Lecture on quantum entanglement in condensed matter systems

Shinsei Ryu

University of Chicago

January 12, 2018
Overview

- Quantum entanglement as an “order parameter”
  - SPT phases (free systems)
  - (1+1)d CFTs
  - Perturbed CFTs
  - (2+1)d topologically ordered phases
  - ...

- Developing theoretical/computational tools:
  - DMRG, MPS, PEPS, MERA, and other tensor networks

- Other applications – ETH and many-body localization, thermalization and chaos in dynamical systems, etc.

- Applications to physics of spacetime
Phases of matter

- Symmetry protected topological phases (SPT phases)
- Topologically ordered phases
  - Long-range entangled states
  - Trivial phases
    - Short-range entangled states (a.k.a "invertible" states)
  - Symmetry
    - Symmetry protected topological phases (SPT phases)
- Quantum disordered phases
  - Gapped quantum disordered phases
  - Gapped quantum critical disordered phases and critical points
  - Symmetry
    - Symmetry enriched topological phases (SET phases)
- Ordered phases
  - Continuous symmetry-broken phases
  - Discrete symmetry-broken phases
- No spontaneous symmetry breaking
  - Gapped quantum disordered phases
Entanglement and entropy of entanglement

• (0) States of your interest, e.g., \( \rho_{\text{tot}} = |\Psi\rangle\langle \Psi| \).

• (i) Bipartition Hilbert space \( \mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \).

• (ii) Partial trace:

\[
\rho_A = \text{Tr}_B |\Psi\rangle\langle \Psi| = \sum_j p_j |\psi_j\rangle_A \langle \psi_j|_A \quad (\sum_j p_j = 1) \tag{1}
\]

• (iii) von Neumann Entanglement entropy:

\[
S_A = -\text{Tr}_A [\rho_A \ln \rho_A] = -\sum_j p_j \ln p_j \tag{2}
\]

• (iv) Entanglement spectrum \( \rho_A \propto \exp(-H_e)/Z \):

\[
\{\xi_i\} \quad \text{where} \quad p_i =: \exp(-\xi_i)/Z \tag{3}
\]
• Mutual information:

\[ I_{A:B} \equiv S_A + S_B - S_{A \cup B} \]  \hspace{1cm} (4)

• Rényi entropy:

\[ R_A^{(q)} = \frac{1}{1 - q} \ln(\text{Tr} \rho_A^q). \]  \hspace{1cm} (5)

Note that \( S_A = \lim_{q \to 1} R_A^{(q)}. \) \( \{ R_A^{(q)} \} = \text{entanglement spectrum}. \)

• The Rényi mutual information:

\[ I_{A:B}^{(q)} \equiv R_A^{(q)} + R_B^{(q)} - R_{A \cup B}^{(q)} \]  \hspace{1cm} (6)

• Other entanglement measures, e.g., entanglement negativity.
Some key properties

• If $\rho_{\text{tot}}$ is a pure state and $B = \bar{A}$, $S_A = S_B$.

• If $\rho_{\text{tot}}$ is a mixed state (e.g., $\rho_{\text{tot}} = e^{-\beta H}$), $S_A \neq S_B$ even when $B = \bar{A}$,

• If $B = \emptyset$, $S_A = S_{\text{thermal}}$.

• Subadditivity:

$$S_{A+B} \leq S_A + S_B.$$  \hspace{1cm} (7)

i.e., the positivity of the mutual information:

$I_{A:B} = S_A + S_B - S_{A+B} \geq 0$.

• Strong subadditivity

$$S_B + S_{ABC} \leq S_{AB} + S_{BC}$$ \hspace{1cm} (8)

By setting $C = \emptyset$, we obtain the subadditivity relation.
ES in non-interacting systems

- Consider the ground states $|GS\rangle$ of free (non-interacting) systems, and bipartitioning $\mathcal{H} = \mathcal{H}_L \otimes \mathcal{H}_R$.

- When $\rho_{tot} = |GS\rangle\langle GS|$ is a Gaussian state, $H_e$ is quadratic [Pesche (02)].

\[
H_e = \sum_{I,J \in L} \psi_I^\dagger K_{IJ} \psi_J, \quad I = r, \sigma, i, \ldots
\]  

(9)

- $H_e$ can be reconstructed from 2pt functions: $C_{IJ} := \langle GS | \psi_I^\dagger \psi_J | GS \rangle$.

\[
C = \begin{pmatrix}
    C_L & C_{LR} \\
    C_{RL} & C_R
\end{pmatrix}, \quad C_{RL} = C_{LR}^\dagger
\]  

(10)

- Correlation matrix is a projector:

\[
C^2 = C, \quad Q^2 = 1 \quad (Q_{IJ} := 1 - 2C_{IJ}).
\]  

(11)

- Entanglement Hamiltonian:

\[
H_e = \sum_{I,J \in L} \psi_I^\dagger K_{IJ} \psi_J, \quad K = \ln[(1 - C_L)/C_L].
\]  

(12)
E.g. the integer quantum Hall effect

- A prototype of topological phases
- Characterized by quantized Hall conductance $\sigma_{xy} = (e^2/h) \times \text{(integer)}$.
- Gapped bulk, gapless edge
- Robust against disorder and interactions
- Chiral edge states in ES

Figure: Physical v.s. entanglement spectra of a Chern insulator [SR-Hatsugai (06)]
E.g. the SSH model

- 1d lattice fermion model:

\[ H = t \sum_i (a_i^\dagger b_i + h.c.) + t' \sum_i (b_i^\dagger a_{i+1} + h.c.) \]  

(13)

- Phase diagram:

- Physical spectrum, entanglement spectrum, entanglement entropy.

Figure: [SR-Hatsugai (06)]
Symmetry-protected degeneracy in ES

- Robust zero mode in ES; 2-fold degeneracy for each level.

\[ S_A = A \log \frac{\xi}{a_0} + \log 2 \]

- Degeneracy is symmetry-protected; Symmetry: \( a_i \rightarrow a_i^\dagger \), \( b_i \rightarrow -b_i^\dagger \). (Class D or AIII/BDI topological insulator)

- Symmetry-protected degeneracy is an indicator of symmetry-protected topological (SPT) phases. [Pollmann-Berg-Turner-Oshikawa (10)]
Symmetry-protected topological phases (SPT phases)

- "Deformable" to a trivial phase (state w/o entanglement) in the absence of symmetries.

- (Unique ground state on any spatial manifold – "invertible")

- But sharply distinct from trivial state, once symmetries are enforced.

Example: SSH model, time-reversal symmetric topological insulators, the Haldane phase

Symmetry-breaking paradigm does not apply: no local order parameter

\[ \langle M \rangle \neq 0 \quad \langle M \rangle = 0 \]
• How about symmetry?
• Corr. matrix inherits symmetries of the Hamiltonian

\[ \psi_I \rightarrow U_{IJ} \psi_J, \quad H_{phys} \rightarrow U^\dagger H_{phys} U = H_{phys}, \]
\[ Q \rightarrow U^\dagger QU = Q \]  \hspace{1cm} (14)

• Non-spatial symmetry, the sub block of corr. matrix inherits symmetries:

\[ Q_L \rightarrow U^\dagger Q_L U = Q_L \]  \hspace{1cm} (15)

So does the entanglement Hamiltonian. This may result in degeneracy in the ES.
Another example

- Spin-1 Antiferromagnetic spin chain

\[ H = \sum_j S_j \cdot S_{j+1} + U_{zz} \sum_j (S_j^z)^2 \]  

(16)

Figure: [Pollmann-Berg-Turner-Oshikawa (10)]
View from Matrix product states

- Matrix product state representation:

\[
\Psi(s_1, s_2, \cdots) = \sum_{\{i_n=1,\cdots\}} A_{s_1 i_1 i_2}^s A_{s_2 i_2 i_3}^s A_{s_3 i_3 i_4}^s \cdots \quad s_a = -1, 0, 1
\]

- Symmetry action: for \( g, h \in \text{Symmetry group} \), we have \( U(g) \) acting on physical Hilbert space:

\[
U(g)U(h) = U(gh) \\
U(g)^{s'} A^s = V^{-1}(g) A^{s'} V(g) e^{i\theta g} \quad (17)
\]

- Symmetry acts on the “internal” space projectively:

\[
V(g)V(h) = e^{i\alpha(g,h)} V(gh) \quad (18)
\]

[Chen et al (11), Pollmann et al (10-12), Schuch et al (11)]
(Entanglement spec)$^2$ and SUSY QM

- From $C^2 = C$:

\[
\begin{align*}
C_L^2 - C_L &= -C_{LR}C_{RL}, \\
Q_L C_{LR} &= -C_{LR}Q_R, \\
C_{RL} Q_L &= -Q_R C_{ReL}, \\
C_R^2 - C_R &= -C_{RL}C_{LR} 
\end{align*}
\]  
\tag{19}

- Introduce:

\[
S = 1 - \begin{pmatrix} Q_L^2 & 0 \\ 0 & Q_R^2 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 2C_{LR} \\ 0 & 0 \end{pmatrix}, \quad Q^+ = \begin{pmatrix} 0 & 0 \\ 2C_{RL} & 0 \end{pmatrix}.
\]  
\tag{20}

- SUSY algebra

\[
[S, Q] = [S, Q^+] = 0,
\]
\[
\{Q, Q^+\} = S, \quad \{Q, Q\} = \{Q^+, Q^+\} = 0
\]  
\tag{21}
Entanglement spec. and spatial symmetries

- \( L/R = \) “fermionic”/”bosonic” sector; \( C_{L,R} \) intertwines the two sectors:

\[
\mathcal{H}_L \overset{C_{LR}}{\underset{C_{RL}}{\leftrightarrow}} \mathcal{H}_R \tag{22}
\]

- Spatial symmetry \( \mathcal{O} \): choose bipartitioning s.t.

\[
\mathcal{O} : \mathcal{H}_L \leftrightarrow \mathcal{H}_R \tag{23}
\]

\[
\begin{pmatrix}
0 & O_{LR} \\
O_{RL} & 0
\end{pmatrix}, \quad O_{LR} O_{LR}^\dagger = O_{RL} O_{RL}^\dagger = 1 \tag{24}
\]

- Symmetry of entanglement Hamiltonian:

\[
Q_L C_{LR} O_{LR}^\dagger = C_{LR} O_{LR}^\dagger Q_L^* \tag{25}
\]

[Turner-Zhang-Vishwanath (10), Hughes-Prodan-Bernevig (11), Fang-Gilbert-Bernevig (12-13), Chang-Mudry-Ryu (14)]
Graphene with Kekule order

- Kekule distortion in graphene

- Degeneracy protected by inversion

- Entanglement spec. is more useful than physical spec.
Short notes: Conformal field theory in (1+1)d

- Scale invariance in (1+1)d $\rightarrow$ conformal symmetry (Polchinski)
  
- Conformal symmetry is infinite dimensional. Holomorphic-anti-holomorphic factorization
  
- Infinite symmetry generated by stress energy tensor

$$T(z) = \sum_{n=-\infty}^{+\infty} L_n z^{-n-2}, \quad \bar{T}(\bar{z}) = \sum_{n=-\infty}^{+\infty} \bar{L}_n \bar{z}^{-n-2},$$  \hspace{1cm} (26)

- Virasoro algebra

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m,-n}$$  \hspace{1cm} (27)

- Characterized by a number $c$ “central charge” (among others)
Short notes: CFT in (1+1)d

- Structure of the spectrum: “tower of states”:

\[
|h, N; j\rangle \otimes |\bar{h}, \bar{N}; \bar{j}\rangle,
\]

\[
L_0|h, N; j\rangle = (h + N)|h, N; j\rangle.
\]

\[
\bar{L}_0|\bar{h}, \bar{N}; \bar{j}\rangle = (\bar{h} + \bar{N})|\bar{h}, \bar{N}; \bar{j}\rangle.
\]

- In other words:

\[
\mathcal{H} = \bigoplus_{h, \bar{h}} n_{h, \bar{h}} \mathcal{V}_h \otimes \overline{\mathcal{V}_{\bar{h}}},
\]

\(n_{h, \bar{h}}\): the number of distinct primary fields with conformal weight \((h, \bar{h})\).

(For simplicity, we only consider the diagonal CFTs with \(n_{h, \bar{h}} = \delta_{h, \bar{h}}\).)
Central charge

- $c = \text{Weyl anomaly}; \text{ at critical points, there are emergent scale invariance, but this emergent symmetry is broken by an anomaly.}$
- $c \simeq (\text{number of degrees of freedom})$
- $c$ shows up in free energy and specific heat, etc:
  \[ c_V = \frac{\pi c}{3v\beta} \]  
  \hspace{1cm} (30)

  Note: $v$ is non-universal.
- Can be extracted from the entanglement entropy scaling:
  \[ S_A = \frac{c}{3} \log R + \cdots \]  
  \hspace{1cm} (31)
- RG monotone. (Zamolodchikov $c$-function; entropic $c$-function)
Radial and angular quantization

- $w(z) = \log z$

\[
z = x + iy
\]

- CFT on a plane $\leftrightarrow$ CFT on a cylinder
- Radial evolution $\leftrightarrow$ Hamiltonian
- Angular evolution (Entanglement or Rindler Hamiltonian) $\leftrightarrow$ Hamiltonian with boundary
Radial flow – Finite size scaling

- CFT on a cylinder of circumference $L$

$$H = \frac{1}{2\pi} \int_0^L dv T_{uu}(u_0, v)$$

$$= \frac{1}{2\pi} \oint_{C_w} dw T(w) + \text{(anti-hol)} \quad (32)$$

- Conformal map: cylinder $\rightarrow$ plane $w = \frac{L}{2\pi} \log z$

$$\oint_{C_w} dw T(w) = \oint_{C_z} dz \frac{dw}{dz} \left(\frac{2\pi}{L}\right)^2 \left[ z^2 T(z) - \frac{c}{24} \right]$$

$$= \oint_{C_z} dz \left(\frac{L}{2\pi}\right) \left[ zT(z) - \frac{c}{24} \frac{1}{z} \right] \quad (33)$$

- CFT Hamiltonian on a cylinder can be written in terms of dilatation operator $L_0 + \bar{L}_0$ on a plane:

$$H = \frac{2\pi}{L} \left( L_0 + \bar{L}_0 - \frac{c}{24} \right) \quad (34)$$
• Gives relation between stress tensor (on $z$-plane) to a “physical” Hamiltonian on a finite cylinder.

• Level spacing scales as $1/L$.

• Levels are equally spaced (within a tower)

• The $c/24 \times 1/L$ part allows us to determine $c$ (numerically). (the extensive part $A \times L$ has to be subtracted.)

• Degeneracy $\rightarrow$ full identification of the theory
Radial flow – Numerics

- XX model: \( H = \sum_j \left( S^x_j S^x_{j+1} + S^y_j S^y_{j+1} \right) \)

\[ \frac{E - E_{GS}}{E_1 - E_{GS}} \]

- For a given tower, all levels are equally spaced.
- Level spacing scales as \( 1/L \).
Angular flow

\[ z = (x + iy) = \exp(w) = \exp(u + iv) \]

\[ u_2 = \ln(R_2/a) \]

\[ L = u_2 - u_1 \]

\[ u_1 = \ln(R_1/a) \]
Angular flow – Corner transfer matrix

- Corner transfer matrix $A_{\sigma | \sigma'}$ and partition function $Z = \text{Tr} A^4$

[Baxter (80's); Figures: Wikipedia]
Angular flow = Entanglement (Rindler) Hamiltonian

- In Euclidean signature, \( z = x + iy = e^w = e^{u+iv} \)
  maps the complex \( z \)-plane to a cylinder.

- In Minkowski signature: \((t, x) \rightarrow (u, v)\) (Rindler coordinate):
  \[
  x = e^u \cosh v,
  \quad t = e^u \sinh v.
  \]

- In the Rindler coord., the half of the 2d spacetime is inaccessible ("traced out").

- Radial evolution in the complex \( z \)-plane \( \rightarrow u \)-evolution in the cylinder

- Angular evolution in the complex \( z \)-plane \( \rightarrow v \)-evolution in the cylinder
  \[= \text{entanglement (or Rindler) Hamiltonian}\]
Rindler Hamiltonian

- **Constant $u$ trajectories** = World-lines of observer with constant acceleration $a$ where $a = 1$ in our case. 
  Accelerated observer in Minkowski space = Static observer in Rindler space

- **Unruh effect**: Vacuum is observer dependent. Observer in an accelerated frame (Rindler observer) sees the vacuum of the Minkowski vacuum as a thermal bath with Unruh temperature

  \[ T = \frac{a}{2\pi} = \frac{1}{2\pi} \]  
  \hspace{1cm} (35)

  - This is due to a “Rindler horizon” and inability to access the other part of spacetime. Rinder coordinates covers with metric

  \[ ds^2 = e^{2au}(-dv^2 + du^2) \]  
  \hspace{1cm} (36)

  only covers $x > |t|$ (the right Rindler wedge).

- Left Rindler wedge is defined by

  \[ x = e^u \cosh v, \]
  
  \[ t = -e^u \sinh v. \]
Entanglement Hamiltonian for finite interval

- \( w(z) = \ln(z + R)/(z - R) \)

- Entanglement Hamiltonian on finite interval \([-R, +R] \rightarrow \text{Hamiltonian with boundaries}\)

- Transforming from strip to plane:

\[
H = \int du \, T_{vv}|_{v_0=\pi} = \int_{-R}^{+R} \, dx \, \left( \frac{x^2 - R^2}{2R} \right) T_{yy}|_{y=0}
\]  

(37)

- Entanglement spec: \( 1/\log(R) \) scaling

E.g., Casini-Huerta-Myers (11), Cardy-Tonni (16)
SSH chain

- Entanglement spectrum of CFT GS: \( H^E = \text{const.} \frac{L_0}{\log(R/a)} \)

\[
H = t \sum_i \left( a_i \dagger b_i + h.c. \right) + t' \sum_i \left( b_i \dagger a_{i+1} + h.c. \right)
\]  \quad (38)

with \( t = t' \)

\[ \text{Figure: [Cho-Ludwig-Ryu (16)]} \]
Numerics

Figure: [Lauchli (13)]
Remarks:

- What is an analogue of the radial direction?
- It is related to the so-called sine-square deformation (SSD).
  [Gendiar-Krcmar-Nishino (09), Hikihara-Nishino (11), ...]
- Evolution operator:

  \[ H = \int_0^\pi dv T_{uu}(u_0, v) = r_0^2 \int_0^{2\pi} d\theta \frac{\cos \theta + \cosh u_0}{\sinh u_0} T_{rr}(r, \theta) \quad (39) \]

- In the limit \( R \rightarrow 0 \),

  \[ H \sim \int_0^L ds \sin^2 \left( \frac{\pi s}{L} \right) T_{rr} \left( \frac{L}{2\pi}, \frac{2\pi s}{L} \right) \quad (40) \]

  [Ishibashi-Tada (15-16); Okunishi (16); Wen-Ryu-Ludwig (16)]
Perturbed CFT

- Add a relevant perturbation

\[ S = S_* + g \int d^2 z \phi(z, \bar{z}) \]  \hspace{1cm} (41)

and go into a massive phase; Consider the entanglement Hamiltonian for half space.

- The above conformal map leads to an exponentially growing potential

\[ S_* + g \int_{u_1}^{u_2} du \int_0^{2\pi} dv e^{yu} \Phi(w, \bar{w}) \]  \hspace{1cm} (42)

with length scale \( \log(\xi/a) \).
Entanglement Spectrum

- Entanglement spectrum for gapped phases is given by a CFT with boundaries (Boundary CFT in short) of a nearby CFT

\[ Z_{AB} = \text{Tr}_{AB} e^{-H_e} \]  \hspace{1cm} (43)

Here, \( A = \text{vacuum} \) and \( B = \text{SPT} \). [‘RG domain wall’ idea:]

- Spectrum is given by half of the full CFT:

\[ H_e = \text{const.} \frac{L_0}{\log(\xi/a)} \]
Spectrum depends on type of boundaries (type of SPTs): There is symmetry-protected degeneracy in the topological phase.
Entanglement spectrum for gapped phases is given by BCFT.

When the gapped phase is an SPT, the topological invariant can also be computed from BCFT. [Cho-Shiozaki-Ryu-Ludwig (16)]

Switching space and time,

\[ Z = \text{Tr} e^{-\beta/\ell L_0} = \langle A| e^{-\ell/\beta(L_0+\bar{L}_0)} |B\rangle \]  \hspace{1cm} (44)

we introduce boundary states \(|A\rangle\) and \(|B\rangle\):

\[(L_n - \bar{L}_{-n})|B\rangle = 0, \hspace{0.5cm} \forall n \in \mathbb{Z} \]  \hspace{1cm} (45)

From \(|B\rangle\), the corresponding SPT phase can be identified by the phase

\[ g|B\rangle_h = \varepsilon_B(g|h)|B\rangle_h, \hspace{0.5cm} g, h \in G \]

where \(|B\rangle_h\) is the boundary state in \(h\)-twisted sector. This phase is called the discrete torsion phase \(\varepsilon_B(g|h) \in H^2(G,U(1))\).
Boundary states as gapped states

- Conformally invariant boundary states, \((L_n - \bar{L}_{-n})|B\rangle = 0\).

- Boundary states \(|B\rangle\) do not have real-space correlations:

\[
\langle B| e^{-\epsilon H} O_1(x_1) \cdots O_n(x_n) e^{-\epsilon H} |B\rangle / \langle B| e^{-2\epsilon H} |B\rangle
\]

where \(x_1, \cdots, x_n\) refer to different spatial positions. In the limit \(\epsilon \to 0\) with \(x_i \neq x_j\) the correlation function factorizes and does not depend on \(x_i - x_j\).

- Boundary states represent a highly excited state within the Hilbert space of a gapless conformal field theory and can be viewed as gapped ground states. [Miyaji-Ryu-Takayanagi-Wen (14), Cardy (17), Konechny (17)]
Free fermion example

• A massive free massive Dirac fermion in (1+1)d:

\[ H = \int dx \left[ -i \psi^\dagger \sigma_z \partial_x \psi + m \psi^\dagger \sigma_x \psi \right], \quad \psi = (\psi_L, \psi_R)^T \]

• The ground state of this Hamiltonian is given by

\[ |GS\rangle = \exp \left[ \sum_{k>0} \frac{m}{\sqrt{m^2 + k^2 + k}} \left( \psi_{Lk}^\dagger \psi_{Rk} + \psi_{R-k}^\dagger \psi_{L-k} \right) \right] |G_L\rangle \otimes |G_R\rangle \]

where \( \psi_{L,Rk} \) is the Fourier component of \( \psi_{L,R}(x) \), and \( |G_{L,R}\rangle \) is the Fock vacuum of the left- and right-moving sector. In the limit \( m \to \infty \) \( (m/(v_F k) \to \infty) \), \( |GS\rangle \) reduces to the boundary states of the free massless fermion theory.
More details

• SPT phases in (1+1)d are classified by group cohomology $H^2(G, U(1))$. [Chen-Gu-Liu-Wen (02)] Recall:

\[ V(g)V(h) = e^{i\alpha(g,h)}V(gh) \]  

(46)

• CFT context: Discrete torsion phases in CFT [Vafa (86) ...] and in BCFT [Douglas (98) ...].

• Discrete torsion phases and entanglement spectrum (symmetry-protected degeneracy):

Twisted partition function:

\[ Z^h_{AB} = \text{Tr}_{\mathcal{H}_{AB}} \left[ \hat{h} e^{-\beta H^\text{open}_{AB}} \right] \]

vanishes when $A \neq B$. (symmetry-enforced vanishing of partition function).

Exchange time and space, $Z^h_{AB} = \hbar \langle A|e^{-\frac{\ell}{2} H^\text{closed}_{AB}}|B\rangle_h$ and insert $g$ to show

\[ [\varepsilon_B(g|h) - \varepsilon_A(g|h)]Z^h_{AB} = 0 \]
RG and entanglement: entropic c-theorem

- **Entropic $c$-function** [Casini (04)]:

  \[ c_E(R) := 3R \frac{dS(R)}{dR} \]  
  \hspace{1cm} (47)

- At critical points, $c_E = c$ (central charge).
- From strong subadditivity:

  \[ S_A + S_B \geq S_{A\cap B} + S_{A\cup B} \]  
  \hspace{1cm} (48)

  can argue that $S$ is concave w.r.t. $\log R$:

  \[ 2S(\sqrt{rR}) \geq S(R) + S(r) \]  
  \hspace{1cm} (49)

  Taking the limit: $r \to R$:

  \[ \frac{c'_E(R)}{3} = S''(R) + RS'''(R) \leq 0 \]  
  \hspace{1cm} (50)
Remark: F-theorem

- Is there an analogue of $c$ and $c$-theorem in $(2+1)d$? (No weyl anomaly in $(2+1)d$)
- EE of a disc $D$ of radius $R$ [Ryu-Takayanagi (06), Myers-Sinha (10)]:

$$S_D(R) = \alpha \frac{2\pi R}{\epsilon} - F(R) \quad (51)$$

$F$ at the critical point is a universal constant. C.f. topological entanglement entropy.

- $F$ is related to the partition functions on a sphere $S^3$, $F = -\log Z(S^3)$ [Casini-Huerta-Myers (11)].

F-theorem: [Jafferis et al (11), Klebanov et al (11)]:

$$F_{UV} \geq F_{IR}$$

- Entropic $\mathcal{F}$ function: [Liu-Mezei (13)]

$$\mathcal{F}(R) = \left( R \frac{\partial}{\partial R} - 1 \right) S_D(R) \quad (52)$$

$\mathcal{F}(R)|_{CFT} = F$ and $\mathcal{F}'(R) \leq 0$ [Casini-Huerta (12)]

- Applications [Grover (12), ...] Stationarity?
Topological phases of matter

- Topologically ordered phases: phases which support anyons (∼ support topology dependent ground state degeneracy)

- E.g., fractional quantum Hall states, \( \mathbb{Z}_2 \) quantum spin liquid, etc.

- Quantum phases which are not described by the symmetry-breaking paradigm. (I.e., Landau-Ginzburg type of theories)

- Instead, characterized by properties of anyons (fusion, braiding, etc.) (I.e., topological quantum field theories)
Algebraic theory of anyons

- (Bosonic) topological orders are believed to be fully characterized by a unitary modular tensor category (UMTC).
- (i) Finite set of anyons \( \{1, a, b, \ldots\} \) equipped with quantum dimensions \( \{1, d_a, d_b, \ldots\} \) \((d_a \geq 1)\). Total quantum dimension \(D\):

\[
D = \sqrt{\sum_a d_a^2}\tag{53}
\]

- (ii) Fusion \(a \times b = \sum_c N_{ab}^c c\).
- (iii) The modular \(T\) matrix, \(T = \text{diag} (1, \theta_a, \theta_b, \ldots)\) where \(\theta_a = \exp 2\pi i h_a\) is the self-statistical angle of \(a\) with \(h_a\) the topological spin of \(a\).
- (iv) The modular \(S\) matrix encodes the braiding between anyons, and given by (“defined by”)

\[
S_{ab} = \frac{1}{D} \sum_c N_{ab}^c \frac{\theta_c}{\theta_a \theta_b} d_c .\tag{54}
\]
There may be topologically ordered phases with the same braiding properties, but different values of $c$, the chiral central charge of the edge modes.

Albeit the same braiding properties, they cannot be smoothly deformed to each other without closing the energy gap.

Topological order is conjectured to be fully characterized by $(S, T, c)$.
Ground states and $S$ and $T$

- Ground state degeneracy depending on the topology of the space (topological ground state degeneracy), related to the presence of anyons [Wen (90)]
- Ground state degeneracy on a spatial torus, $\{ |\Psi_i\rangle \}$.
- $S$ and $T$ are extracted from the transformation law of $\{ |\Psi_i\rangle \}$ [Wen (92)]
Topologically ordered phases and quantum entanglement

- Consider: the reduced density matrix $\rho_A$ obtained from a ground state $|GS\rangle$ of a topologically-ordered phase by tracing out half-space.

$$\rho_A \propto \text{Tr}_R e^{-\epsilon H} |B.S.\rangle \langle B.S.| e^{-\epsilon H}$$

[Qi-Katsura-Ludwig (12), Fliss et al. (17), Wong (17)]

- Different (Ishibashi) boundary states correspond to different ground states

- With this explicit form of the reduced density matrix, various entanglement measures can be computed: [Wen-Matsuura-SR]
  - the entanglement entropy
  - the mutual information
  - the entanglement negativity
Bulk-boundary correspondence

- Bulk anyon ↔ twisted boundary conditions at edge:
  \[ \Psi_i \leftrightarrow b_{\text{bound} \text{ary partition function}} \]

- Bulk \( \Psi_i \) ↔ boundary partition function \( \chi_i \)

- Bulk \( S \) and \( T \) matrices acting on \( \Psi_i \) on spatial torus
  \[ \leftrightarrow \] 
  \( S \) and \( T \) matrices acting on boundary partition function \( \chi_i \) on spacetime torus  [Cappelli (96), ...]

\[
\chi_a(e^{-\frac{4\pi \beta}{l}}) = \sum_{a'} S_{aa'}\chi_{a'}(e^{-\frac{\pi l}{\beta}}) \tag{55}
\]
• Conformal BC: $L_n|b\rangle = \bar{L}_{-n}|b\rangle \ (\forall n \in \mathbb{Z})$

• Ishibashi boundary state:

$$|h_a\rangle \equiv \sum_{N=0}^{\infty} \sum_{j=1}^{d_{h_a}(N)} |h_a, N; j\rangle \otimes |\bar{h}_a, N; j\rangle$$  \hspace{1cm} (56)$$

• Topological sector dependent normalization (regularization):

$$|\bar{h}_a\rangle = \frac{e^{-\epsilon H}}{\sqrt{n_a}} |h_a\rangle \text{ so that } \langle \bar{h}_a | h_b \rangle = \delta_{ab}. \hspace{1cm} (57)$$

• More generically, one can consider a superposition $|\psi\rangle = \sum_a \psi_a |h_a\rangle$

• Reduced density matrix:

$$\rho_{L,a} = \text{Tr}_R(|\bar{h}_a\rangle \langle \bar{h}_a|)$$

$$= \sum_{N,j} \frac{1}{n_a} e^{-\frac{8\pi}{l}(h_a + N - \frac{c}{24})} |h_a, N; j\rangle \langle h_a, N; j|. \hspace{1cm} (58)$$
• Trance of the reduced density matrix:

$$\text{Tr}_L (\rho_{L,a})^n = \frac{1}{n_{\alpha}^n} \chi_\alpha (e^{-\frac{8\pi n\epsilon}{l}}) = \frac{\chi_\alpha (e^{-\frac{8\pi n\epsilon}{l}})}{\chi_\alpha (e^{-\frac{8\pi \epsilon}{l}})^n}$$  \hspace{1cm} (59)

• Modular transformation

$$\chi_\alpha (e^{-\frac{8\pi n\epsilon}{l}}) = \sum_{\alpha'} S_{\alpha \alpha'} \chi_{\alpha'} (e^{-\frac{\pi l}{2n\epsilon}})$$

$$\rightarrow S_{\alpha 0} \times e^{\frac{\pi cl}{48n\epsilon}} \quad (l/\epsilon \rightarrow \infty),$$  \hspace{1cm} (60)

i.e., only the identity field $I$, labeled by “0” here, survives the limit.

• Hence, in the thermodynamic limit $l/\epsilon \rightarrow \infty$:

$$\text{Tr}_L (\rho_{L,a})^n = \frac{\sum_{\alpha'} S_{\alpha \alpha'} \chi_{\alpha'} (e^{-\frac{\pi l}{2n\epsilon}})}{\left[ \sum_{\alpha'} S_{\alpha \alpha'} \chi_{\alpha'} (e^{-\frac{\pi l}{2\epsilon}}) \right]^n} \rightarrow e^{\frac{\pi cl}{48\epsilon}} \left( \frac{1}{n} - n \right) (S_{\alpha 0})^{1-n},$$  \hspace{1cm} (61)
Final result:

\[ S_L^{(n)} = \frac{1 + n}{n} \cdot \frac{\pi c}{48} \cdot \frac{l}{\epsilon} - \ln D + \frac{1}{1 - n} \ln d_a^{1-n} \]

\[ S_L^{\text{N}} = \frac{\pi c}{24} \cdot \frac{l}{\epsilon} - \ln D + \ln d_a \]  \hspace{1cm} (62)

where

\[ S_{a0} = d_a / D \]  \hspace{1cm} (63)

is the quantum dimension.
Lessons

- Entanglement cut may be more useful than physical cut.
- Entanglement and universal information of many-body systems.
- Entanglement can tell the direction of the RG flow.
- Entanglement and spacetime physics
- Entanglement has a topological interpretation in particular in topological field theories.
- ...
