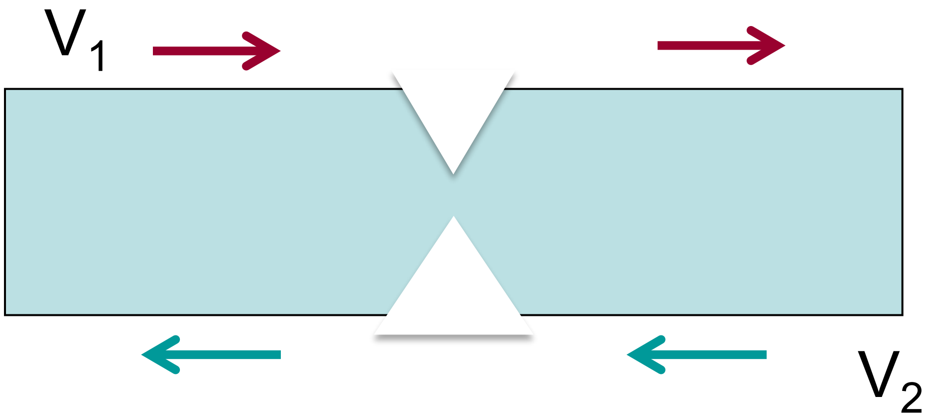
- So states with charge $\pm \frac{e}{3}$ are eigenstates of H_0.

Fractional Charge (Continued)

- For FQH state with $\nu = p/q$, Laughlin's argument says there must be quasiparticles with charge ep/q. But these are not generally the smallest charges. If p and q have no common divisor, you can find integers n, n' with nq-n'p=1. Then combination of n electrons and n' positive quasiparticles has charge $Q=e/q$.
- These are the smallest charges in the most prominent odd-denominator fractions (Jain states).
- For quantum Hall states with even denominator q, it has been shown that there must exist quasiparticles with charge $e/2q$. [Levin and Stern, 2009]

How to see fractional charge

- Shot noise experiments: measure current noise when quasiparticles tunnel across a narrow constriction between opposite edges of an FQH device.



Using a statistical analysis, back out charge of quasiparticles tunneling from one edge to another. (Not necessarily the smallest charge.)

Shot noise results

- Glattli group 1997: Heiblum group, 1997-2010:
- Charges e/3 reported for ν = 1/3, 2/3, 4/3, 5/3, 8/3
- Charges e/5 reported for ν = 2/5; e/7 at ν =3/7; e/4 at ν = 5/2.
- But at ν =2/5, charge increased at lowest temperatures from e/5 to 2e/5. Similarly at ν =3/7, 2/3.
- Possible Explanation (Ferraro et al, 2008): Relative tunneling rates depend on T. Renormalization group suggests 2e/5 should dominate tunneling in limit of low T.

Charge Sensing Experiments

- Charge senser (single-electron transistor) on a scanning tip is place just above the 2DEG.

- Voltage applied to tip can establish a potential well beneath the tip.
- Senser can measure jumps in charge when individual quasiparticles enter or leave the well.

- Can be measured far from boundary.

- Should measure unambiguously the smallest quasiparticle charge that is thermodynamically allowed.

Results of charge sensing experiments

- J. Martin et al (Yacoby group, 2004) measured $e/3$ at $\nu = 1/3$ and $\nu = 2/3$.
- V. Venkatachalam et al (Yacobi group, 2011) measured $e/4$ at $\nu = 5/2$.

Necessary condition for nonzero Hall conductance

By Onsager theorem: In order to get non-zero value of Hall conductance, system must have broken time-reversal symmetry.

Usually, provided by applied magnetic field

But quantized Hall states can also occur at B=0 for certain systems where symmetry is spontaneously broken by magnetic order. “Chern Insulators”.

Topological Aspects

Recall: Hall conductance cannot change unless mobility gap vanishes in the bulk, or there is a first order transition.

Hamiltonians depend on parameters that can vary continuously. But Hamiltonians with an energy gap at the Fermi level can be divided into discrete classes, indexed by quantized Hall conductance.

By definition: Topological Classification.

Chern Invariants

For non-interacting electrons in a 2D periodic potential with broken time reversal symmetry, with or without an applied magnetic field:

The variation over the Brillouin zone of the Bloch wave functions for a given electron band can be characterized by a topological invariant known as a Chern number.

If E_F is in an energy gap, there will be a quantized Hall conductance with ν = to the sum of the Chern numbers for all bands below E_F.

System on a Torus

For a two-dimensional electron system on a torus with a non-degenerate ground state and an energy gap to all excited states, with or without interactions, one can define a Chern number from the behavior of the many-body ground-state wave function as one alternately inserts and removes one quantum of magnetic flux through the two holes in the torus.

The Hall conductance (averaged over the values of the fluxes) is determined by this Chern number.

For fractional QH states, the ground state on a torus must have a degeneracy (= integer multiple of the denominator of ν).