The quest for Majorana I

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Outline

• Some general remarks on Majorana fermions in condensed matter

• Toy models capturing Majoranas (continued)
  - Quick reminder of “Kitaev chain”
  - 2D topological superconductivity

• Select plausible experimental realizations
Majorana fermions: high-energy vs. cond. matter

Originally “invented” Majorana fermions
(fermionic particles that are their own antiparticle)

Neutrinos?

Dark Matter?

Ettore Majorana (1906-1938?)

Majorana fermions: high-energy vs. cond. matter

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Neutrinos?

Dark Matter?


Condensed matter physicists mainly seek Majorana fermion zero-modes (which are not really particles!)

\[ \gamma = \gamma \]
Majorana fermions in condensed matter?
Majorana fermions in condensed matter?

\[ c^\dagger |\psi\rangle \] (Adds an electron)
Majorana fermions in *condensed matter*?

\[ c^\dagger |\psi\rangle \quad \text{(Adds an electron)} \]
\[ c |\psi\rangle \quad \text{(Adds a hole)} \]
Majorana fermions in condensed matter?

\[ c^\dagger |\psi\rangle \quad \text{(Adds an electron)} \]

\[ c |\psi\rangle \quad \text{(Adds a hole)} \]

\[ C^\dagger \neq C \]

Majorana appears only through \textit{emergent} excitations
Majorana fermions in **condensed matter?**

**Equation:**

\[ c^\dagger |\psi\rangle \quad \text{(Adds an electron)} \]

\[ c |\psi\rangle \quad \text{(Adds a hole)} \]

\[ c^\dagger \neq C \]

Majorana appears only through **emergent** excitations

**Superconductors**

are natural platforms

\[ f^\dagger \sim uc^\dagger + vc \]
Majorana fermions in **condensed matter**?

\[
c^\dagger |\psi\rangle \quad \text{(Adds an electron)}
\]

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\[
c^\dagger \neq c
\]

Majorana appears only through **emergent** excitations

**Superconductors** are natural platforms but must be **topological**

\[
f^\dagger \sim uc^\dagger + vc
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D spinless p-wave superconductc

\[ \sum - \sum \Delta \phi \]

\[ \pi = 0 \]
\[ t = \Delta \]

(Kitaev 2001)
D spinless p-wave superconductor

\[ \sum - \sum \Delta \phi \]

\[ \pi = 0 \]
\[ t = \Delta \]

- - \( \phi \) \( \gamma \) \( \gamma \)

- \( \sum \) \( \gamma \) \( \gamma \)

Unpaired Majorana fermion zero-modes!

(Kitaev 2001)
Majoranas encode a **two-fold ground-state degeneracy**, consisting of states w/ even and odd fermion #. The zero-modes are unpaired Majorana fermion zero-modes.

(Kitaev 2001)
2D spinless p+ip superconductor

\[
H = \int \frac{d^2 k}{(2\pi)^2} \left[ (\epsilon_k - \pi) \psi_k \psi_k + (\Delta_k \psi_k \psi_{-k} + h.c) \right]
\]

\[
\Delta_k \propto k_x + ik_y
\]

(Read & Green 2000)
2D spinless p+ip superconductor

\[ H = \int \frac{d^2k}{(2\pi)^2} \left[ (\epsilon_k - \pi)\psi_k \psi_k^* + (\Delta_k \psi_k \psi_{-k}^* + h.c.) \right] \]

\[ \Delta_k \propto k_x + ik_y \]

To expose topological structure, rewrite as:

\[ H = \frac{1}{2} \int \frac{d^2k}{(2\pi)^2} \left[ \psi_k^\dagger, \psi_{-k} \right] \begin{bmatrix} \epsilon_k - \mu & \Delta_k^* \\ \Delta_k & -\epsilon_k + \mu \end{bmatrix} \begin{bmatrix} \psi_k \\ \psi_{-k}^\dagger \end{bmatrix} \]

\[ \vec{h}_k \quad \vec{\sigma} \]

(Read & Green 2000)
2D spinless \textit{p+ip} superconductor

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\[ \vec{h}_k \quad \vec{\sigma} \]

Upon sweeping all of k-space, how many times does \( \vec{h}_k \) cover the unit sphere? Defines \textit{“Chern number”}.

\[ C = \int \frac{d^2k}{4\pi} [\hat{h} \cdot (\partial_{k_x} \hat{h} \times \partial_{k_y} \hat{h})] \]

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\]

\[
\vec{h}_k \cdot \vec{\sigma}
\]

Upon sweeping all of \( k \)-space, how many times does \( \vec{h}_k \) cover the unit sphere? Defines “Chern number”.

\[
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\]

(Read & Green 2000)
Boundary physics & vortices

Cut torus into planar geometry:

Topological spinless $p+ip$ SC

Vacuum

(Read & Green 2000)
Boundary physics & vortices

Cut torus into planar geometry:

Gapless chiral Majorana edge modes = half of an integer quantum Hall edge state

$$H_{\text{edge}} = \int du \gamma_{\text{edge}} (-iv \partial_u) \gamma_{\text{edge}}$$

Topological spinless p+ip SC

Vacuum
Boundary physics & vortices

Cut torus into planar geometry:

Gapless chiral Majorana edge modes = half of an integer quantum Hall edge state

$$H_{\text{edge}} = \int du \gamma_{\text{edge}} (-iv\partial_u) \gamma_{\text{edge}}$$

Vortices bind Majorana zero-modes...

$$f_A = \gamma_1 + i\gamma_2$$
$$f_B = \gamma_3 + i\gamma_4$$

(Read & Green 2000)
Boundary physics & vortices

Cut torus into planar geometry:

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Vortices bind Majorana zero-modes...

\[ f_A = \gamma_1 + i\gamma_2 \]
\[ f_B = \gamma_3 + i\gamma_4 \]

...and exhibit non-Abelian statistics!

(Read & Green 2000; Ivanov 2001)
Summary of toy models

1D spinless p-wave SC in topological phase

End Majorana fermion zero-modes

\( \gamma_1 \)

2D spinless p+ip SC in topological phase

Majorana fermion zero-modes bind to vortices...

\( \gamma_1, \gamma_2, \gamma_3, \gamma_4 \)

...and edge supports gapless chiral Majorana mode
1. For the Kitaev chain, find an expression for the topological invariant distinguishing the trivial and non-trivial phases of the model. Does it take on integer values, or is it a $\mathbb{Z}_2$ index?

2. Find the Hamiltonian describing low-energy physics at the phase transition between topological and trivial phases of the Kitaev chain. Compare to the edge Hamiltonian for a $p+ip$ superconductor.

3. For a 2D spinless $p+ip$ superconductor, derive the chiral Majorana edge Hamiltonian. Does the Majorana operator satisfy periodic or antiperiodic boundary conditions? How does your answer change if the bulk has vortices?

$$H_{\text{edge}} = \int du \gamma_{\text{edge}} (-iv \partial_u) \gamma_{\text{edge}}$$

4. Using your answer to 3, argue that $\hbar/2e$ vortices bind Majorana zero-modes as claimed. (Hint: think about the vortex core as a small puncture in the system.) Are there other finite-energy modes bound to the vortex? If so what are their energies?
Outline

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  - 2D topological superconductivity

• Select plausible experimental realizations
The basic challenges

“Spinless” 1D, 2D p-wave superconductivity is hard to find.

1. We live in 3D
The basic challenges

“Spinless” 1D, 2D p-wave superconductivity is hard to find.

1. We live in 3D

(“Intrinsic” 1D and 2D superconductors also do not exhibit LRO at finite T, unlike toy models where this is assumed.)
The basic challenges

“Spinless” 1D, 2D p-wave superconductivity is hard to find.

1. We live in 3D

2. Electrons carry spin
The basic challenges

“Spinless” 1D, 2D p-wave superconductivity is hard to find.

1. We live in 3D

2. Electrons carry spin

3. Vast majority of superconductors form spin-singlet Cooper pairs
Two ways forward

1. Search for new compounds with exotic superconductivity
Two ways forward

1. Search for new compounds w/exotic superconductivity

Matthias’s 6th rule: Stay away from theorists!
Two ways forward

1. Search for new compounds w/exotic superconductivity

Matthias’s 6th rule: Stay away from theorists!

2. “Engineer” topological superconductivity from available materials

Theorists can be useful, particularly if methods involve weakly interacting electrons

(Approach originally pioneered by Fu & Kane.)
General strategy for “engineering” topological superconductors (and other exotic phases)

Suppose you want a system with properties X and Y which seem incompatible...
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Very useful concept, likely with lots of untapped potential!

General strategy for “engineering” topological superconductors (and other exotic phases)
Hybrid device can exhibit both X and Y!

Suppose you want a system with properties X and Y which seem incompatible...

Here, one subsystem will support 1D or 2D modes with strong spin-orbit coupling...

...the other will be a 3D s-wave superconductor (with LRO).

General strategy for “engineering” topological superconductors (and other exotic phases)
Experimental Routes to 1D topological superconductivity
1D topological superconductivity via edge states

I. Gapless as long as time-reversal, U(1) particle conservation are present

II. By construction 1D & “spinless”

III. Easy to make superconducting

Fu & Kane 2009
1D topological superconductivity via edge states

I. Gapless as long as time-reversal, U(1) particle conservation are present

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Fu & Kane 2009
1D topological superconductivity via edge states

s-wave superconductor

2D Topological Insulator

Fu & Kane 2009
1D topological superconductivity via edge states

\[ H_{\text{edge}} = \int dx \left[ -\pi (\psi_R \psi_R + \psi_L \psi_L) - \mathrm{i} \hbar v (\psi_R \partial_x \psi_R - \psi_L \partial_x \psi_L) \right] + H_{\text{hybridization}} \]

Can then “integrate out” gapped superconductor degrees of freedom

Fu & Kane 2009
1D topological superconductivity via edge states

\[ s\text{-wave superconductor} \]

\[ 2\text{D Topological Insulator} \]

\[ H_{\text{edge}} = \int dx \left[ -\pi (\psi_R \psi_R + \psi_L \psi_L) - ihv (\psi_R \partial_x \psi_R - \psi_L \partial_x \psi_L) \right] + \int dx \Delta (\psi_R \psi_L + \text{H.c.}) \]

Describes a 1D topological superconductor (on a ring)!

Fu & Kane 2009
Generating Majoranas at the edge I

Majoranas arise, but are immobile. Look for a modified setup that allows more control.

Fu & Kane 2009
Generating Majoranas at the edge II

2D Topological Insulator

Gapped due to B-field

Alicea & Lindner
Generating Majoranas at the edge II

2D Topological Insulator

Gapless despite the B-field
Generating Majoranas at the edge II

s-wave superconductor

2D Topological Insulator

Gapped due to B-field

Alicea & Lindner
Generating Majoranas at the edge II

s-wave superconductor

2D Topological Insulator

Gapped due to superconductivity

Alicea & Lindner
Generating Majoranas at the edge II

2D Topological Insulator

Gapped due to superconductivity

Gapped due to B-field

Alicea & Lindner
Majorana fermions in 1D wires

(Lutchyn, Sau, Das Sarma 2010; Oreg, Refael, von Oppen 2010)

\[ H = \int dx \psi \left[ -\frac{\partial^2 x}{2m} - \pi \right] \psi \]
Majorana fermions in 1D wires

$Lutchyn, Sau, Das Sarma 2010; Oreg, Refael, von Oppen 2010$

1D spin-orbit-coupled wire (e.g. InSb)

\[ H = \int dx \psi \left[ -\frac{\partial^2}{2m} - \pi - i\hbar v \partial_x \sigma^y \right] \psi \]

“Rashba spin-orbit coupling” \((E \times P) \cdot \sigma\)

(Lutchyn, Sau, Das Sarma 2010; Oreg, Refael, von Oppen 2010)
Majorana fermions in 1D wires

\[ H = \int dx \psi \left[ -\frac{\partial_x^2}{2m} - \pi - i\hbar v \partial_x \sigma^y - \frac{g\pi B B}{2} \sigma^z \right] \psi \]

(Lutchyn, Sau, Das Sarma 2010; Oreg, Refael, von Oppen 2010)
Majorana fermions in 1D wires

\[ H = \int dx \psi \left[ -\frac{\partial_x^2}{2m} - \pi - i\hbar v \partial_x \sigma^y - \frac{g\pi B B}{2} \sigma^z \right] \psi \]

\[ + (\Delta \psi^\uparrow \psi^\downarrow + \hbar c) \]

Generates a 1D ‘spinless’ p-wave superconducting state with Majorana zero-modes!

(Lutchyn, Sau, Das Sarma 2010; Oreg, Refael, von Oppen 2010)
1. Show that integrating out the parent superconductor’s degree of freedom indeed generates pairing terms for the edge modes/wire. Do other parameters in the Hamiltonian also get renormalized due to the hybridization?

2. Rewrite the wire Hamiltonian in terms of operators that add excitations to the upper/lower bands. Project out the upper band and compare the resulting effective Hamiltonian with the toy model for a spinless p+ip superconductor.

\[
H = \int dx \psi \left[ -\frac{\partial_x^2}{2m} - \pi - i\hbar v \partial_x \sigma^y - \frac{g\pi B^2}{2} \sigma^z \right] \psi
+ (\Delta \psi_\uparrow \psi_\downarrow + \hbar c)
\]

3. Find the parameter range (i.e., magnetic field, pairing energy, and chemical potential) over which the topological phase occurs in 1D wires.
1D wire vs. 2D topological insulator setups

- Semiconductor technology well advanced
- Required ingredients demonstrated long ago
- Need to fine-tune chemical potential within (small) Zeeman gap
- Disorder poses more serious issue

- Not much tuning required
- Built-in resilience against disorder (Anderson’s theorem)
- Few materials (but situation is improving)
Other promising realizations

**3D topological insulator nanowires**


**Magnetic-atom chains on a superconductor**

Experimental Routes to 2D topological superconductivity
An “intrinsic” realization

Composite fermions form a “spinless” metal

Composite Fermi sea is unstable towards $p+ip$ pairing!

\[
\Psi_{Pf} = Pf\left(\frac{1}{z_i - z_j}\right) \prod_{i<j} (z_i - z_j)^2
\]

Willet, Eisenstein, et al. (1987)
Moore & Read (1991)
Bonderson, Kitaev, Shtengel (2006)
Stern & Halperin (2006)
W. Kang et al. (2011)
Majorana fermions in a 3D topological insulator

3D topological insulators: inert bulk but **odd # of Dirac cones on the surface**

\[ H = \int d^2 \mathbf{r} \psi \left( -i v \vec{\sigma} \cdot \nabla - \pi \right) \psi \]

Surface looks “**spinless**”! (i.e., only one Fermi surface rather than two)

(Fu, Kane, & Mele 2006; Moore & Balents 2006; Roy 2006; Fu & Kane 2008)
Majorana fermions in a 3D topological insulator

\[ H = \int d^2 r \left[ \psi \left( -i v \vec{\sigma} \cdot \nabla - \pi \right) \psi + \left( \Delta \psi_\uparrow \psi_\downarrow + h c \right) \right] \]

Surface inherits spin-singlet pairing...

(Fu, Kane, & Mele 2006; Moore & Balents 2006; Roy 2006; Fu & Kane 2008)
Majorana fermions in a 3D topological insulator

\[ H = \int d^2 r \left[ \psi (-i v \vec{\sigma} \cdot \nabla - \pi) \psi + (\Delta \psi^\uparrow \psi^\downarrow + h c) \right] \]

...but spin is not conserved, so this is NOT a simple s-wave superconductor

Surface inherits spin-singlet pairing...

(Fu, Kane, & Mele 2006; Moore & Balents 2006; Roy 2006; Fu & Kane 2008)
Majorana fermions in a 3D topological insulator

Interface realizes 2D topological SC supporting Majorana zero-modes at vortices!

\[ H = \int d^2k \left\{ \epsilon_+(k) \psi_+ \psi_+ + \epsilon_-(k) \psi_- \psi_- \right\} \]

\[ + \Delta \left[ \left( \frac{k_x + ik_y}{2k} \right) \left[ \psi_-(k) \psi_-(k) - \psi_+(k) \psi_+(k) \right] + h c \right] \]

(Pairing is \( p+ip \) in this basis!

(Fu, Kane, & Mele 2006; Moore & Balents 2006; Roy 2006; Fu & Kane 2008)
Majorana fermions in semiconductor devices

(Sau, Lutchyn, Tewari, & Das Sarma 2009)
Majorana fermions in semiconductor devices

\[ H = \int d^2 r \psi \left[ -\frac{\nabla^2}{2m} - \pi - i\alpha (\sigma^x \partial_y - \sigma^y \partial_x) \right] \psi \]

Rashba spin-orbit coupling in 2D

(Sau, Lutchyn, Tewari, & Das Sarma 2009)
Majorana fermions in semiconductor devices

2DEG looks “spinless” if chemical potential is here

\[ H = \int d^2 \mathbf{r} \psi \left[ -\frac{\nabla^2}{2m} - \pi - i\alpha (\sigma^x \partial_y - \sigma^y \partial_x) + V_z \sigma^z \right] \psi \]

(Sau, Lutchyn, Tewari, & Das Sarma 2009)
Majorana fermions in semiconductor devices

\[ H = \int d^2 r \psi \left[ -\frac{\nabla^2}{2m} - \pi - i\alpha (\sigma^x \partial_y - \sigma^y \partial_x) + V_z \sigma^z \right] \psi + \int d^2 r (\Delta \psi_\uparrow \psi_\downarrow + h.c.) \]

Realizes 2D topological SC supporting Majorana zero-modes (when chemical potential is tuned)

(Sau, Lutchyn, Tewari, & Das Sarma 2009)
Summary so far

1D spinless p-wave SC

Key advance!

2D Topological Insulator

among other promising realizations!

Common theme: spin-orbit coupling + proximity effects

2D spinless p+ip SC

3D Topological Insulator

FM insulator

s-wave SC

2DEG with Rashba spin-orbit coupling

s-wave SC
Next time: Majorana detection schemes, experimental status, and applications