Quantum Hall transitions: history, recent developments, and challenges
Part II

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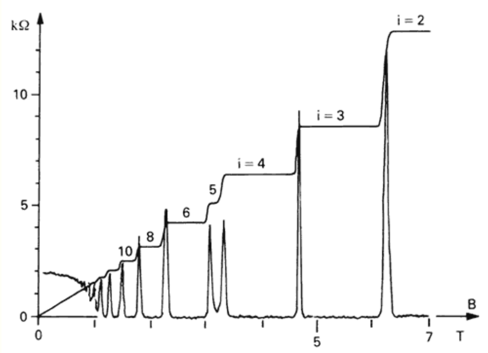
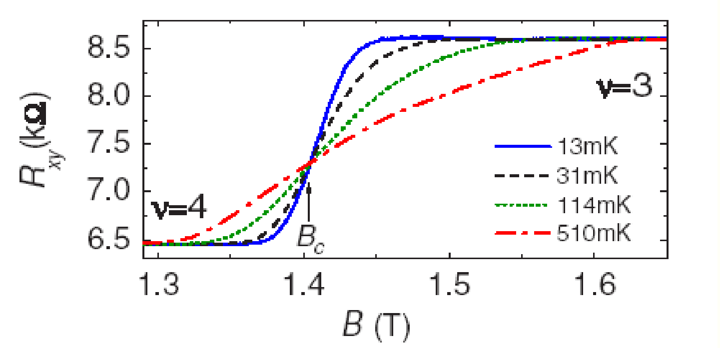
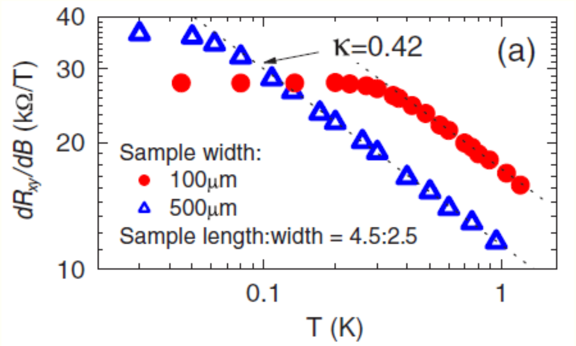
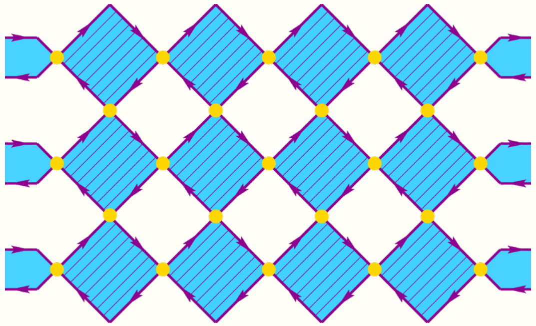
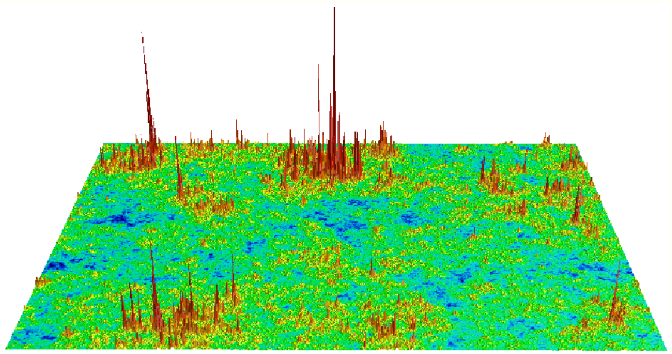
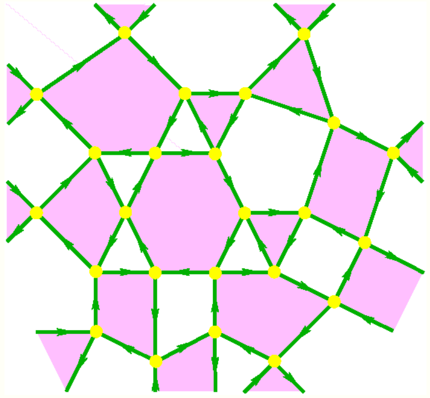






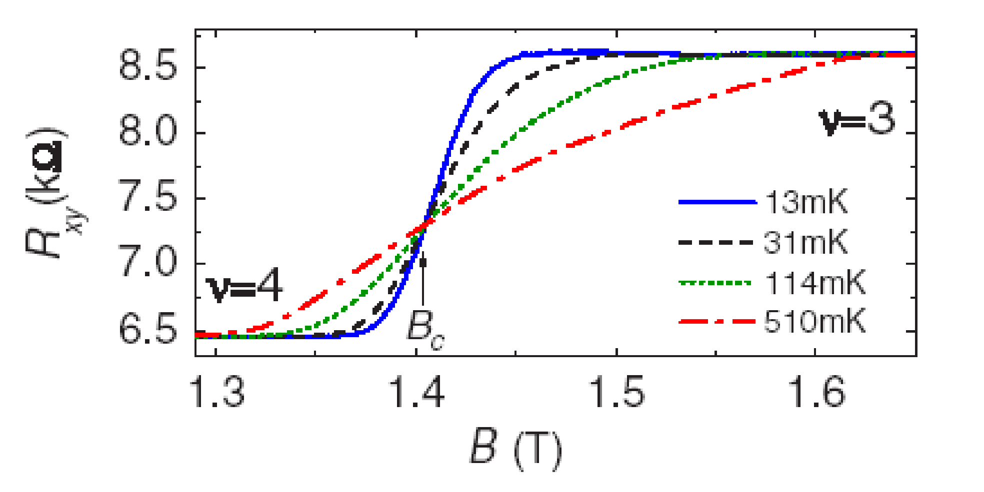


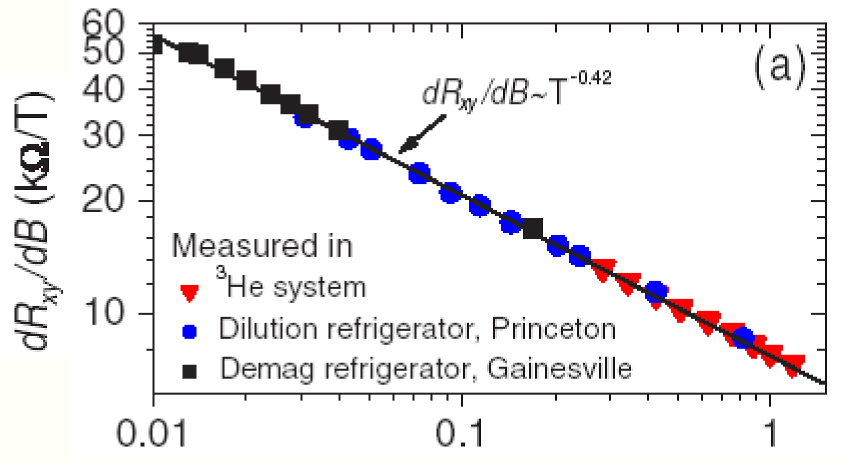




Summary of part I

- Quantum Hall transitions – quantum phase transitions





- Critical scaling with temperature, frequency, current,…

$$\left.\frac{d\rho_H}{dB}\right|_{B_c} \sim T^{-\kappa}, \quad \kappa = \frac{1}{z\nu}, \quad \nu \approx 2.38, \quad z = 1$$

- Universality: same critical exponents for different transitions and systems

- (Ir)relevance of interactions (short-range vs long-range) at the noninteracting fixed point

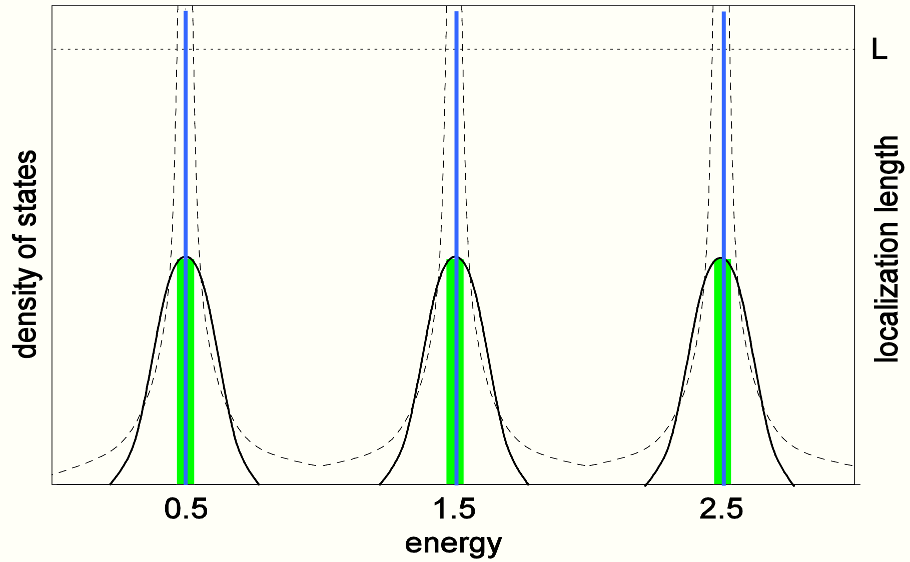
IQH and localization in strong magnetic field

- Single electron in a magnetic field and a random potential

$$H = \frac{1}{2m}\Big(-i\hbar\nabla + \frac{e}{c}\mathbf{A}\Big)^2 + U(\mathbf{r})$$

- Without disorder: Landau levels
- Disorder broadens the levels and localizes most states
- Extended states near E_c (green)
- IQH transition upon varying E_F or B
- Diverging scale is the localization length

$$\xi(E) \propto |E - E_c|^{-\nu}$$



- An Anderson (localization-delocalization) transition: a non-interacting quantum phase transition

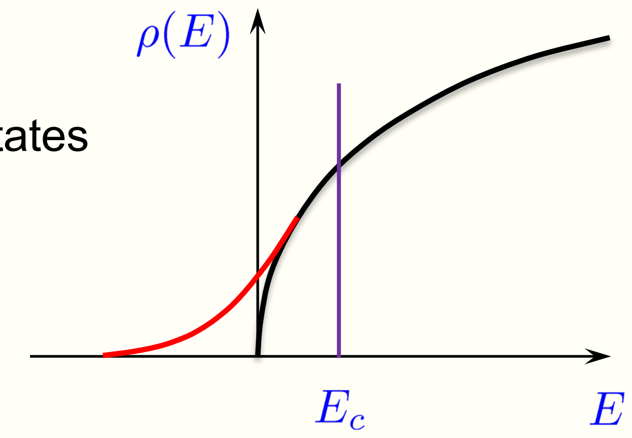
Anderson transitions

- Single electron in a random potential: $H = -\frac{\hbar^2}{2m}\nabla^2 + U(\mathbf{r}), \quad H\psi = E\psi$ P. W. Anderson `58

- Ensemble of disorder realizations: statistical treatment $P[U(\mathbf{r})] \propto \exp\left(-\frac{1}{2\gamma}\int d^d r\, U^2(\mathbf{r})\right)$

- Eigenstates are extended for $E > E_c$ and localized for $E < E_c$

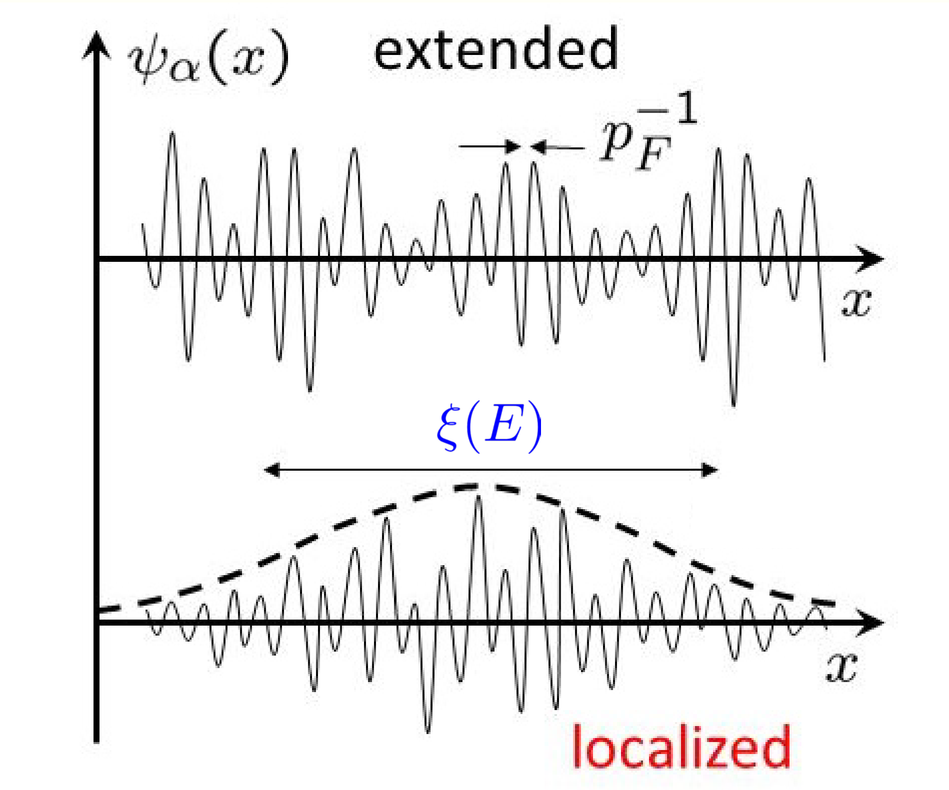
- Insulator: localized states



- Metal: extended states

Anderson transitions

- Eigenstates are extended for $E > E_c$ and localized for $E < E_c$
- Localization length diverges $\xi(E) \propto |E - E_c|^{-\nu}$
- Interesting to look at the spatial structure of critical wave functions exactly at $E = E_c$. These are important for the RG analysis of (ir)relevance of interactions at ATs



Symmetries and AZ classes

A. Altland, M. Zirnbauer '96

Conventional (Wigner-Dyson) classes

	T	spin rot.	chiral	p-h	symbol
GOE	+	+	−	−	AI
GUE	−	+/−	−	−	A
GSE	+	−	−	−	AII

Chiral classes

	T	spin rot.	chiral	p-h	symbol
ChOE	+	+	+	−	BDI
ChUE	−	+/−	+	−	AIII
ChSE	+	−	+	−	CII

Bogoliubov-de Gennes classes

	T	spin rot.	chiral	p-h	symbol
	+	+	−	+	CI
	−	+	−	+	C
	+	−	−	+	DIII
	−	−	−	+	D

- Integer quantum Hall effect (A)
- Spin quantum Hall effect (exact results) (C)
- Thermal quantum Hall effect (D)

Anderson transitions: random critical points

- All observables are random, a complete theory would describe their distributions
  - Functional RG?
- At least try to compute moments. Disorder average: replica limit or supersymmetry

$$\overline{\langle \mathcal{O} \rangle} = \int \mathcal{D}U \, \langle \mathcal{O} \rangle = \int \mathcal{D}U \, \frac{\int \mathcal{D}\psi \, \mathcal{O}(\psi) \, e^{-S[\psi,U]}}{\int \mathcal{D}\psi \, e^{-S[\psi,U]}} = \int \mathcal{D}U \int \mathcal{D}\psi \mathcal{D}\phi \, \mathcal{O}(\psi) e^{-S[\psi,U]-S[\phi,U]}$$

- Partition function $Z[U] = \int \mathcal{D}\psi \mathcal{D}\phi \, e^{-S[\psi,U]-S[\phi,U]} = 1$
- Is there conformal symmetry at Ats and other random critical points, similarly to conventional critical points? (Scale vs conformal invariance)
- If yes, ATs are described by (unknown) non-unitary CFTs (In 2D: CFTs with $c = 0$)

Models, methods, and recent results in the theory of Anderson transitions

- Field theory: non-linear sigma models — F. Wegner, K. Efetov, A. Pruisken,…
  - Fixed points at strong coupling, except in $2+\varepsilon$ dimensions
- A lot of intuition comes from network models amenable to numerics — J. T. Chalker, P. D. Coddington `88
- Recent advances include
  - High-precision numerics (irrelevant operators) — K. Slevin, T. Ohtsuki `09; W. Nuding, A. Klümper, A. Sedrakyan `15; F. Evers et al., T. Vojta et al., R. Roemer et al.… `18-`25
  - Sigma-model-based symmetry analysis of multifractal (MF) critical wave functions — N. Charles, IAG, J. F. Karcher, A. W. W. Ludwig, A. D. Mirlin, M. R. Zirnbauer `11-`24
  - Constraints from conformal symmetry on MF spectra — R. Bondesan, D. Wieczorek, M. R. Zirnbauer `14-`19; J. Padayasi, IAG `23
  - Mapping to classical models, statistical mechanics and CFT — E. Bettelheim, IAG, A. W. W. Ludwig `12; IAG, J. F. Karcher, A. D. Mirlin `22
  - Random networks and quantum gravity — H. Topchyan, IAG, W. Nuding, A. Klümper, A. Sedrakyan `17-`25; A. Mukherjee, IAG, V. Kazakov `25; E. Bettelheim, IAG, E. F. M. Ramirez `25

Critical wave functions at Anderson transitions

- Interesting to look at the spatial structure of critical wave functions exactly at $E = E_c$
  - Critical wave functions are neither localized nor truly extended

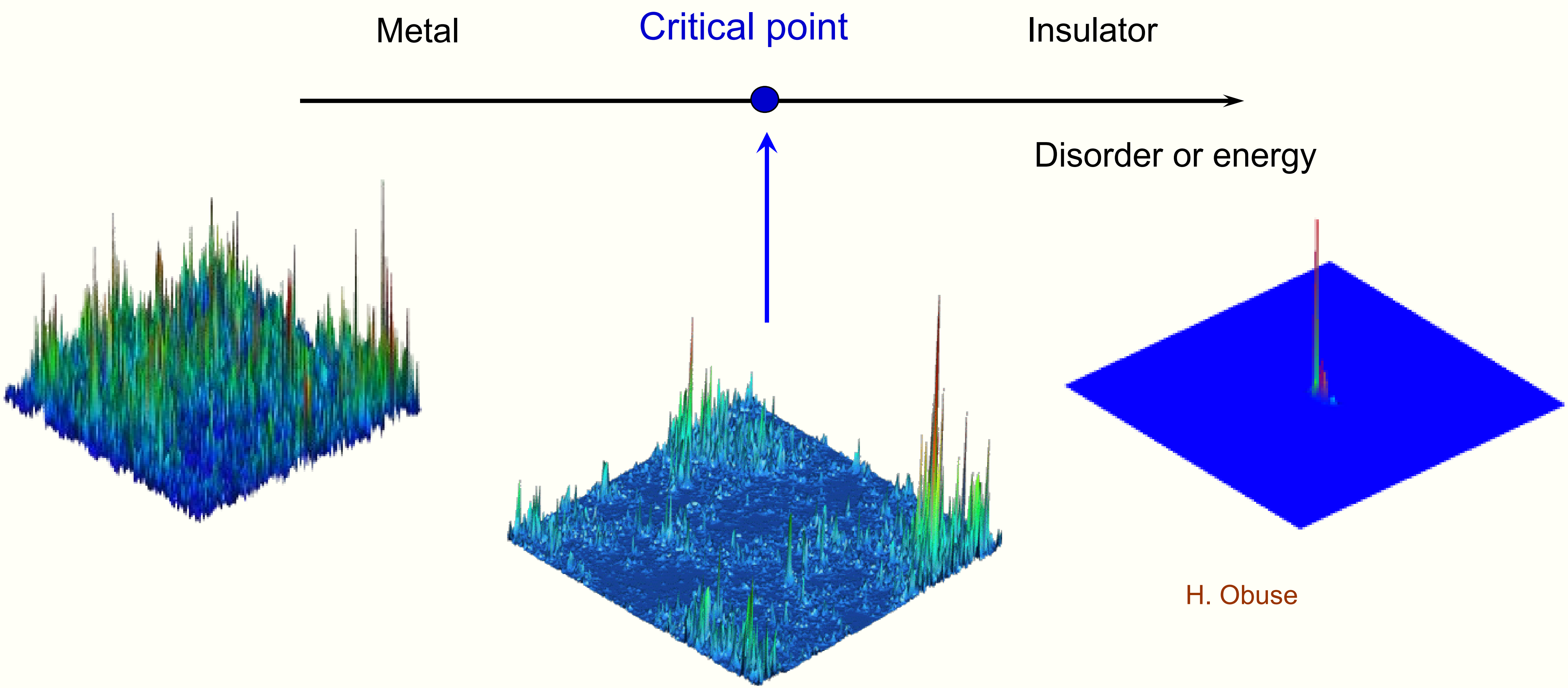
    F. Wegner `80
  - Complicated statistically scale-invariant multifractals
  - Characterized by a continuum of exponents: the multifractal spectrum

- Partial analytical results are available for some Anderson transitions, a lot of numerical results

- We derive strong constraints on MF spectra assuming
  - A nonlinear sigma model description of ATs
  - Conformal invariance at ATs

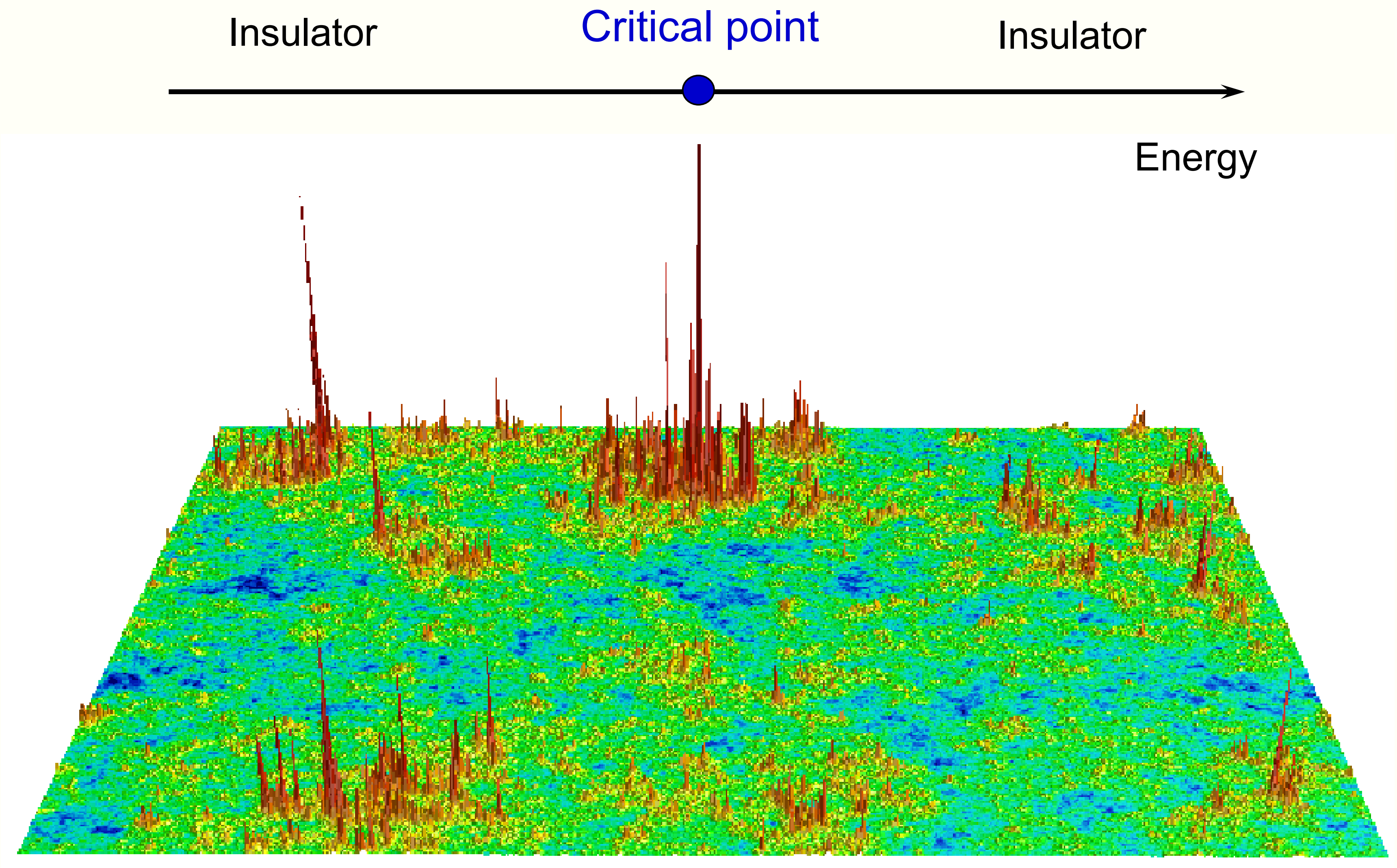
Numerics: wave functions across Anderson transitions

- Metal-insulator transition in 2D



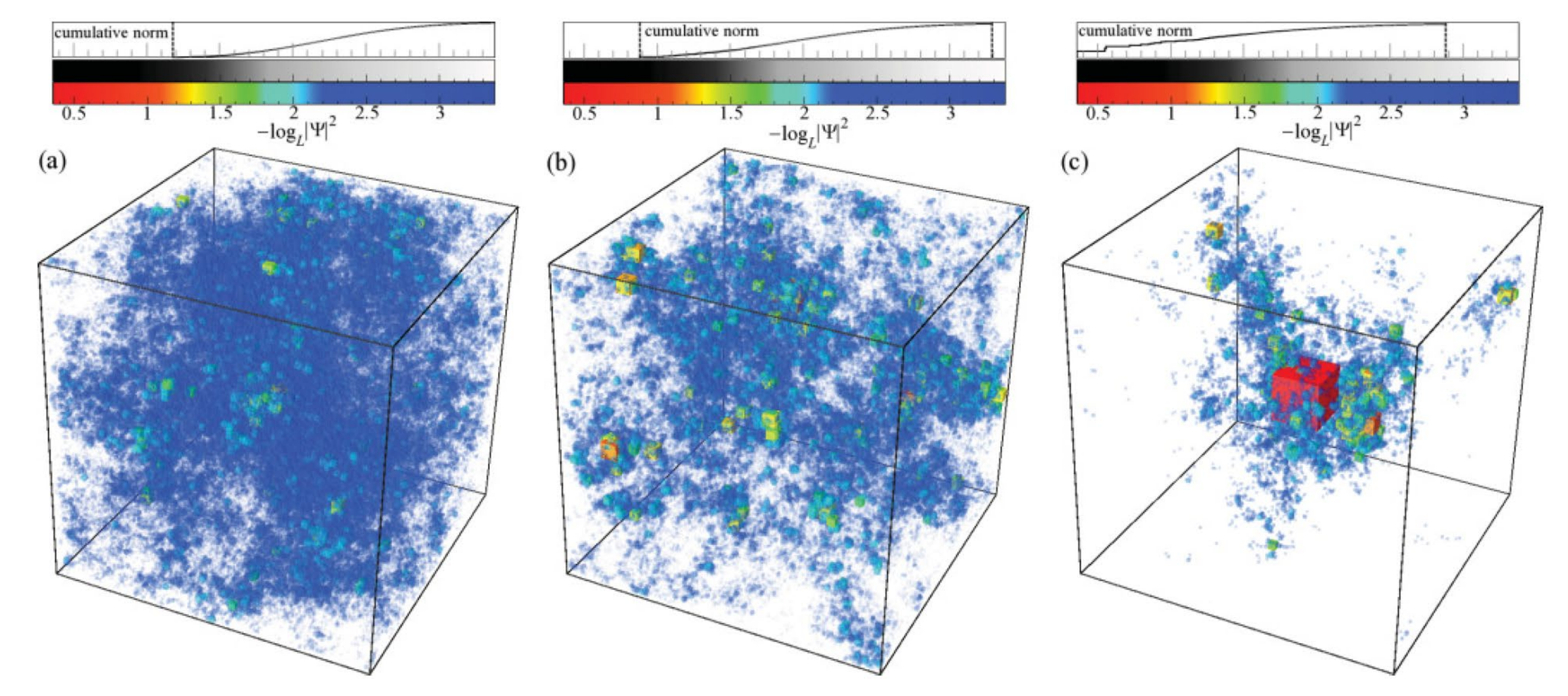
H. Obuse

Numerics: critical wave function at the IQH transition



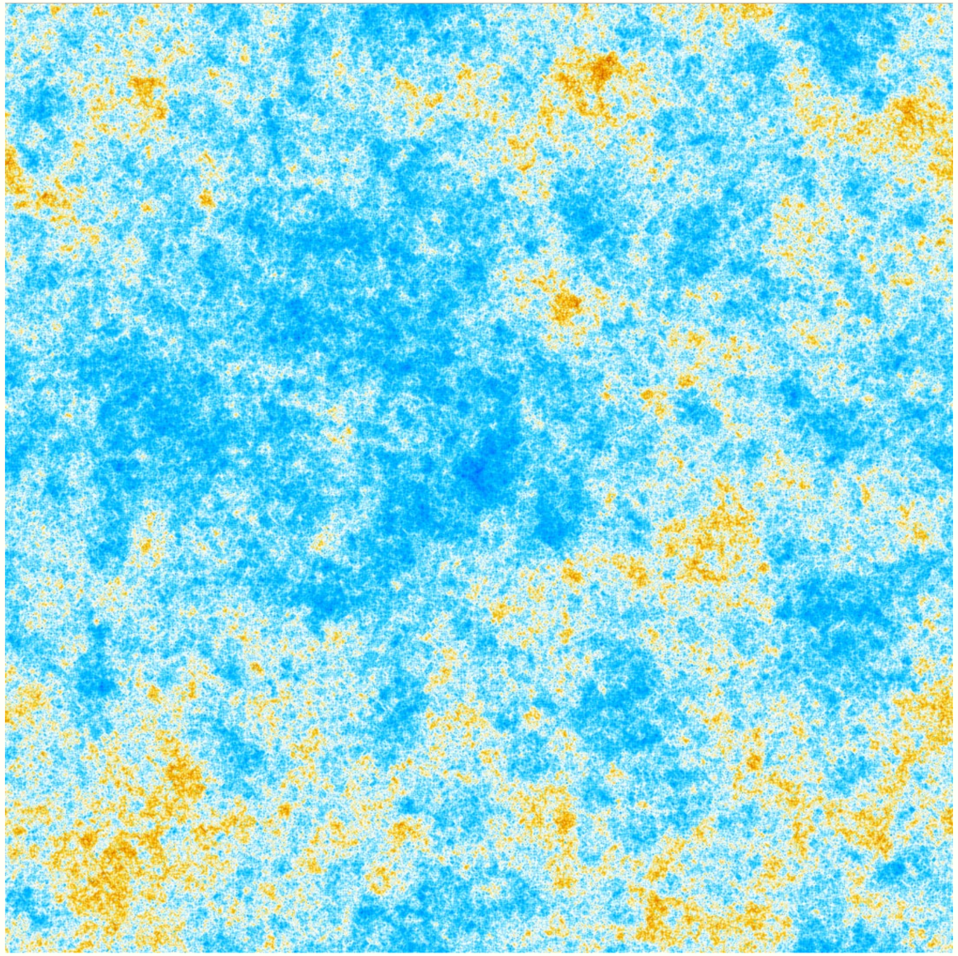
A. Mildenberger

Numerics: wave functions across a 3D Anderson transition



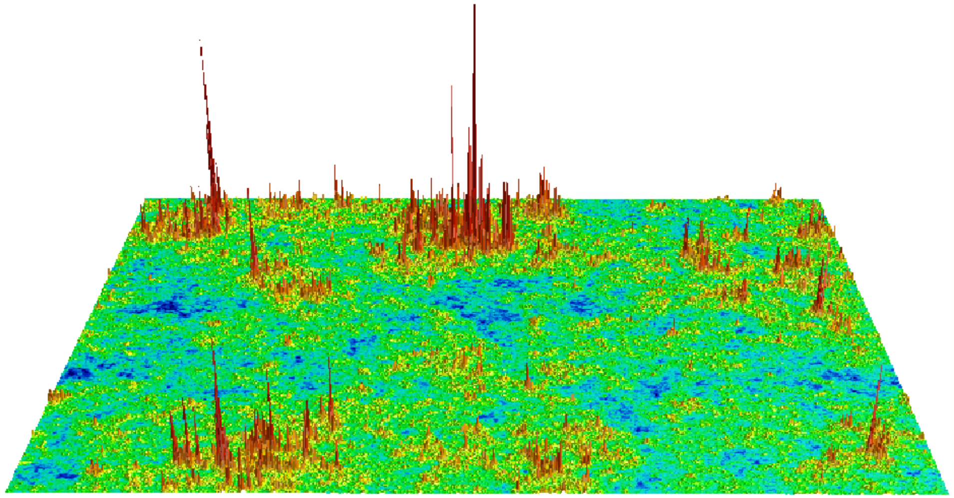
A. Rodriguez et al. *"Multifractal finite-size scaling and universality at the Anderson transition"* PRB 84,134209 (2011)

Wave function at the IQH transition



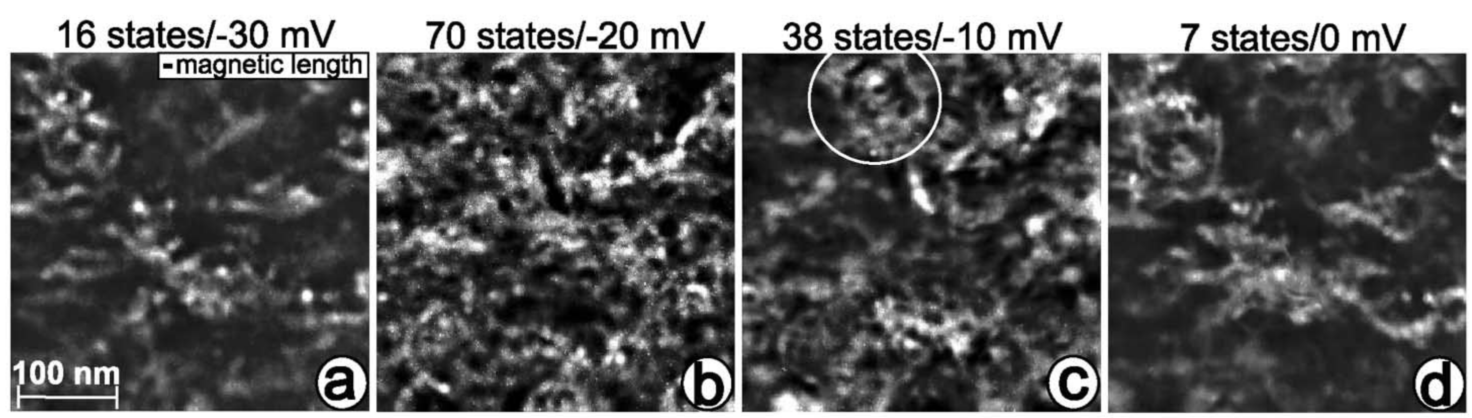
M. Puschmann, unpublished `22

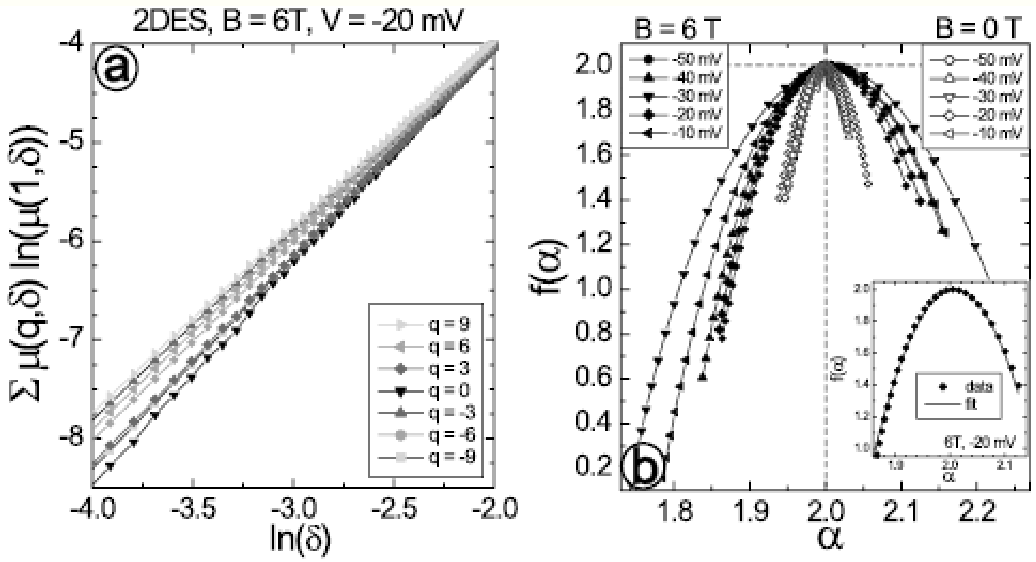
- Color = logarithm of $|\psi(\mathbf{r})|^2$
- Multifractal: sets of points with a given $|\psi(\mathbf{r})|^2$ are fractals with different fractal dimensions
- Self-similarity and scale invariance
- Histogram for $|\psi(\mathbf{r})|^2$
- Continuum of critical exponents



A. Mildenberger

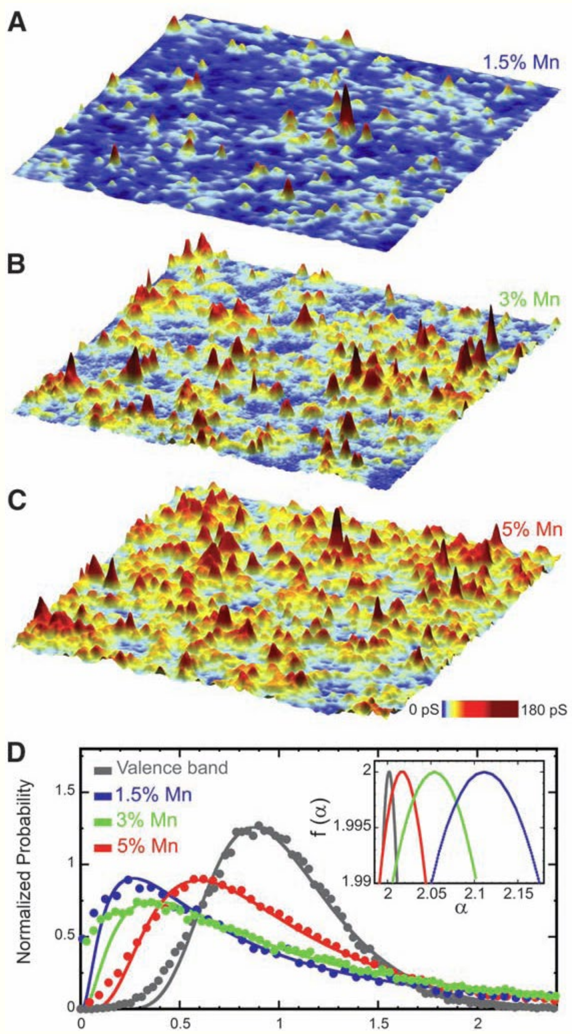
Experimental multifractality: IQH





M. Morgenstern et al, ‘03

Experimental multifractality: MIT



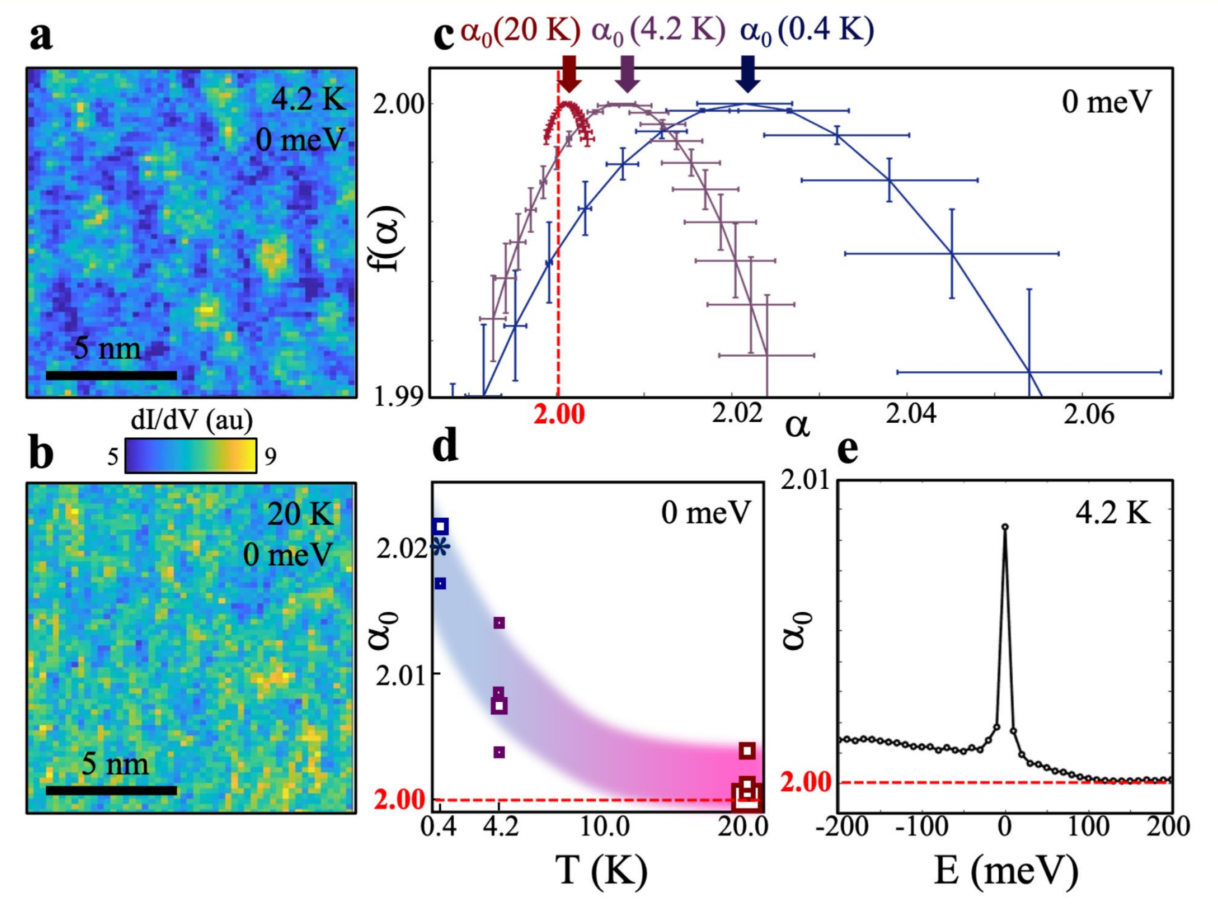
Profiles of the local density of states (LDOS) obtained by STM in $Ga_{1-x}Mn_xAs$

A. Richardella et al, ‘10

Multifractal spectra

Experimental multifractality: MIT

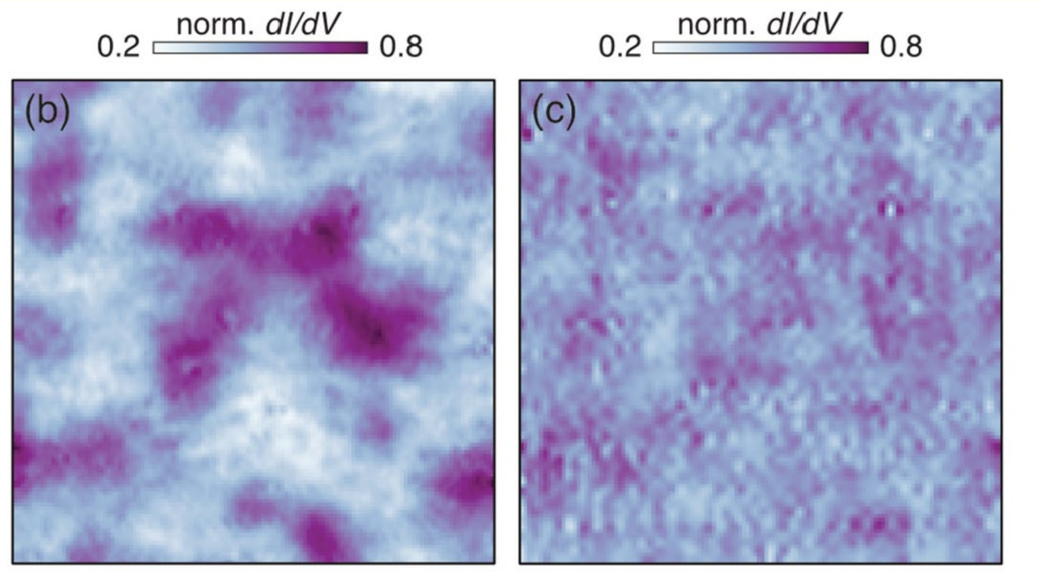
- LDOS of the ferromagnetic nodal line semimetal Fe_3GeTe_2

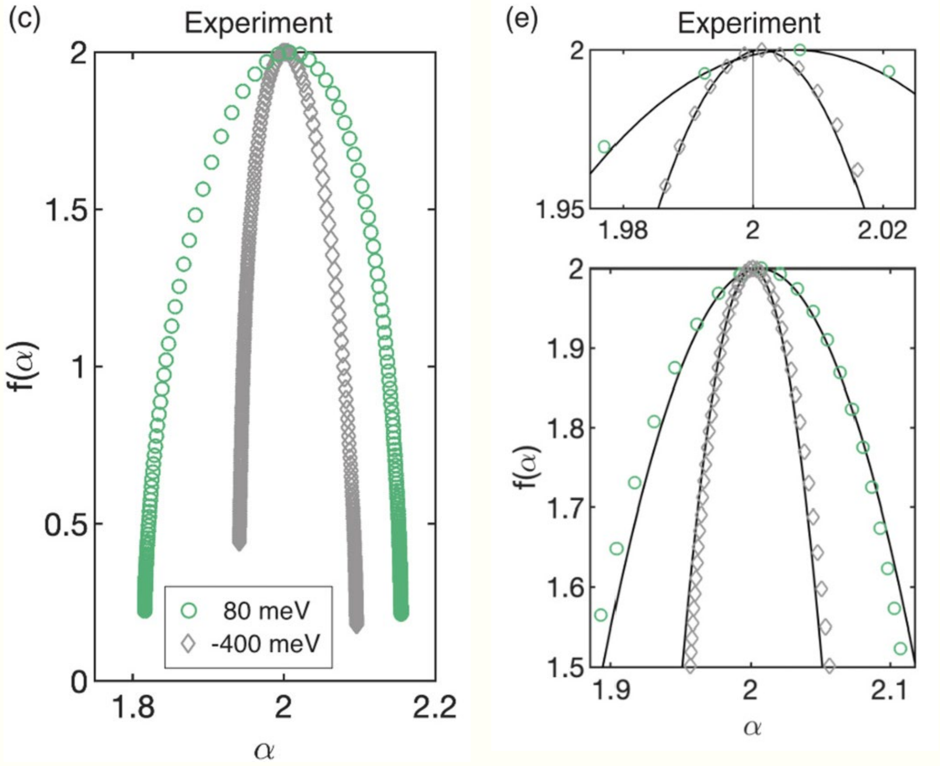


S. Mathimalar et al. *"Concurrent Multifractality and Anomalous Hall Response in the Nodal Line Semimetal* Fe_3GeTe_2 *Near Localization"* arXiv:2503.04367

Experimental multifractality: 2D Anderson transition

- LDOS of 2D alloy Bi_xPb_{1-x}/Ag(111)

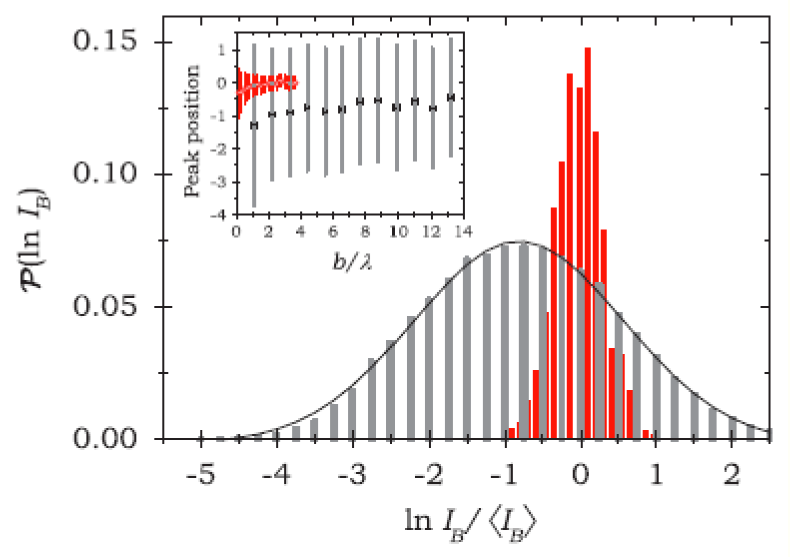




B. Jack et al. *"Visualizing the multifractal wave functions of a disordered two-dimensional electron gas"* PRR 3, 013022 (2021)

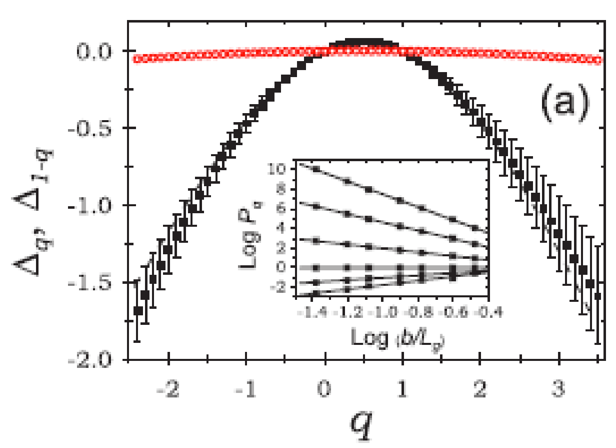
Experimental multifractality: localization of ultrasound

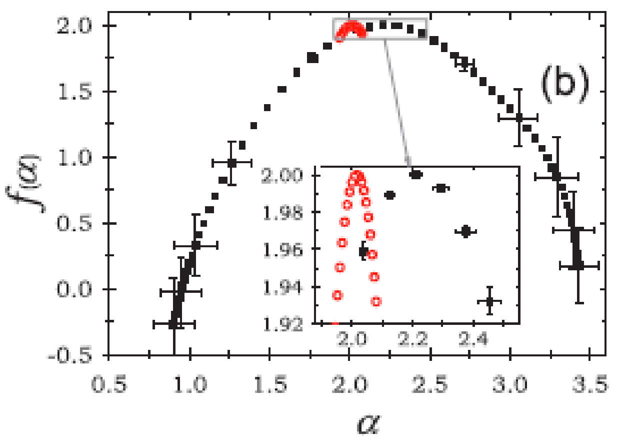
S. Faez et al, '11



Broad distribution of the sound intensity

Multifractal spectra





Multifractal spectrum

F. Wegner `80
C. Castellani and L. Peliti `86

- Critical wave functions are characterized by a continuum of multifractal (MF) exponents

- Moments of the wave function $$P_q = \int d^d r \, \overline{|\psi(r)|^{2q}} = L^d \overline{|\psi(r)|^{2q}}$$

- Scaling with the system size L $$P_q \sim \begin{cases} L^0, & \text{insulator} \\ L^{-\tau_q}, & \text{critical} \\ L^{-d(q-1)}, & \text{metal} \end{cases}$$

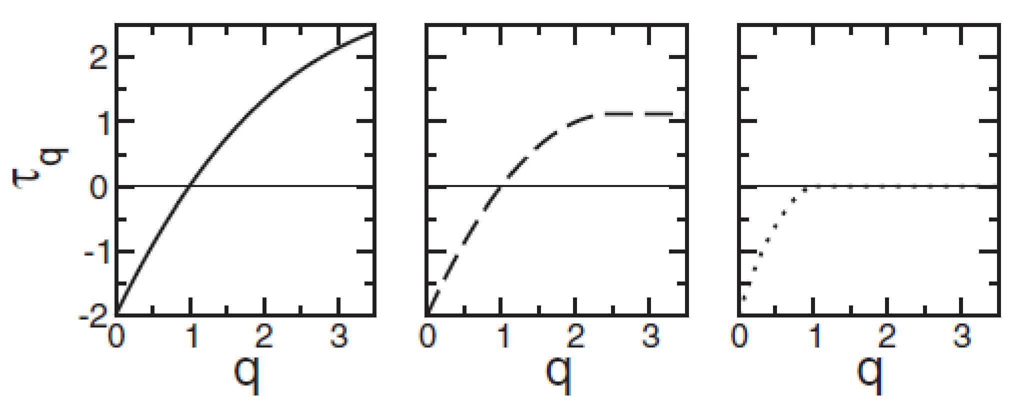
- Anomalous dimensions at the critical point $\tau_q = d(q-1) + \Delta_q$

- Broad probability distribution of critical wave function intensity $\mathcal{P}(|\psi|^2)$

- The wave function amplitude is not self-averaging

Multifractal measures: generalities

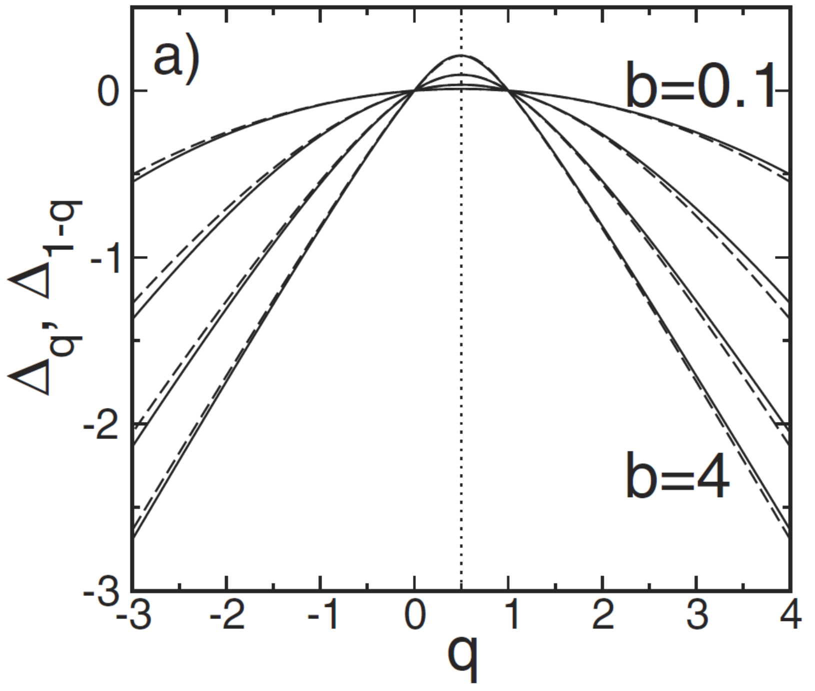
- Multifractals appear in diverse systems across nature
- Probability measure $d\mu$ with support in a cube (torus) of size L
- Divide the cube into N boxes B_i of size a, $N = \left(\frac{L}{a}\right)^d$
- Measure of each box $p_i = \int_{B_i} d\mu(\mathbf{r})$
- (Complex) moments of the measure scale with L/a $\quad P_q = \sum_{i=1}^{N} p_i^q \sim \left(\frac{L}{a}\right)^{-\tau_q}$
- Multifractal spectrum τ_q
  - τ_q is non-decreasing: $\tau_q' \geq 0$
  - τ_q is convex: $\tau_q'' \leq 0$
  - $\tau_0 = -d$ (dimension of the support)
  - $\tau_1 = 0$ (normalization of the measure)



Multifractal spectrum: limiting cases

- Deep in the metal wave functions are extended
  - Uniform measure $p_i = 1/N = (L/a)^{-d}, \quad P_q = N p_i^q = \left(\frac{L}{a}\right)^{-d(q-1)}$
  - MF spectrum is linear: $\tau_q = d(q-1)$
- Deep in the insulator wave functions are strongly localized
  - Measure localized in volume $\xi^d, \quad a < \xi < L$
  - $\left(\frac{\xi}{a}\right)^d$ boxes filled with $p_i = \left(\frac{\xi}{a}\right)^{-d}, \quad P_q = \left(\frac{\xi}{a}\right)^{-d(q-1)}$
  - P_q is independent of $L \quad \Rightarrow \quad \tau_q = 0$
- At the critical point define "anomalous dimensions"

$$\tau_q = d(q-1) + \Delta_q$$



A. Mirlin, Y. Fyodorov, A. Mildenberger, F. Evers `06

Multifractality and field theory

- Moments of the local density of states at the critical energy $\overline{\nu^q} \sim L^{-x_q}$

$$\tau_q = d(q-1) + x_q - qx_1$$

- x_q are scaling dimensions of operators $\mathcal{O}_q(r)$ in a field theory: $\langle \mathcal{O}_q(r) \rangle \sim L^{-x_q}$

- Roughly $\nu^q(r) \sim \mathcal{O}_q(r)$, more precisely $\overline{\nu^{q_1}(r_1) \ldots \nu^{q_n}(r_n)} = \langle \mathcal{O}_{q_1}(r_1) \ldots \mathcal{O}_{q_n}(r_n) \rangle$

- Precise correspondence exists for nonlinear sigma models of ATs

- Caveats

  - Sigma models are derived in the metal at weak disorder

  - Recent proposals suggest that symmetries of sigma models may break down at ATs M. Zirnbauer `23-24

Field theories for ATs: nonlinear sigma models

- Supersymmetric nonlinear sigma model

$$S[Q] \propto -\int d^d\mathbf{r}\,\mathrm{Str}[D(\nabla Q)^2 + 2i\omega\Lambda Q]$$

- $Q \in G/K$ (super)coset space: $Q^2 = 1$

- 10 Altland-Zirnbauer classes

- Critical points are strongly coupled!

- High degree of symmetry helps

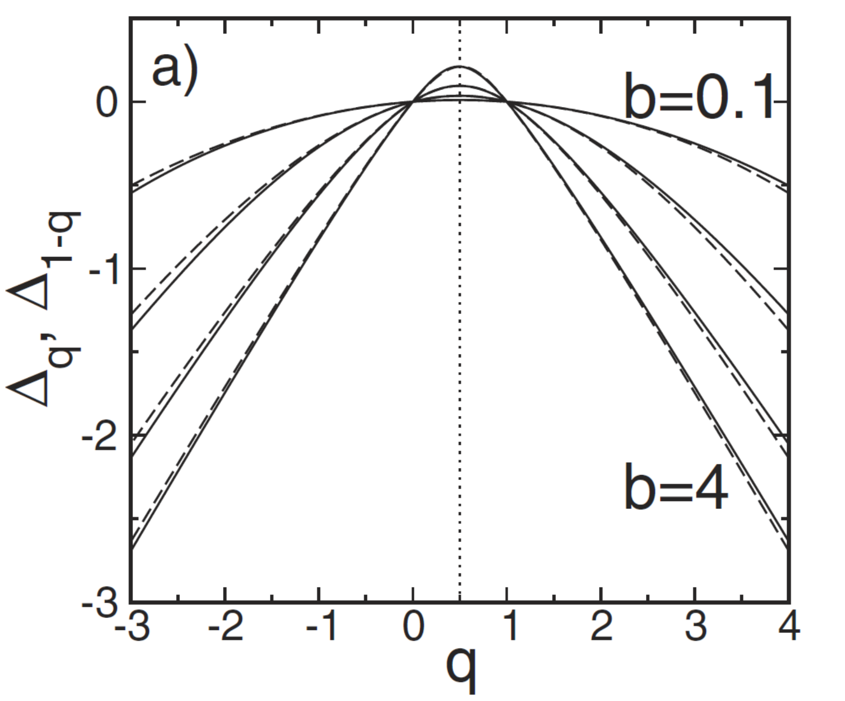
Symmetry Class	NLσM (n-c|c)	Compact (fermionic) space	Non-compact (bosonic) space
A (UE)	AIII|AIII	$\mathrm{U}(2N)/\mathrm{U}(N)\times\mathrm{U}(N)$	$\mathrm{U}(N,N)/\mathrm{U}(N)\times\mathrm{U}(N)$
AI (OE)	BDI|CII	$\mathrm{Sp}(4N)/\mathrm{Sp}(2N)\times\mathrm{Sp}(2N)$	$\mathrm{SO}(N,N)/\mathrm{SO}(N)\times\mathrm{SO}(N)$
AII (SE)	CII|BDI	$\mathrm{SO}(2N)/\mathrm{SO}(N)\times\mathrm{SO}(N)$	$\mathrm{Sp}(2N,2N)/\mathrm{Sp}(2N)\times\mathrm{Sp}(2N)$
AIII (chUE)	A|A	$\mathrm{U}(N)$	$\mathrm{GL}(N,\mathbb{C})/\mathrm{U}(N)$
BDI (chOE)	AI|AII	$\mathrm{U}(2N)/\mathrm{Sp}(2N)$	$\mathrm{GL}(N,\mathbb{R})/\mathrm{O}(N)$
CII (chSE)	AII|AI	$\mathrm{U}(N)/\mathrm{O}(N)$	$\mathrm{GL}(N,\mathbb{H})/\mathrm{Sp}(2N) \equiv \mathrm{U}^*(2N)/\mathrm{Sp}(2N)$
C (SC)	DIII|CI	$\mathrm{Sp}(2N)/\mathrm{U}(N)$	$\mathrm{SO}^*(2N)/\mathrm{U}(N)$
CI (SC)	D|C	$\mathrm{Sp}(2N)$	$\mathrm{SO}(N,\mathbb{C})/\mathrm{SO}(N)$
BD (SC)	CI|DIII	$\mathrm{O}(2N)/\mathrm{U}(N)$	$\mathrm{Sp}(2N,\mathbb{R})/\mathrm{U}(N)$
DIII (SC)	C|D	$\mathrm{O}(N)$	$\mathrm{Sp}(2N,\mathbb{C})/\mathrm{Sp}(2N)$

Exact results for MF operators

IAG, A. Ludwig, A. Mirlin, M. Zirnbauer `11, `13; J. Karcher, N. Charles, IAG, A. ; Mirlin `21

- $\mathcal{O}_q$ are exact scaling operators at critical points, with dimensions x_q: $\langle \mathcal{O}_q(r) \rangle \sim L^{-x_q}$
- There is $q_* > 0$ such that $x_{q_*} = 0$, non-trivial $\mathcal{O}_{q_*}$ with $\langle \mathcal{O}_{q_*} \rangle = 1$ (DOS in class A)
- Exact symmetry of MF spectra: $x_q = x_{q_* - q}$
- *Abelian fusion:* $\mathcal{O}_{q_1}\mathcal{O}_{q_2} \sim \mathcal{O}_{q_1+q_2} + \dots$

  (the ellipses denote derivatives also called *descendants*)
- These results
  - Follow from symmetries of sigma models
  - Do not require conformal invariance
  - Are fully supported by numerics



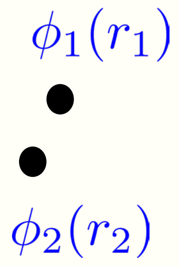
A. Mirlin, Y. Fyodorov, A. Mildenberger, F. Evers `06

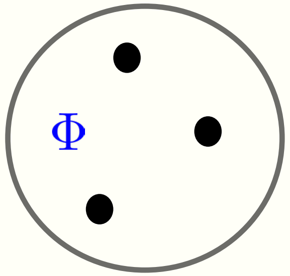
Operator product expansion

- Operator product expansion (OPE)

$$\langle \phi_1(r_1)\phi_2(r_2)\Phi\rangle = \sum_k f_{12k}\langle \phi_k(r)\Phi\rangle$$

$$\phi_1(r_1)\phi_2(r_2) = \sum_k f_{12k}\,\phi_k(r)$$





- Fusion rules: $\phi_1 \times \phi_2 \sim \sum_k \phi_k$
  - Ising model: $\sigma \times \sigma \sim I + \epsilon$ (microscopically, $\sigma_i^2 = 1$ and $-J\sigma_i\sigma_j =$ bond energy)
- OPEs (fusion rules) are strongly restricted by global symmetries
- OPEs are especially useful in conformal field theories

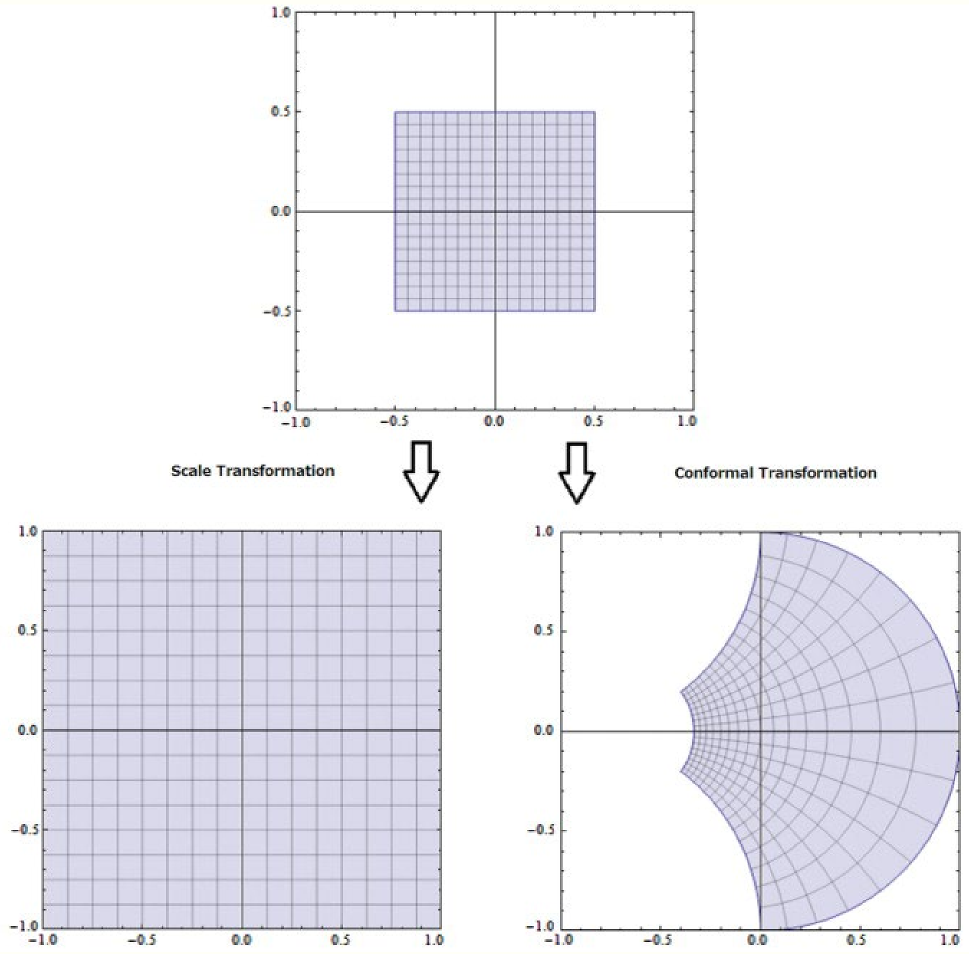
Scale versus conformal invariance

- Often (but not always) scale invariance is promoted to conformal invariance

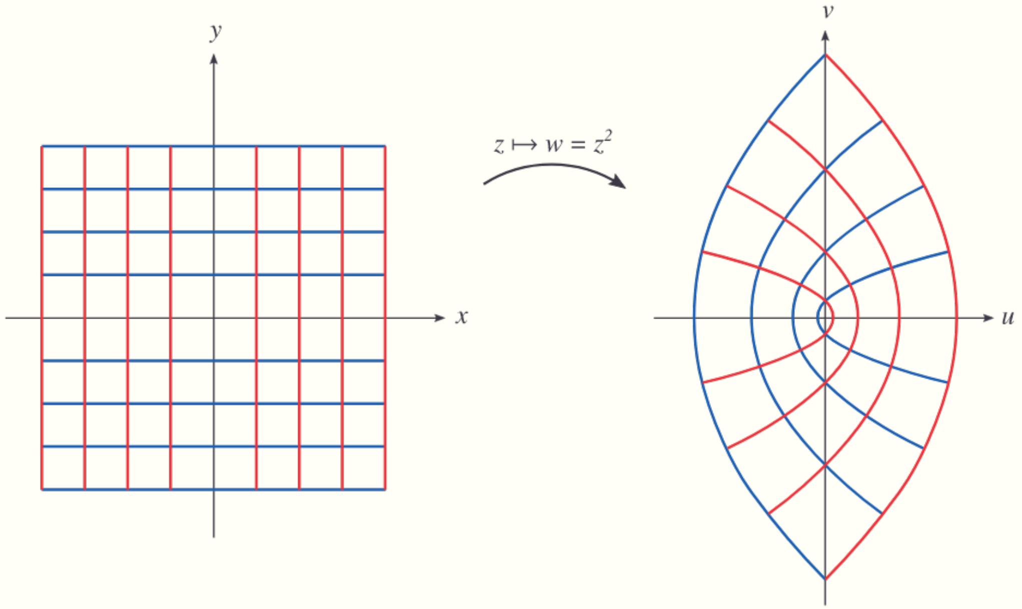
- This happens at conventional second-order phase transitions

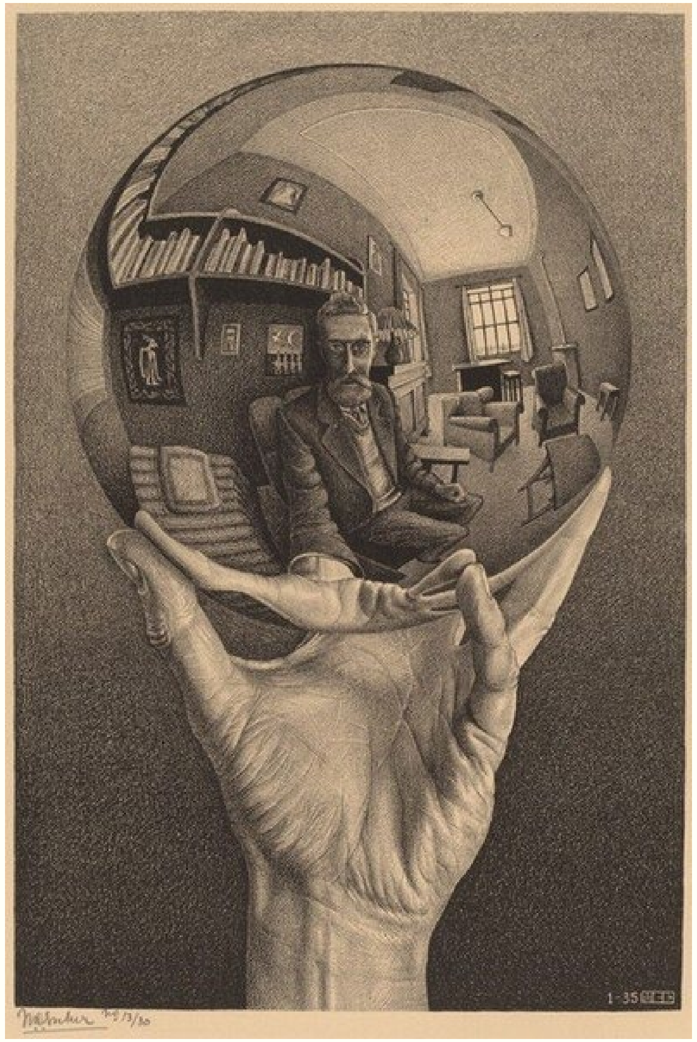
- Conformal transformations $r \to r'$ preserve angles: local rescaling and rotation

- Conformal invariance is very powerful



Conformal transformations in two and higher dimensions





- In 2D, any analytic function gives a conformal map
- Infinite-dimensional Virasoro symmetry
- In $d > 2$, conformal transformations form a finite-dimensional group, including inversions in spheres
- Basis of the conformal bootstrap method in CFT

CFT correlators

- *Primary* operators $\phi_i(r)$ with scaling dimensions x_i
  - Simple transformations under conformal transformations: $\phi_i'(r') = \Omega(r)^{-x_i} \phi_i(r)$
- Conformal symmetry constrains correlation functions of *primary* operators
  - One-point functions $\langle \phi_i(r) \rangle_{\mathrm{CFT}} = \delta_{x_i,0}$
  - Two-point functions $\langle \phi_1(r_1)\phi_2(r_2) \rangle_{\mathrm{CFT}} = \dfrac{\delta_{x_1,x_2}}{r_{12}^{2x_1}}$
  - Three-point functions $\langle \phi_1(r_1)\phi_2(r_2)\phi_3(r_3) \rangle_{\mathrm{CFT}} = \dfrac{f_{123}}{r_{12}^{x_1+x_2-x_3} r_{13}^{x_1+x_3-x_2} r_{23}^{x_2+x_3-x_1}}$
- CFT data: scaling dimensions x_i and structure constants f_{123} uniquely specify a CFT
- What about higher-point functions?

Conformal OPE and conformal blocks

- Conformal OPE is a series with a finite radius of convergence

$$\phi_i(r_1)\phi_j(r_2) = \sum_k f_{ijk}\hat{C}(r_1, r_2, r, \partial_r)\phi_k(r)$$

- Conformal symmetry fully determines the differential operator $\hat{C}(r_1, r_2, r, \partial_r)$

- Apply the OPE to the pairs 12 and 34

- Conformal block expansion $\langle \phi_1(r_1) \ldots \phi_4(r_4) \rangle_{\mathrm{CFT}} = \sum_k f_{12k} f_{34k} G_k^{(s)}(r_i)$

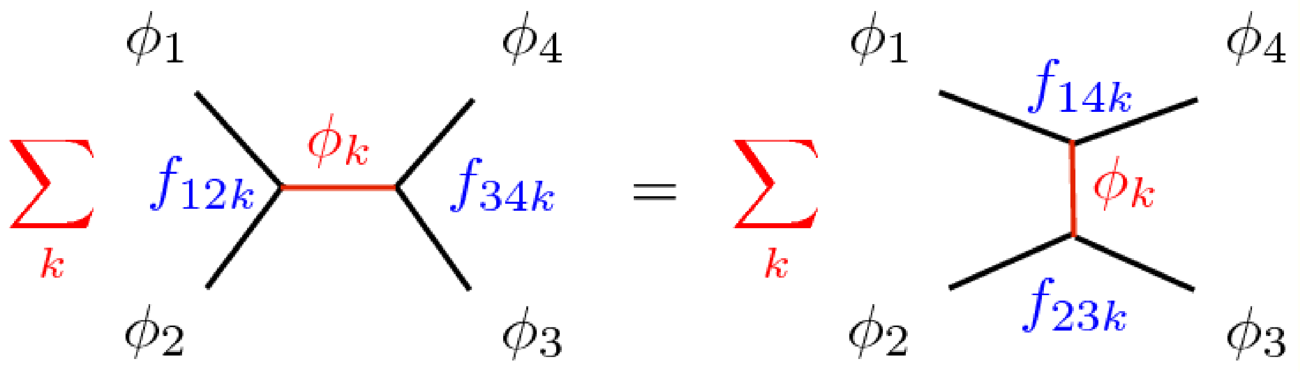
- Conformal blocks $G_k^{(s)}(r_i) = \hat{C}(r_1, r_2, r, \partial_r)\hat{C}(r_3, r_4, r', \partial_{r'})\langle \phi_k(r)\phi_k(r') \rangle$

  are completely fixed by conformal symmetry as long as the CFT data (x_i, f_{123}) are known

Solving a CFT: crossing symmetry

- The OPE can be done on different pairs of operators, leading to the crossing symmetry

$$\sum_k f_{12k} f_{34k} G_k^{(s)}(r_i) = \sum_k f_{14k} f_{23k} G_k^{(t)}(r_i)$$



- Basis for a non-perturbative bootstrap approach

- Any set of x_i, f_{ijk} consistent with crossing, defines a valid CFT

Two dimensions versus higher dimensions

- Two-dimensional CFTs are special: conformal symmetry is infinite-dimensional (Virasoro algebra)

A. Belavin, A. Polyakov, A. Zamolodchikov `84

- This allows to solve crossing equations exactly in many cases
  - Rational CFTs: a finite number of *Virasoro* conformal blocks in all 4-point functions
  - Abelian CFT: a single conformal block in each 4-point function

- In higher dimensions the group of conformal transformations is finite-dimensional
  - Infinity of *global* conformal blocks in any 4-point function

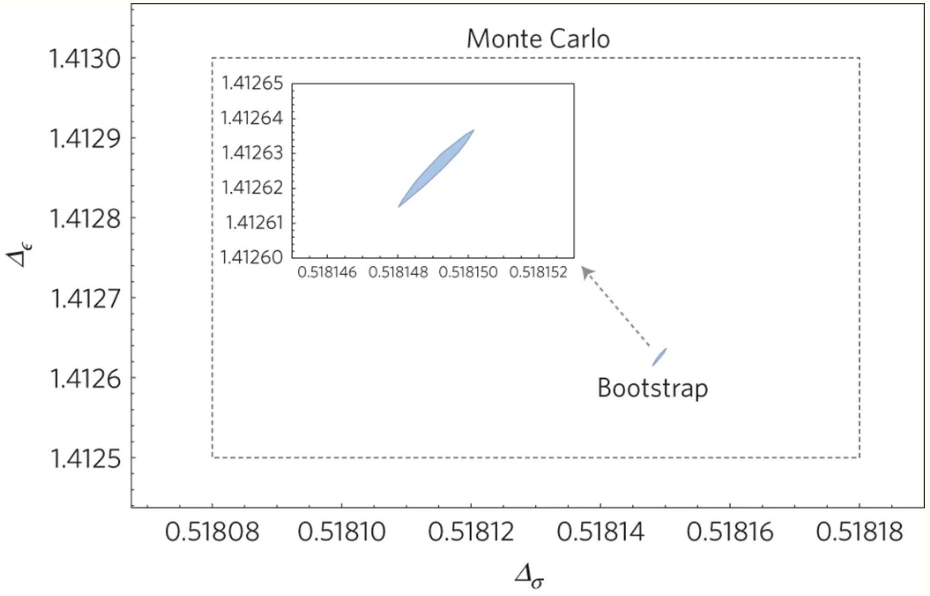
- Progress in higher-dimensional CFTs was slow until 2008

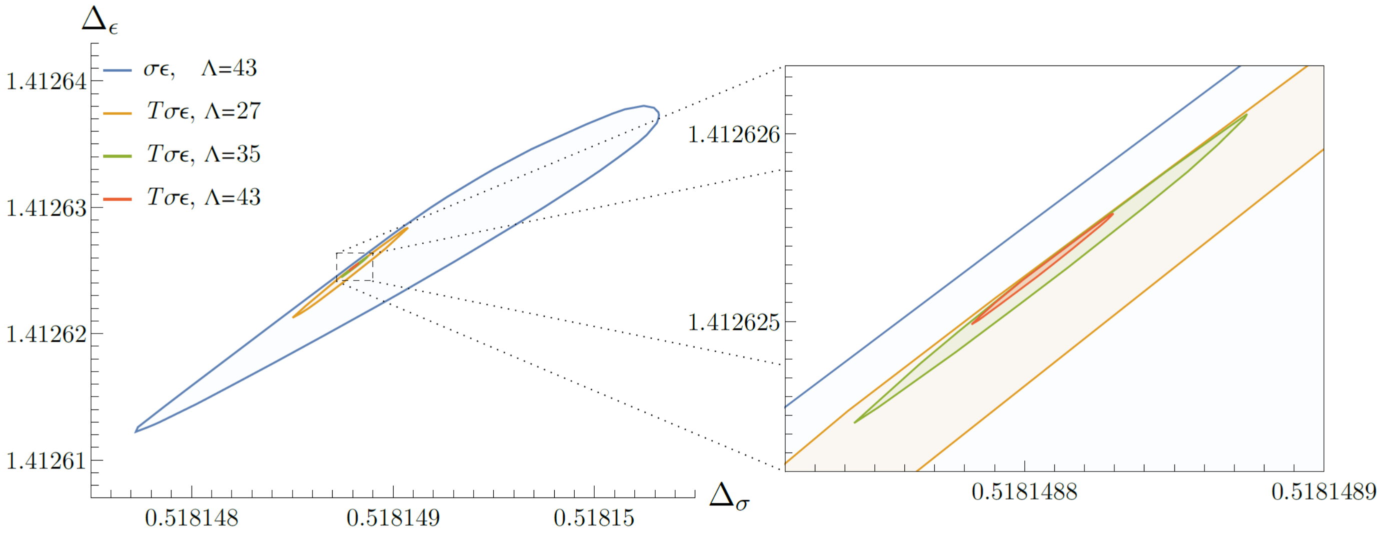
- Conformal bootstrap revival: highly efficient numerical and analytical methods

S. Rychkov et al `08-present

Conformal bootstrap successes

- Precise and rigorous bounds for critical exponents
- 3D Ising model





Kos, Poland, Simmons-Duffin, Vichi \`16

Chang et al. \`24

- 2016: $\Delta_\sigma = 0.5181489(10)$, $\Delta_\epsilon = 1.412625(10)$
- 2024: $\Delta_\sigma = 0.518148806(24)$, $\Delta_\epsilon = 1.41262528(29)$

Conformal bootstrap successes

- Precise and rigorous bounds for critical exponents
- 3D O(2)-model

Conformal bootstrap and the λ-point specific heat experimental anomaly

Carving out OPE space and precise O(2) model critical exponents
Authors: S. M. Chester, W. Landry, J. Liu, D. Poland, D. Simmons-Duffin, N. Su, A. Vichi
arXiv:1912.03324

Recommended with a Commentary by Slava Rychkov, IHES

$\nu^{\mathrm{MC}} = 0.67169(7)$

$\nu^{\mathrm{EXP}} = 0.6709(1)$

$\nu^{\mathrm{CB}} = 0.67175(10)$

$\nu^{\mathrm{RG}} = 0.6703(15)$

- Can we apply the power of CFT to Anderson transitions?

Crossing symmetry and Abelian OPE in 2D

- *Assumptions:* 1) conformal invariance, 2) $\mathcal{O}_q$ are *Virasoro* primaries, 3) Abelian OPE

- Only one Virasoro conformal block appears, crossing equations simplify a lot and lead to

$$x_{q_1+q_2+q_3} - x_{q_1+q_2} - x_{q_1+q_3} - x_{q_2+q_3} + x_{q_1} + x_{q_2} + x_{q_3} = 0$$

D. C. Lewellen `89

- This implies *exactly parabolic* multifractal spectra: $x_q = bq(q_* - q)$

R. Bondesan, D. Wieczorek, M. Zirnbauer `16

- $\mathcal{O}_q$ are vertex operators of a Gaussian free field, their correlators are those of a *Coulomb gas* CFT

- If we can demonstrate (analytically or numerically) that x_q is not parabolic, one of the assumptions must be wrong

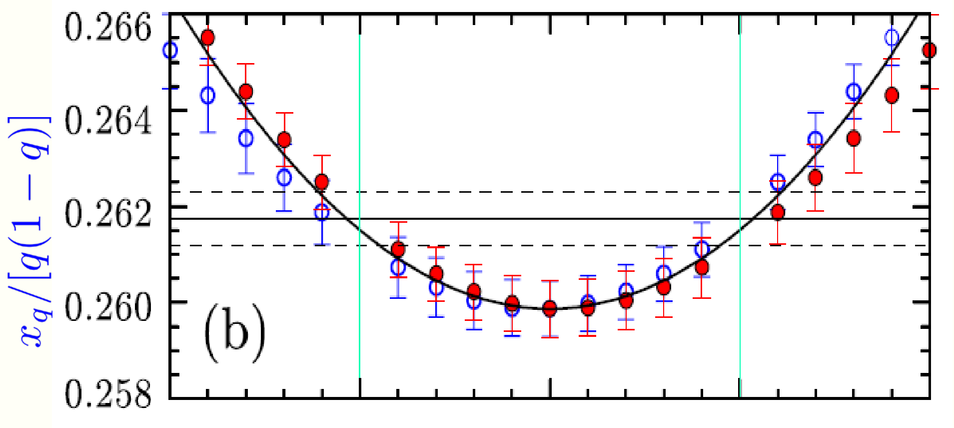
Recent results: IQH transition

M. Zirnbauer `19, 21

- Proposed CFT for the IQH transition: $\mathrm{GL}(r|r)_{n=4}$ WZW model with a marginal perturbation

- Includes a sector with a free boson, and predicts $x_q = \frac{1}{4}q(1-q)$

- Unconventional critical properties, most numerical results do not agree with this proposal



H. Obuse et al `08

- A lot of ongoing work on IQH transition

- Recent results rule out exact parabolicity

F. Evers et al `22-26

Multifractality at the SQH transition

- Class C (SQH transition): mapping to classical percolation

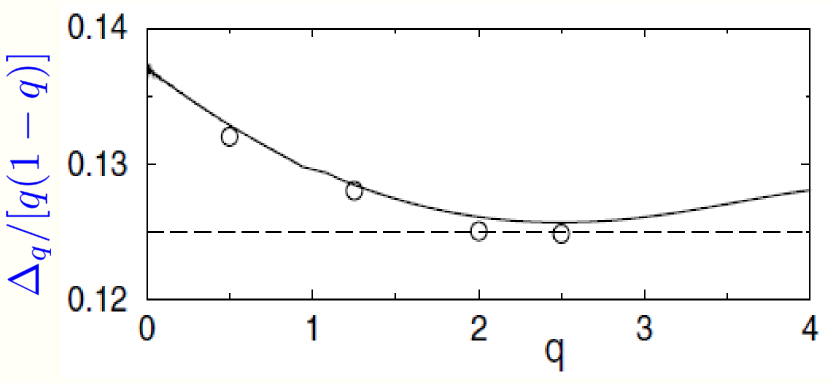
IAG, A. Ludwig, N. Read `99
A. Mirlin, F. Evers, A. Mildenberger `03

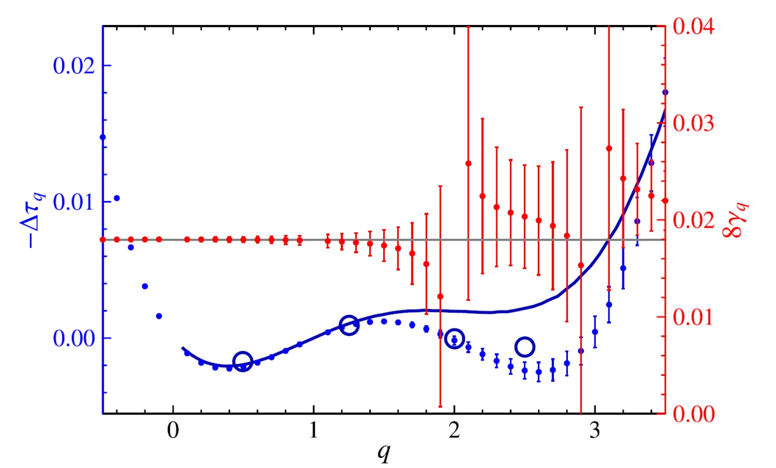
- Exact MF exponents $x_0 = x_3 = 0, \quad x_1 = x_2 = 1/4$

- These four values lie on the parabola $x_q = q(3-q)/8$

A. Mirlin, F. Evers, A. Mildenberger `03
M. Puschmann et al `21

- Numerical results seem to rule out exact parabolicity





Exact results: generalized MF scaling operators

D. Höf and F. Wegner `86-87; IAG, A. Mirlin, M. Zirnbauer `13

- Class A: determinants of critical wave functions at close points

$$P_{(1^p)}(r_1,\ldots,r_p) = \left|\det\begin{pmatrix}\psi_1(r_1) & \cdots & \psi_1(r_p)\\ \vdots & \ddots & \vdots \\ \psi_p(r_1) & \cdots & \psi_p(r_p)\end{pmatrix}\right|^2$$

- Generalized MF observables $\mathcal{P}_\lambda = P_{(1)}^{q_1-q_2} P_{(1^2)}^{q_2-q_3} \cdots P_{(1^{n-1})}^{q_{n-1}-q_n} P_{(1^n)}^{q_n}, \quad \lambda = (q_1, q_2, \ldots, q_n)$

- Sigma-model scaling operators $\mathcal{P}_\lambda \leftrightarrow \mathcal{O}_\lambda = d_1^{q_1-q_2} d_2^{q_2-q_3} \ldots d_n^{q_n}$

- d_m are are constructed via the Iwasawa decomposition of the sigma-model field Q

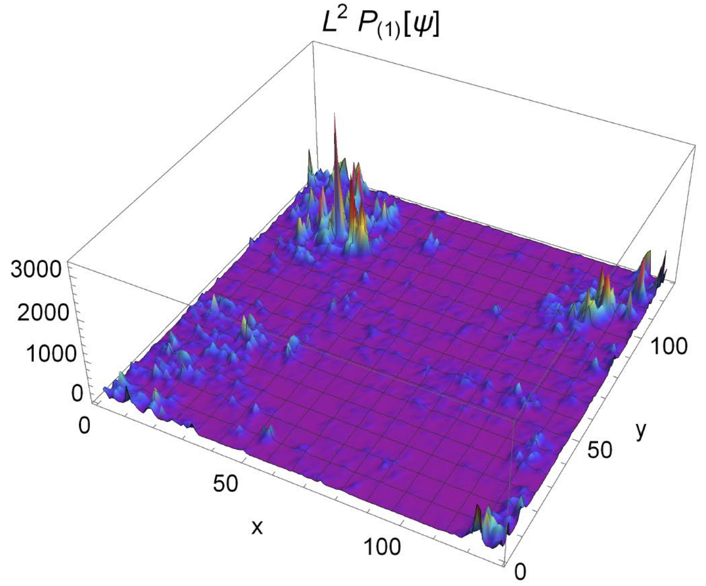
- General symmetry relation $x_\lambda = x_{w(\lambda)}$

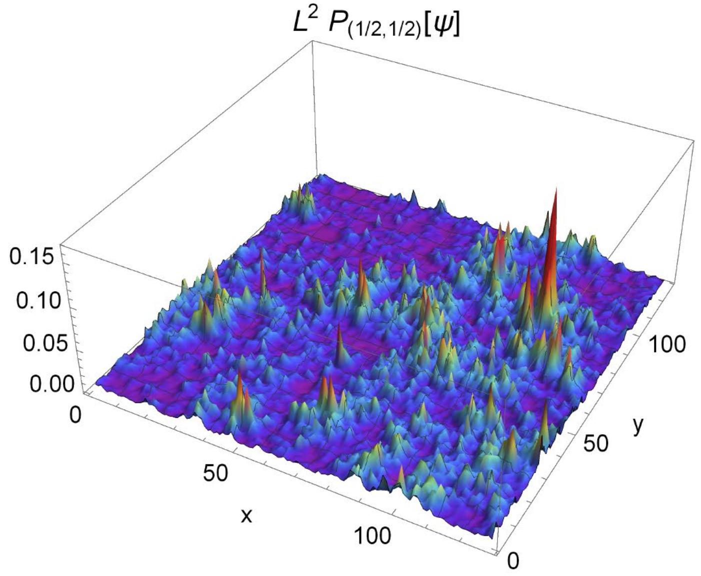
J. Karcher, N. Charles, IAG, A. Mirlin `21
J. Karcher, IAG, A. Mirlin `22

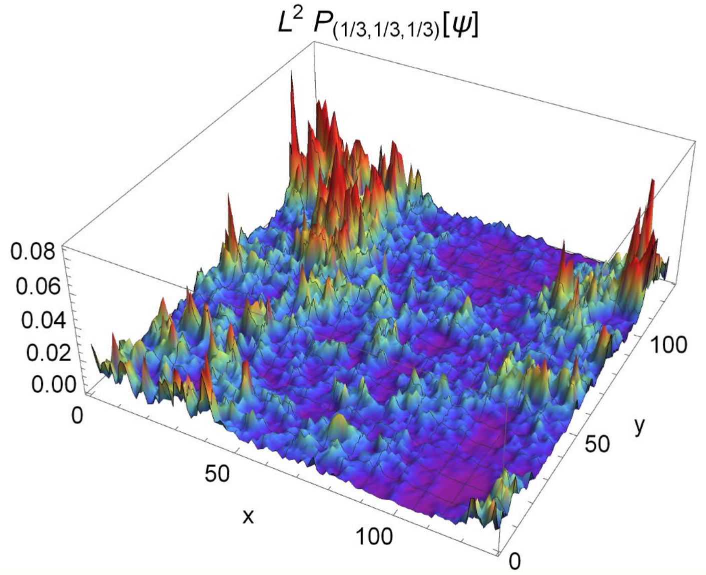
- Recent construction for all AZ symmetry classes

Generalized multifractals

J. Karcher `22







Example of spatial distribution of building blocks
$L^2 P_{(1)}[\psi]$, $\quad L^2 \left(P_{(1,1)}[\psi]\right)^{1/2}$,
and $\; L^2 \left(P_{(1,1,1)}[\psi]\right)^{1/3}$
for 2D metal-insulator transition of class AII

Generalized MF as test of conformal invariance

J. Karcher, N. Charles, IAG, A. Mirlin `21

- Generalized MF operators still satisfy Abelian fusion $\mathcal{O}_\lambda \mathcal{O}_{\lambda'} \sim \mathcal{O}_{\lambda+\lambda'} + \dots$

- Assuming conformal invariance, in 2D get the generalized parabolicity

$$x_\lambda = -b \textstyle\sum_i q_i (q_i + c_i)$$

- Numbers c_i have a group-theoretic origin, known for all AZ classes

- Compare with exact analytical and numerical results for the SQH transition

Generalized MF at SQH transition via percolation

J. Karcher, IAG, A. Mirlin `22

- All generalized MF averages $\langle \mathcal{O}_\lambda \rangle$ with $|\lambda| \leq 3$ can be obtained from mapping to percolation
- Get $x_{(1)}, x_{(2)}, x_{(1,1)}, x_{(3)}, x_{(2,1)}, x_{(1,1,1)}$ in terms of known *n*-hull exponents
- Also obtain the most irrelevant scaling operators $\langle \mathcal{O}_\lambda \rangle$ with $\lambda = (1^n)$
- This gives $x_{(1^n)} = x_n^{\text{hull}} = (4n^2 - 1)/12$
- The generalized parabolicity would give $x_{(1^n)}^{\text{para}} = n^2/4 \neq x_{(1^n)}$
- All results are in perfect agreement with Weyl symmetry relations and extensive numerics
- Strong violation of generalized parabolicity!

Generalized MF at SQH transition

J. Karcher, IAG, A. Mirlin `22

λ	x_λ^{perc}	x_λ^{qn}	x_λ^{para}
(1)	$x_1^{\text{h}} = 1/4 = 0.25$	—	$1/4$
(2)	$x_1^{\text{h}} = 1/4 = 0.25$	0.249 ± 0.001	$1/4$
(1,1)	$x_2^{\text{h}} = 5/4 = 1.25$	1.251 ± 0.001	1
(3)	0	0.004 ± 0.004	0
(2,1)	$x_2^{\text{h}} = 5/4 = 1.25$	1.249 ± 0.002	1
(1^3)	$x_3^{\text{h}} = 35/12 \simeq 2.917$	2.915 ± 0.002	$9/4$
(4)	—	-0.49 ± 0.02	$-1/2$
(3,1)	—	0.985 ± 0.007	$3/4$
(2,2)	—	1.865 ± 0.006	$3/2$
(2,1,1)	$x_3^{\text{h}} = 35/12 \simeq 2.917$	2.911 ± 0.005	$9/4$
(1^4)	$x_4^{\text{h}} = 21/4 = 5.25$	5.242 ± 0.004	4
(5)	—	-1.19 ± 0.06	$-5/4$
(4,1)	—	0.48 ± 0.03	$1/4$
(3,2)	—	1.59 ± 0.02	$5/4$
(3,1,1)	—	2.64 ± 0.02	2
(2,2,1)	—	3.50 ± 0.02	$11/4$
$(2, 1^3)$	$x_4^{\text{h}} = 21/4 = 5.25$	5.23 ± 0.01	4
(1^5)	$x_5^{\text{h}} = 33/4 = 8.25$	8.16 ± 0.01	$25/4$

- Excellent agreement of numerical values x_λ^{qn} with analytical results x_λ^{perc} (from mapping to percolation)
- Weyl symmetry holds nicely
- Generalized parabolicity (x_λ^{para}, last column) strongly violated

Crossing symmetry and Abelian OPE in any dimension

J. Padayasi, IAG `23

- In any dimension, we *assume:* 1) conformal invariance, 2) $\mathcal{O}_\lambda$ are *global* primaries

- Infinity of global conformal blocks in any OPE requires us to generalize Abelian fusion

- Get the Lewellen constraint for $\lambda_2 = \lambda_1$

$$x_{2\lambda_1+\lambda_3} - x_{2\lambda_1} - 2x_{\lambda_1+\lambda_3} + 2x_{\lambda_1} + x_{\lambda_3} = 0$$

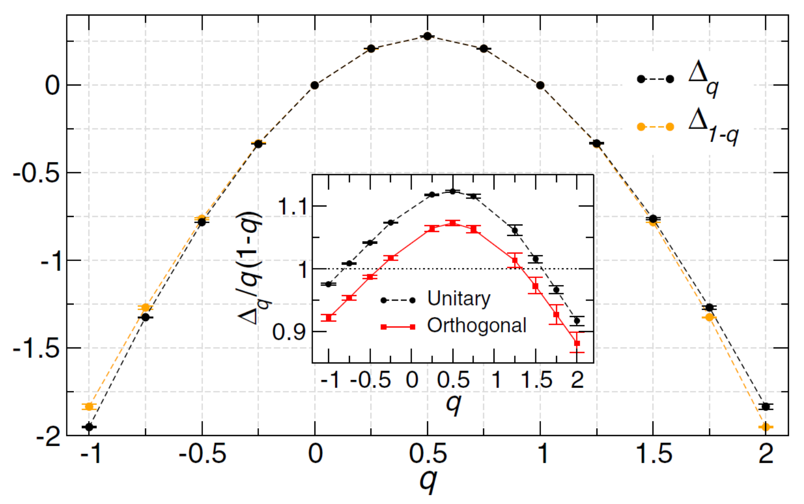
- This implies *exactly parabolic* multifractal spectrum:

$$x_q = bq(q_* - q), \qquad x_\lambda = -b\sum_i q_i(q_i + c_i)$$

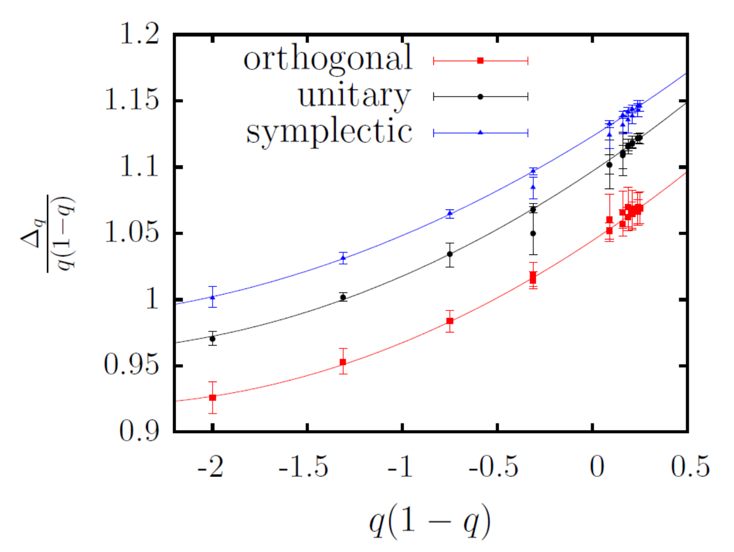
- If we can demonstrate (analytically or numerically) that x_λ is not parabolic, one of the assumptions must be wrong

Known results deviate from parabolicity

- Numerical multifractal spectra in 3, 4, 5, and 6 dimensions are non-parabolic



J. Lindinger and A. Rodríguez `17



L. Ujfalusi and I. Varga `15

- Analytical results in $d = 2 + \epsilon$ are not parabolic

$$x_q = q(1-q)\epsilon + \frac{\zeta(3)}{4} q(1-q)[q(1-q) - 1]\epsilon^4 + O(\epsilon^5)$$

- Analytical results for $d \gg 1$ are strongly non-parabolic (triangle) $x_q \approx \frac{1}{2} d(q_* - |2q - q_*|)$

Conundrum

- Nonlinear sigma models and their symmetries lead to Abelian fusion
- Together with (assumed) conformal invariance at ATs this implies parabolic MF spectra in all dimensions
- Known results for MF spectra (numerical and analytical) rule out exact parabolicity
- Conventional description of ATs using sigma models leads to results that are inconsistent with conformal invariance at the transitions!
- Dichotomy:

  1. ATs are scale invariant but not conformally invariant (logically possible)
  2. ATs are conformally invariant but not described by conventional sigma models. Sigma model symmetries may be spontaneously broken close to or at the fixed point

Conclusions and outlook

- Anderson transitions (including IQH transition) remain interesting and mysterious

- Sigma model description of Anderson transitions leads to results that seem to be inconsistent with conformal invariance

- Alternative approaches (more microscopic and exact) are necessary to resolve the puzzle

Thank you!