Topological Quantum Computing

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Early Digital Memory

Stone
Early Digital Memory

\[ = 0 \]
Early Digital Memory

= 1
Early Digital Memory

The iStone

= 1
Early Digital Memory

The iStone: 1 bit
Early Digital Memory

The iStone 5: \( \sim 20 \text{ bits} \)
Modern Digital Memory

The iPhone 5: $\sim 5.5 \times 10^{11}$ bits
Modern Digital Memory

The iPod: $\sim 1.4 \times 10^{12}$ bits
Modern Digital Memory

http://en.wikipedia.org/wiki/Hard_disk_drive
Magnetic Order

A spin-1/2 particle:

“spin up”

“spin down”
Magnetic Order

A spin-1/2 particle:

“spin up”

“spin down”

= 0
Magnetic Order

A spin-1/2 particle: 

“spin up”

“spin down”

= 1
Another Kind of Order

A valence bond:

\[ \frac{1}{\sqrt{2}} (\uparrow\downarrow - \downarrow\uparrow) \]
Another Kind of Order

A valence bond:

\[ \frac{1}{\sqrt{2}} (\uparrow\downarrow - \downarrow\uparrow) \]
Another Kind of Order

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\]
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Another Kind of Order

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Quantum superposition of valence-bond states. A “spin liquid.”
Another Kind of Order

A valence bond:

\[
\frac{1}{\sqrt{2}} (\uparrow\downarrow - \downarrow\uparrow)
\]
Another Kind of Order

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Another Kind of Order

A valence bond:

$$\frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)$$
Another Kind of Order

A valence bond:

\[ \frac{1}{\sqrt{2}} (\uparrow\downarrow - \downarrow\uparrow) \]
Another Kind of Order

A valence bond:

\[ = \frac{1}{\sqrt{2}} (\uparrow\downarrow - \downarrow\uparrow) \]
Another Kind of Order

A valence bond:

\[
\begin{align*}
\quad & = \frac{1}{\sqrt{2}} (\uparrow\downarrow - \downarrow\uparrow) \\
\end{align*}
\]
Is it a 0 or a 1?
Is it a 0 or a 1?
Is it a $|0\rangle$ or a $|1\rangle$?
Is it a $|0\rangle$ or a $|1\rangle$?
Is it a $|0\rangle$ or a $|1\rangle$?
Topological Order (Wen & Niu, PRB 41, 9377 (1990))

Conventionally Ordered States: Multiple “broken symmetry” ground states characterized by a locally observable order parameter.

\[ m = \langle S_z \rangle = + \frac{1}{2} \]

Topologically Ordered States: Multiple ground states on topologically nontrivial surfaces with no locally observable order parameter.

Nature’s classical error correction

Nature’s quantum error correction
Topological Order: Excitations
Breaking a bond creates an excitation with $S_z = 1$.
Topological Order: Excitations

Breaking a bond creates an excitation with $S_z = 1$
Breaking a bond creates an excitation with $S_z = 1$
$S_z = 1$ excitation *fractionalizes* into two $S_z = \frac{1}{2}$ excitations
A two dimensional gas of electrons in a strong magnetic field $\mathbf{B}$. 
An incompressible quantum liquid can form when the Landau level filling fraction $\nu = n_{\text{elec}}(hc/eB)$ is a rational fraction.
When an electron is added to a FQH state it can be fractionalized --- i.e., it can break apart into fractionally charged quasiparticles.
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Topological Degeneracy

As in our spin-liquid example, FQH states on topologically nontrivial surfaces have degenerate ground states which can only be distinguished by global measurements.

For the $\nu = 1/3$ state:

Degeneracy

<table>
<thead>
<tr>
<th>N</th>
<th>Degeneracy</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>N</td>
<td>$3^N$</td>
</tr>
</tbody>
</table>
“Non-Abelian” FQH States (Moore & Read ’91)

Essential features:

A degenerate Hilbert space whose dimensionality is exponentially large in the number of quasiparticles.

States in this space can only be distinguished by global measurements provided quasiparticles are far apart.

A perfect place to hide quantum information!
Exchanging Particles in 2+1 Dimensions

Particle “world-lines” form **braids** in 2+1 (=3) dimensions
Exchanging Particles in 2+1 Dimensions

Particle “world-lines” form **braids** in 2+1 (=3) dimensions
Many Non-Abelian Anyons

\[ \Psi_f = \begin{pmatrix} a_{11} & \cdots & a_{1M} \\ \vdots & \ddots & \vdots \\ a_{M1} & \cdots & a_{MM} \end{pmatrix} \Psi_i \]
Many Non-Abelian Anyons

\[ |\Psi_f\rangle = \left( \begin{array}{ccc} a_{11} & \cdots & a_{1M} \\ \vdots & \ddots & \vdots \\ a_{M1} & \cdots & a_{MM} \end{array} \right) |\Psi_i\rangle \]

Matrix depends only on the topology of the braid swept out by anyon world lines!

Robust quantum computation?
Possible Non-Abelian FQH States

\( \nu = 12/5 \): Possible Read-Rezayi "Parafermion" state. Read & Rezayi, ‘99

Charge \( e/5 \) quasiparticles are **Fibonacci anyons**. Slingerland & Bais ’01

\( \nu = 5/2 \): Probable Moore-Read Pfaffian state.

Charge \( e/4 \) quasiparticles are **Majorana fermions**. Moore & Read ‘91

J.S. Xia et al., PRL (2004).
Enclosed “charge”
0 or 1
\[ \alpha |0 \rangle + \beta |1 \rangle \]
$\alpha |0\rangle + \beta |1\rangle$

2 dimensional Hilbert space
Quantum states are protected from environment if particles are kept far apart.

Need to measure all the way around both particles to determine what state they are in.
3 dimensional Hilbert space
3 dimensional Hilbert space
a can be 0 or 1
a can be 0 or 1

b can be 0 or 1
The F Matrix

\[ F = \begin{pmatrix} F_{00} & F_{01} \\ F_{10} & F_{11} \end{pmatrix} \]
The F Matrix

\[ F = \begin{pmatrix} F_{00} & F_{01} \\ F_{10} & F_{11} \end{pmatrix} \]
Equivalent to
Equivalent to
Equivalent to
Equivalent to

$F$

$F$
The Pentagon Equation
The Pentagon Equation
The Pentagon Equation
The Pentagon Equation
The Pentagon Equation

Diagram of the Pentagon Equation with green circles and arrows indicating connections.
The Pentagon Equation
The Pentagon Equation
The Pentagon Equation
The Pentagon Equation
The Pentagon Equation
The Pentagon Equation
The Pentagon Equation
The Pentagon Equation
The Pentagon Equation

\[ F_{0b} \]
The Pentagon Equation

\[ F_{0b} \]
The Pentagon Equation
The Pentagon Equation
The Pentagon Equation

\[ F_{0b} \quad F_{b0} \]
The Pentagon Equation

\[ 1 = \sum_b F_{0b} F_{b0} = F_{00} F_{00} + F_{01} F_{10} \]
The Pentagon Equation
The Pentagon Equation
The Pentagon Equation
The Pentagon Equation
The Pentagon Equation
The Pentagon Equation
The Pentagon Equation
The Pentagon Equation

$F_{01}$
The Pentagon Equation

$F_{01}$
The Pentagon Equation
The Pentagon Equation

\[ F_{01} \]
The Pentagon Equation
The Pentagon Equation
The Pentagon Equation

\[ F_{01} \]

\[ F_{0b} \]
The Pentagon Equation
The Pentagon Equation
The Pentagon Equation

$F_{00}$

$F_{01}$
The Pentagon Equation
The Pentagon Equation

\[ F_{01} \]

\[ F_{0b} = \delta_{b,0} + \delta_{b,1} F_{11} \]
The Pentagon Equation

\[ F_{01} = \delta_{b,0} + \delta_{b,1} F_{11} \]

\[ F_{0b} \]
The Pentagon Equation

\[ F_{0b} = \delta_{b,0} + \delta_{b,1} F_{11} \]

\[ F_{01} \]
The Pentagon Equation

\[ F_{01} = \sum_b F_{0b} \left( \delta_{b,0} + \delta_{b,1} F_{11} \right) F_{b1} = F_{00} F_{01} + F_{01} F_{11}^2 \]
The Pentagon Equation

Unique unitary solution (up to irrelevant phase factors):

\[
F = \begin{pmatrix}
\phi^{-1} & \phi^{-1/2} \\
\phi^{-1/2} & -\phi^{-1}
\end{pmatrix}
\]

\[\phi = \frac{\sqrt{5} + 1}{2} \approx 1.618 \ldots\]

Golden Mean
Hilbert Space Dimensionality
Hilbert Space Dimensionality

Bratteli Diagram
Hilbert Space Dimensionality

States are paths in the fusion diagram

“charge”
Hilbert Space Dimensionality

Here's another one

“charge”
Hilbert Space Dimensionality

- Hilbert space dimensionality grows as the **Fibonacci sequence**!

  ![Fibonacci Anyons](image)

- Exponentially large in the number of quasiparticles \((\text{deg} \sim \phi^N)\), so big enough for quantum computing.

"charge"

```
  1 1 2 3 5 8 13 21
```

```
  1 1 2 3 5 8 13 21
```

```
  1 1 2 3 5 8 13 21
```
Problem 1. Pentagon Equation for Fibonacci Anyons.

For Fibonacci anyons the $2 \times 2$ $F$ matrix,

$$ F = \begin{pmatrix} F_{00} & F_{01} \\ F_{10} & F_{11} \end{pmatrix}, $$

describes the following basis change,

$$ a_1 \rightarrow \sum_b F_{ab} b_1. $$

The **pentagon equation** then equates the results of two distinct ways of using the $F$ matrix to express a four anyon state from the basis,

$$ a_c b \rightarrow a c b, $$
as a superposition of states from the basis,

$$ a b c. $$

For each of the seven pentagon diagrams that follow, use the fact that the two paths (top and bottom) should yield the same amplitude for the contribution of the rightmost state in the expansion of the leftmost state to derive seven polynomial equations for $F_{00}$, $F_{11}$ and the product $F_{01}F_{10}$.

Now solve these equations. You should find two solutions, only one of which yields a unitary $F$ matrix if you take $F_{01} = F_{10} = \sqrt{F_{01}F_{10}}$. Find this $2 \times 2$ unitary $F$ matrix.

You may find it convenient to express your answer in terms of $\tau = (\sqrt{5} - 1)/2 \approx 0.62$, where $\tau$ is the inverse of the golden mean $\phi = (\sqrt{5} + 1)/2 \approx 1.62$. 

\[ \text{Diagram: pentagon equations and solutions.} \]
Problem 2. Hexagon Equation for Fibonacci Anyons.

For Fibonacci anyons the $2 \times 2$ $R$ matrix,

$$R = \begin{pmatrix} R_0 & 0 \\ 0 & R_1 \end{pmatrix},$$

describes the phase factor acquired when anyons with a given total topological charge are exchanged in a clockwise manner,

$$a \quad \begin{array}{c}{R}_{a} \\ \end{array} \begin{array}{c}{R}_{a} \\ \end{array}.$$  

The hexagon equation then describes two different ways to use the $F$ and $R$ matrices to compute the effect of moving two anyons around a third.

For each of the four hexagon diagrams that follow, use the fact that the two paths (top and bottom) should yield the same amplitude for the contribution of the rightmost state to the expansion of the state obtained by carrying out the anyon exchanges on the leftmost state to derive four polynomial equations for $R_0$ and $R_1$. (In this calculation you should use the unitary $F$ matrix obtained in Problem 1).

Again you should find two solutions, this time corresponding to the two possible choices for the ‘handedness’ of the anyons.

The following identities might be useful in simplifying your results for $R_0$ and $R_1$,

$$\sin \left( \frac{3\pi}{5} \right) = \frac{\sqrt{10 - 2\sqrt{5}}}{4}, \quad \cos \left( \frac{3\pi}{5} \right) = \frac{1 - \sqrt{5}}{4}.$$