A non-Fermi liquid: Quantum criticality of metals near the Pomeranchuk instability

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Quantum criticality of Ising-nematic ordering

Fermi surface with full square lattice symmetry
Quantum criticality of Ising-nematic ordering

Spontaneous elongation along $x$ direction:
Quantum criticality of Ising-nematic ordering

Spontaneous elongation along $y$ direction:
Ising-nematic order parameter

\[ \phi \sim \int d^2 k \left( \cos k_x - \cos k_y \right) c_{k\sigma}^\dagger c_{k\sigma} \]

Measures spontaneous breaking of square lattice point-group symmetry of underlying Hamiltonian
Quantum criticality of Ising-nematic ordering

or

\[ \langle \phi \rangle \neq 0 \]

or

\[ \langle \phi \rangle = 0 \]

Pomeranchuk instability as a function of coupling \( r \)
Quantum criticality of Ising-nematic ordering

Phase diagram as a function of $T$ and $r$
Quantum criticality of Ising-nematic ordering

Phase diagram as a function of $T$ and $r$
Quantum criticality of Ising-nematic ordering

Phase diagram as a function of $T$ and $r$

Quantum critical

Classical $d=2$ Ising criticality

$\langle \phi \rangle \neq 0$

$T_{I-n}$

$D=2+l$ Ising criticality?

$\langle \phi \rangle = 0$

Tuesday, January 8, 13
Quantum criticality of Ising-nematic ordering

Phase diagram as a function of $T$ and $r$

$T$

$T_{I-n}$

Classical $d=2$ Ising criticality

$\langle \phi \rangle \neq 0$

$D=2+1$ Ising criticality?

$\langle \phi \rangle = 0$

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Quantum criticality of Ising-nematic ordering

Phase diagram as a function of $T$ and $\mathcal{r}$
Quantum criticality of Ising-nematic ordering

Phase diagram as a function of $T$ and $r$
Effective action for Ising order parameter

$$S_\phi = \int d^2r d\tau \left[ (\partial_\tau \phi)^2 + c^2 (\nabla \phi)^2 + (\lambda - \lambda_c) \phi^2 + u\phi^4 \right]$$
Quantum criticality of Ising-nematic ordering

Effective action for Ising order parameter

\[ S_{\phi} = \int d^2r d\tau \left[ (\partial_\tau \phi)^2 + c^2(\nabla \phi)^2 + (\lambda - \lambda_c)\phi^2 + u\phi^4 \right] \]

Effective action for electrons:

\[ S_c = \int d\tau \sum_{\alpha=1}^{N_f} \left[ \sum_i c_{i\alpha}^\dagger \partial_\tau c_{i\alpha} - \sum_{i<j} t_{ij} c_{i\alpha}^\dagger c_{i\alpha} \right] \]

\[ \equiv \sum_{\alpha=1}^{N_f} \sum_k \int d\tau c_{k\alpha}^\dagger \left( \partial_\tau + \varepsilon_k \right) c_{k\alpha} \]
Quantum criticality of Ising-nematic ordering

Coupling between Ising order and electrons

\[ S_{\phi c} = -g \int d\tau \sum_{\alpha=1}^{N_f} \sum_{k,q} \phi_q (\cos k_x - \cos k_y) c_{k+q/2,\alpha}^\dagger c_{k-\mathbf{q}/2,\alpha} \]

for spatially dependent \( \phi \)

\[ \langle \phi \rangle > 0 \quad \text{and} \quad \langle \phi \rangle < 0 \]
Quantum criticality of Ising-nematic ordering

\[ S_\phi = \int d^2r d\tau \left[ (\partial_\tau \phi)^2 + c^2(\nabla \phi)^2 + (\lambda - \lambda_c)\phi^2 + u\phi^4 \right] \]

\[ S_c = \sum_{\alpha=1}^{N_f} \sum_{k} \int d\tau c_{k\alpha}^\dagger (\partial_\tau + \varepsilon_k) c_{k\alpha} \]

\[ S_{\phi c} = -g \int d\tau \sum_{\alpha=1}^{N_f} \sum_{k,q} \phi_q (\cos k_x - \cos k_y) c_{k+q/2,\alpha}^\dagger c_{k-q/2,\alpha} \]

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• $\phi$ fluctuation at wavevector $\vec{q}$ couples most efficiently to fermions near $\pm \vec{k}_0$. 
Quantum criticality of Ising-nematic ordering

- \( \phi \) fluctuation at wavevector \( \vec{q} \) couples most efficiently to fermions near \( \pm \vec{k}_0 \).
- Expand fermion kinetic energy at wavevectors about \( \pm \vec{k}_0 \) and boson (\( \phi \)) kinetic energy about \( \vec{q} = 0 \).
Quantum criticality of Ising-nematic ordering

\[ \mathcal{L}[\psi_\pm, \phi] = \psi_+^\dagger \left( \partial_\tau - i\partial_x - \partial_y^2 \right) \psi_+ + \psi_-^\dagger \left( \partial_\tau + i\partial_x - \partial_y^2 \right) \psi_- 
- \phi \left( \psi_+^\dagger \psi_+ + \psi_-^\dagger \psi_- \right) + \frac{1}{2g^2} \left( \partial_y \phi \right)^2 \]

Quantum criticality of Ising-nematic ordering

\[ \mathcal{L} = \psi_+^\dagger (\partial_\tau - i \partial_x - \partial_y^2) \psi_+ + \psi_-^\dagger (\partial_\tau + i \partial_x - \partial_y^2) \psi_- - \phi \left( \psi_+^\dagger \psi_+ + \psi_-^\dagger \psi_- \right) + \frac{1}{2g^2} (\partial_y \phi)^2 \]

One loop $\phi$ self-energy with $N_f$ fermion flavors:

\[ \Sigma_\phi(\vec{q}, \omega) = N_f \int \frac{d^2k}{4\pi^2} \frac{d\Omega}{2\pi} \frac{1}{[-i(\Omega + \omega) + k_x + q_x + (k_y + q_y)^2] [-i\Omega - k_x + k_y^2]} \]

\[ = \frac{N_f}{4\pi} \frac{|\omega|}{|q_y|} \]

Landau-damping
Quantum criticality of Ising-nematic ordering

\[ \mathcal{L} = \psi_+^\dagger (\partial_\tau - i\partial_x - \partial_y^2) \psi_+ + \psi_-^\dagger (\partial_\tau + i\partial_x - \partial_y^2) \psi_- \\
- \phi (\psi_+^\dagger \psi_+ + \psi_-^\dagger \psi_-) + \frac{1}{2g^2} (\partial_y \phi)^2 \]

Electron self-energy at order \(1/N_f\):

\[ \Sigma(\vec{k}, \Omega) = -\frac{1}{N_f} \int \frac{d^2q}{4\pi^2} \frac{d\omega}{2\pi} \frac{1}{[-i(\omega + \Omega) + k_x + q_x + (k_y + q_y)^2] \left[ \frac{q_y^2}{g^2} + \frac{|\omega|}{|q_y|} \right]} \]

\[ = -i \frac{2}{\sqrt{3}N_f} \left( \frac{g^2}{4\pi} \right)^{2/3} \text{sgn}(\Omega)|\Omega|^{2/3} \]
Quantum criticality of Ising-nematic ordering

\[ \mathcal{L} = \psi_+^\dagger (\partial_\tau - i \partial_x - \partial_y^2) \psi_+ + \psi_-^\dagger (\partial_\tau + i \partial_x - \partial_y^2) \psi_- \\
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Electron self-energy at order $1/N_f$:

\[ \Sigma(\vec{k}, \Omega) = -\frac{1}{N_f} \int \frac{d^2q \ d\omega}{4\pi^2 2\pi} \frac{1}{[-i(\omega + \Omega) + k_x + q_x + (k_y + q_y)^2] \left[ \frac{q_y^2}{g^2} + \frac{\left|\omega\right|}{|q_y|} \right]} \]

\[ = -i \frac{2}{\sqrt{3}N_f} \left( \frac{g^2}{4\pi} \right)^{2/3} \text{sgn}(\Omega) |\Omega|^{2/3} \sim |\Omega|^{d/3} \text{ in dimension } d. \]
Quantum criticality of Ising-nematic ordering

\[ \mathcal{L} = \psi_+^\dagger \left( \partial_\tau - i \partial_x - \partial_y^2 \right) \psi_+ + \psi_-^\dagger \left( \partial_\tau + i \partial_x - \partial_y^2 \right) \psi_- - \phi \left( \psi_+^\dagger \psi_+ + \psi_-^\dagger \psi_- \right) + \frac{1}{2g^2} (\partial_y \phi)^2 \]

Schematic form of \( \phi \) and fermion Green's functions in \( d \) dimensions

\[
D(\vec{q}, \omega) = \frac{1/N_f}{q_\perp^2 + \frac{|\omega|}{|q_\perp|}}, \quad G_f(\vec{q}, \omega) = \frac{1}{q_x + q_\perp^2 - i \text{sgn}(\omega) |\omega|^{d/3}/N_f}
\]

In the boson case, \( q_\perp^2 \sim \omega^{1/z_b} \) with \( z_b = 3/2 \).
In the fermion case, \( q_x \sim q_\perp^2 \sim \omega^{1/z_f} \) with \( z_f = 3/d \).

Note \( z_f < z_b \) for \( d > 2 \) \( \Rightarrow \) Fermions have higher energy than bosons, and perturbation theory in \( g \) is OK.
Strongly-coupled theory in \( d = 2 \).
Quantum criticality of Ising-nematic ordering

\[ \mathcal{L} = \psi_+^\dagger \left( \partial_\tau - i \partial_x - \partial_y^2 \right) \psi_+ + \psi_-^\dagger \left( \partial_\tau + i \partial_x - \partial_y^2 \right) \psi_- \\
- \phi \left( \psi_+^\dagger \psi_+ + \psi_-^\dagger \psi_- \right) + \frac{1}{2g^2} \left( \partial_y \phi \right)^2 \]

Schematic form of \( \phi \) and fermion Green’s functions in \( d = 2 \)

\[ D(q, \omega) = \frac{1/N_f}{q_y^2 + \frac{\omega}{|q_y|}} \quad , \quad G_f(q, \omega) = \frac{1}{q_x + q_y^2 - i \text{sgn}(\omega)|\omega|^{2/3}/N_f} \]

In both cases \( q_x \sim q_y^2 \sim \omega^{1/z} \), with \( z = 3/2 \). Note that the bare term \( \sim \omega \) in \( G_f^{-1} \) is irrelevant.

Strongly-coupled theory without quasiparticles.
Quantum criticality of Ising-nematic ordering

\[ \mathcal{L} = \psi_+^\dagger (\partial_\tau - i\partial_x - \partial_y^2) \psi_+ + \psi_-^\dagger (\partial_\tau + i\partial_x - \partial_y^2) \psi_- \\
- \phi \left( \psi_+^\dagger \psi_+ + \psi_-^\dagger \psi_- \right) + \frac{1}{2g^2} (\partial_y \phi)^2 \]

Simple scaling argument for \( z = 3/2 \).
Quantum criticality of Ising-nematic ordering

\[
\mathcal{L}_{\text{scaling}} = \psi_+^\dagger (-i \partial_x - \partial_y^2) \psi_+ + \psi_-^\dagger (+i \partial_x - \partial_y^2) \psi_-
\]

\[- g \phi \left( \psi_+^\dagger \psi_+ - \psi_-^\dagger \psi_- \right) + (\partial_y \phi)^2 \]

Simple scaling argument for \( z = 3/2 \).
Quantum criticality of Ising-nematic ordering

\[
\mathcal{L}_{\text{scaling}} = \psi_+^\dagger (-i\partial_x - \partial_y^2) \psi_+ + \psi_-^\dagger (+i\partial_x - \partial_y^2) \psi_- \\
- g \phi \left( \psi_+^\dagger \psi_+ - \psi_-^\dagger \psi_- \right) + (\partial_y \phi)^2
\]

Simple scaling argument for \( z = 3/2 \).

Under the rescaling \( x \to x/s \), \( y \to y/s^{1/2} \), and \( \tau \to \tau/s^z \), we find invariance provided

\[
\phi \to \phi s^{(2z+1)/4} \\
\psi \to \psi s^{(2z+1)/4} \\
g \to g s^{(3-2z)/4}
\]

So the action is invariant provided \( z = 3/2 \).
Quantum critical metals near the onset of antiferromagnetism: superconductivity and other instabilities

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BaFe$_2$(As$_{1-x}$P$_x$)$_2$

Resistivity $\sim \rho_0 + AT^n$

Lower $T_c$ superconductivity in the heavy fermion compounds

**Hole-doped cuprates**

G. Grissonnanche et al., preprint
Temperature (K)

Hole doping

0.00 0.06 0.12 0.18 0.24 0.30

Spin order

Charge order

T\textsubscript{max}

YBCO

T\textsubscript{c}

G. Grissonnanche et al., preprint
Figure 4: Fermi surface reconstruction from order that is biaxial within each bilayer and staged perpendicular to the bilayers. (A) A schematic of biaxial order yielding inequivalent sites (depicted by symbols of varying color and size) within the bilayers (considering an example where $\delta_1 \approx \delta_2 \approx 0.25$) resulting in a body-centered tetragonal superstructure. (B) Reconstruction of the Brillouin zone, with one instance of the pocket location indicated at the $\Gamma' X'$ point in relation to the original Fermi surface (purple) and nodes in the superconducting wave function. (C) A three-dimensional rendition of the concentrically arranged Fermi surfaces resulting from bilayer splitting exhibiting a twofold screw warping.

Of the form we experimentally observe occurs at the $X'$ points of the new Brillouin zone, as demonstrated in Fig. 1D. The Fermi surface pockets must therefore be located at the $X'$ point of the new body-centered tetragonal Brillouin zone. In order to locate the Fermi surface pockets in the original Brillouin zone, we note that the $X'$ point of the new reduced Brillouin zone coincides with the nodes in the superconducting wave function in the original Brillouin zone (shown by dotted lines). We are thus able to locate the reconstructed Fermi surface pocket as originating from the nodal regions of the original Brillouin zone. For example, the biaxial reconstruction scheme discussed in Ref. [16] would yield the nodal Fermi surface shown in Fig. 4B. The staggered unidirectional model discussed in Ref. [35] would also result in a body-centred tetragonal transformation, the Fermi surface resulting from such a model would be of interest to explore.

Our finding of a twofold screw symmetry rules out an origin of the pockets from non-conductors [17, 18, 19] (see Fig. 1B)). Materials with a body-centered tetragonal structure (see Fig. 1C), by contrast, exhibit a unique twofold screw symmetry at the Brillouin zone corner $\Gamma' X'$ (see Fig. 1D). Examples of layered materials with Fermi surface warpings exhibiting this symmetry include the ruthenates [7], the overdoped Tl-based cuprates [9] and pnictide superconductors of the $\Gamma' 122'$ composition [8].

The symmetry of the Fermi surface warping is determined by fitting to the angular dependence of the quantum oscillation frequencies and amplitudes in tilted magnetic fields [7, 9]. In the case...
Outline

1. Weak coupling theory of SDW ordering, and d-wave superconductivity

2. Universal critical theory of SDW ordering

3. Emergent pseudospin symmetry, and quadrupolar density wave

4. Quantum Monte Carlo without the sign problem
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The Hubbard Model

\[ H = - \sum_{i<j} t_{ij} c_{i\alpha} \dagger c_{j\alpha} + U \sum_i \left( n_{i\uparrow} - \frac{1}{2} \right) \left( n_{i\downarrow} - \frac{1}{2} \right) - \mu \sum_i c_{i\alpha} \dagger c_{i\alpha} \]

\( t_{ij} \rightarrow \) “hopping”. \( U \rightarrow \) local repulsion, \( \mu \rightarrow \) chemical potential

Spin index \( \alpha = \uparrow, \downarrow \)

\[ n_{i\alpha} = c_{i\alpha} \dagger c_{i\alpha} \]

\[ c_{i\alpha} \dagger c_{j\beta} + c_{j\beta} c_{i\alpha} \dagger = \delta_{ij} \delta_{\alpha\beta} \]

\[ c_{i\alpha} c_{j\beta} + c_{j\beta} c_{i\alpha} = 0 \]
The electron spin polarization obeys

\[ \left\langle \vec{S}(\mathbf{r}, \tau) \rightangle = \varphi(\mathbf{r}, \tau)e^{i\mathbf{K} \cdot \mathbf{r}} \]

where \( \mathbf{K} \) is the ordering wavevector.
The Hubbard Model

Decouple $U$ term by a Hubbard-Stratanovich transformation

\[
S = \int d^2r \, d\tau \left[ \mathcal{L}_c + \mathcal{L}_\varphi + \mathcal{L}_{c\varphi} \right]
\]

\[
\mathcal{L}_c = c_\alpha^\dagger \varepsilon (-i \nabla) c_\alpha
\]

\[
\mathcal{L}_\varphi = \frac{1}{2} (\nabla \varphi_\alpha)^2 + \frac{r}{2} \varphi_\alpha^2 + \frac{u}{4} (\varphi_\alpha^2)^2
\]

\[
\mathcal{L}_{c\varphi} = \lambda \varphi_\alpha e^{i \mathbf{K} \cdot \mathbf{r}} c_\alpha^\dagger \sigma_\alpha^{ab} c_b.
\]

"Yukawa" coupling between fermions and antiferromagnetic order:

$\lambda^2 \sim U$, the Hubbard repulsion
Fermi surface + antiferromagnetism

Metal with electron and hole pockets

Metal with “large” Fermi surface

Increasing interaction

\( \langle \phi \rangle \neq 0 \)

\( \langle \phi \rangle = 0 \)

Theory of quantum criticality in the cuprates

Fluctuating Fermi pockets

Large Fermi surface

Quantum Critical

Spin density wave (SDW)

Underlying SDW ordering quantum critical point in metal at $x = x_m$
Fluctuating Fermi pockets

Large Fermi surface

Quantum Critical

Relaxation and equilibration times $\sim \hbar / k_B T$ are robust properties of strongly-coupled quantum criticality
Theory of quantum criticality in the cuprates

Fluctuating Fermi pockets

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Strange Metal?

Spin density wave (SDW)

Relaxation and equilibration times $\sim \hbar/k_B T$ are robust properties of strongly-coupled quantum criticality
Pairing by SDW fluctuation exchange

We now allow the SDW field $\varphi$ to be dynamical, coupling to electrons as

$$H_{sdw} = - \sum_{k, q, \alpha, \beta} \varphi_k \cdot c^\dagger_{k, \alpha} \bar{\sigma}_{\alpha \beta} c_{k+q, \beta}.$$  

Exchange of a $\varphi$ quantum leads to the effective interaction

$$H_{ee} = -\frac{1}{2} \sum_{q, \alpha, \beta} \sum_{p, \gamma} V_{\alpha \beta, \gamma \delta (q)} c^\dagger_{k, \alpha} c_{k+q, \beta} c^\dagger_{p, \gamma} c_{p-q, \delta},$$

where the pairing interaction is

$$V_{\alpha \beta, \gamma \delta (q)} = \bar{\sigma}_{\alpha \beta} \cdot \bar{\sigma}_{\gamma \delta} \frac{\chi_0}{\xi^{-2} + (q - K)^2},$$

with $\chi_0 \xi^2$ the SDW susceptibility and $\xi$ the SDW correlation length.
**BCS Gap equation**

In BCS theory, this interaction leads to the ‘gap equation’ for the pairing gap \( \Delta_k \propto \langle c_{k\uparrow} c_{-k\downarrow} \rangle \).

\[
\Delta_k = - \sum_p \left( \frac{3 \chi_0}{\xi^{-2} + (p - k - K)^2} \right) \frac{\Delta_p}{2 \sqrt{\xi_p^2 + \Delta_p^2}}
\]

Non-zero solutions of this equation require that \( \Delta_k \) and \( \Delta_p \) have opposite signs when \( p - k \approx K \).
Pairing “glue” from antiferromagnetic fluctuations


\[ \langle c_{\mathbf{k}\alpha}^\dagger c_{-\mathbf{k}\beta}^\dagger \rangle = \varepsilon_{\alpha\beta} \Delta (\cos k_x - \cos k_y) \]

Unconventional pairing at and near hot spots
Theory of quantum criticality in the cuprates

Underlying SDW ordering quantum critical point in metal at $x = x_m$
SDW quantum critical point is unstable to $d$-wave superconductivity. This instability is stronger than that in the BCS theory.

Notice shift between the position of the QCP in the superconductor, and the divergence in effective mass in the metal measured at high magnetic fields.
Spin density wave (SDW) quantum critical point is unstable to \( d \)-wave superconductivity. This instability is stronger than that in the BCS theory.

Theory of quantum criticality in the cuprates

Fluctuating, paired Fermi pockets

Strange Metal

Large Fermi surface

d-wave superconductor

Spin density wave (SDW)

Competition between SDW order and superconductivity moves the actual quantum critical point to $x = x_s < x_m$.

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At stronger coupling, different effects compete:

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At stronger coupling, different effects compete:

- Pairing glue becomes stronger.
- There is stronger fermion-boson scattering, and fermionic quasi-particles lose their integrity.
- Other instabilities can appear *e.g.* to charge density waves/striped order.
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Fermi surface + antiferromagnetism

Metal with “large” Fermi surface
Fermi surfaces translated by $\mathbf{K} = (\pi, \pi)$. 
Fermi surface + antiferromagnetism

“Hot” spots
Electron and hole pockets in antiferromagnetic phase with $\langle \tilde{\phi} \rangle \neq 0$
“Hot” spots
Low energy theory for critical point near hot spots
Low energy theory for critical point near hot spots
Theory has fermions $\psi_{1,2}$ (with Fermi velocities $v_{1,2}$) and boson order parameter $\tilde{\varphi}$, interacting with coupling $\lambda$.
\[ \mathcal{L}_f = \psi_{1\alpha}^{\dagger} (\zeta \partial_{\tau} - iv_1 \cdot \nabla_r) \psi_{1\alpha} + \psi_{2\alpha}^{\dagger} (\zeta \partial_{\tau} - iv_2 \cdot \nabla_r) \psi_{2\alpha} \]

\[ \mathcal{L}_f = \psi_{1\alpha}^\dagger (\zeta \partial_\tau - i \mathbf{v}_1 \cdot \nabla_r) \psi_{1\alpha} + \psi_{2\alpha}^\dagger (\zeta \partial_\tau - i \mathbf{v}_2 \cdot \nabla_r) \psi_{2\alpha} \]

“Hot spot”

“Cold” Fermi surfaces

\[ \mathcal{L}_f = \psi_{1\alpha}^\dagger (\zeta \partial_\tau - i \mathbf{v}_1 \cdot \nabla_r) \psi_{1\alpha} + \psi_{2\alpha}^\dagger (\zeta \partial_\tau - i \mathbf{v}_2 \cdot \nabla_r) \psi_{2\alpha} \]

Order parameter: \[ \mathcal{L}_\varphi = \frac{1}{2} (\nabla_r \bar{\varphi})^2 + \frac{\zeta}{2} (\partial_\tau \bar{\varphi})^2 + \frac{s}{2} \bar{\varphi}^2 + \frac{u}{4} \bar{\varphi}^4 \]

\[ \mathcal{L}_f = \psi_{1\alpha}^\dagger (\zeta \partial_\tau - i \mathbf{v}_1 \cdot \nabla_r) \psi_{1\alpha} + \psi_{2\alpha}^\dagger (\zeta \partial_\tau - i \mathbf{v}_2 \cdot \nabla_r) \psi_{2\alpha} \]

Order parameter: \[ \mathcal{L}_\varphi = \frac{1}{2} (\nabla_r \bar{\varphi})^2 + \frac{\zeta}{2} (\partial_\tau \bar{\varphi})^2 + \frac{s}{2} \varphi^2 + \frac{u}{4} \varphi^4 \]

“Yukawa” coupling: \[ \mathcal{L}_c = -\lambda \bar{\varphi} \cdot \left( \psi_{1\alpha}^\dagger \bar{\sigma}_{\alpha\beta} \psi_{2\beta} + \psi_{2\alpha}^\dagger \bar{\sigma}_{\alpha\beta} \psi_{1\beta} \right) \]

\[ \mathcal{L}_f = \psi_{1\alpha}^\dagger (\zeta \partial_\tau - i \mathbf{v}_1 \cdot \nabla_r) \psi_{1\alpha} + \psi_{2\alpha}^\dagger (\zeta \partial_\tau - i \mathbf{v}_2 \cdot \nabla_r) \psi_{2\alpha} \]

Fermion dispersions: \( \varepsilon_{k1} = \mathbf{v}_1 \cdot \mathbf{k} \) and \( \varepsilon_{k2} = \mathbf{v}_2 \cdot \mathbf{k} \)

Metal with “large” Fermi surface \( \langle \varphi \rangle = 0 \)
Fermion dispersions:

\[ E_{k\pm} = \frac{\varepsilon_{k1} + \varepsilon_{k2}}{2} \pm \sqrt{\left( \frac{\varepsilon_{k1} - \varepsilon_{k2}}{2} \right)^2 + \lambda^2 |\varphi|^2} \]

Metal with hole and electron pockets \( \langle \varphi \rangle \neq 0 \)
Hertz action.

Upon integrating the fermions out, the leading term in the $\bar{\phi}$ effective action is $-\Pi(q, \omega_n)|\bar{\phi}(q, \omega_n)|^2$, where $\Pi(q, \omega_n)$ is the fermion polarizability. This is given by a simple fermion loop diagram

\[
\Pi(q, \omega_n) = \int \frac{d^d k}{(2\pi)^d} \int \frac{d\epsilon_n}{2\pi} \frac{1}{[-i\zeta(\epsilon_n + \omega_n) + \mathbf{v}_1 \cdot (\mathbf{k} + \mathbf{q})][-i\zeta\epsilon_n + \mathbf{v}_2 \cdot \mathbf{k}]}.
\]

We define oblique co-ordinates $p_1 = \mathbf{v}_1 \cdot \mathbf{k}$ and $p_2 = \mathbf{v}_2 \cdot \mathbf{k}$. It is then clear that the integrand in (1) is independent of the $(d-2)$ transverse momenta, whose integral yields an overall factor $\Lambda^{d-2}$ (in $d = 2$ this factor is precisely 1). Also, by shifting the integral...
over $k_1$ we note that the integral is independent of $q$. So we have

$$\Pi(q, \omega_n) = \frac{\Lambda^{d-2}}{|v_1 \times v_2|} \int \frac{dp_1 dp_2 d\epsilon_n}{8\pi^3} \frac{1}{[-i\zeta(\epsilon_n + \omega_n) + p_1][-i\zeta \epsilon_n + p_2]}.$$  

Next, we evaluate the frequency integral to obtain

$$\Pi(q, \omega_n) = \frac{\Lambda^{d-2}}{\zeta |v_1 \times v_2|} \int \frac{dp_1 dp_2}{4\pi^2} \frac{[\text{sgn}(p_2) - \text{sgn}(p_1)]}{-i\zeta \omega_n + p_1 - p_2}$$

$$= -\frac{|\omega_n| \Lambda^{d-2}}{4\pi |v_1 \times v_2|}. \tag{3}$$

In the last step, we have dropped a frequency-independent, cutoff-dependent constant which can absorbed into a redefinition of $r$. Notice also that the factor of $\zeta$ has cancelled.

Inserting this fermion polarizability in the effective action for $\varphi$, we obtain the Hertz action for the SDW transition:

$$S_H = \int \frac{d^d k}{(2\pi)^d} T \sum_{\omega_n} \frac{1}{2} \left[ k^2 + \gamma |\omega_n| + s \right] |\bar{\varphi}(k, \omega_n)|^2$$

$$+ \frac{u}{4} \int d^d x d\tau \left( \varphi^2(x, \tau) \right)^2. \tag{4}$$
Exercise: Perform a tree-level RG rescaling on $S_H$. Now we rescale co-ordinates as $x' = x e^{-\ell}$ and $\tau' = \tau e^{-z\ell}$. Here $z$ is the dynamic critical exponent. Show that the gradient and non-local terms become invariant for $z = 2$ (previous theories considered here had $z = 1$). Then show that the transformation of the quartic term is $u' = u e^{(2-d)\ell}$. This led Hertz to conclude that the SDW quantum critical point was described by a Gaussian theory for the SDW order parameter in $d \geq 2$. 
Fate of the fermions.

Let us, for now, assume the validity of the Hertz Gaussian action, and compute the leading correction to the electronic Green’s function. This is given by the following Feynman graph for the electron self energy, $\Sigma$. At zero momentum for the $\psi_1$ fermion we have

$$\Sigma_1(0, \omega_n) = \lambda^2 \int \frac{d^d q}{(2\pi)^d} \int \frac{d\epsilon_n}{2\pi} \frac{1}{[q^2 + \gamma|\epsilon_n|][-i\zeta(\epsilon_n + \omega_n) + \mathbf{v}_2 \cdot \mathbf{q}]}.$$  

We first perform the integral over the $\mathbf{q}$ direction parallel to $\mathbf{v}_2$, while ignoring the subdominant dependence on this momentum in the boson propagator. The dependence on $\zeta$ immediately
disappears, and we have

\[
\Sigma_1(0, \omega_n) = i \frac{\lambda^2}{|v_2|} \int \frac{d^{d-1}q}{(2\pi)^{d-1}} \int \frac{d\epsilon_n \, \text{sgn}(\epsilon_n + \omega_n)}{2\pi |q|^2 + \gamma|\epsilon_n|} \sqrt{|\omega_n|} \quad , \quad d = 2.
\]

(6)

(7)

(8)

Evaluation of the \( q \) integral shows that

\[
\Sigma_1(0, \omega_n) \sim |\omega_n|^{(d-1)/2}
\]

The most important case is \( d = 2 \), where we have
Strong coupling physics in $d = 2$

The theory so far has the boson propagator

$$
\sim \frac{1}{q^2 + \gamma|\omega|}
$$

which scales with dynamic exponent $z_b = 2$, and now a fermion propagator

$$
\sim \frac{1}{-i\zeta \omega + c_1|\omega|^{(d-1)/2} + v \cdot q}.
$$

First note that for $d < 3$, the bare $-i\zeta \omega$ term is less important than the contribution from the self energy at low frequencies. This indicates that $\zeta$ is *irrelevant* in the critical theory, and we can set $\zeta \to 0$. Fortunately, all the loop diagrams evaluated so far are independent of $\zeta$.

Setting $\zeta = 0$, we see that the fermion propagator scales with dynamic exponent $z_f = 2/(d - 1)$. For $d > 2$, $z_f < z_b$, and so at small momenta the boson fluctuations have lower energy than the fermion fluctuations. Thus it seems reasonable to assume that the
fermion fluctuations are not as singular, and we can focus on an effective theory of the SDW order parameter $\bar{\varphi}$ alone. In other words, the Hertz assumptions appear valid for $d > 2$.

However, in $d = 2$, we have $z_f = z_b = 2$. Thus fermionic and bosonic fluctuations are equally important, and it is not appropriate to integrate the fermions out at an initial stage. We have to return to the original theory of coupled bosons and fermions. This turns out to be strongly coupled, and exhibits complex critical behavior. For more details, see

Perform RG on both fermions and $\bar{\varphi}$, using a local field theory.

$$\mathcal{L}_f = \psi_{1\alpha}^\dagger (\zeta \partial_\tau - i \mathbf{v}_1 \cdot \nabla_r) \psi_{1\alpha} + \psi_{2\alpha}^\dagger (\zeta \partial_\tau - i \mathbf{v}_2 \cdot \nabla_r) \psi_{2\alpha}$$

Order parameter:

$$\mathcal{L}_\varphi = \frac{1}{2} (\nabla_r \bar{\varphi})^2 + \frac{\zeta}{2} (\partial_\tau \bar{\varphi})^2 + \frac{s}{2} \bar{\varphi}^2 + \frac{u}{4} \bar{\varphi}^4$$

“Yukawa” coupling:

$$\mathcal{L}_c = -\lambda \bar{\varphi} \cdot \left( \psi_{1\alpha}^\dagger \bar{\sigma}_{\alpha\beta} \psi_{2\beta} + \psi_{2\alpha}^\dagger \bar{\sigma}_{\alpha\beta} \psi_{1\beta} \right)$$
\[ \mathcal{L}_f = \psi_{1\alpha}^\dagger (\zeta \partial_\tau - i \mathbf{v}_1 \cdot \nabla_r) \psi_{1\alpha} + \psi_{2\alpha}^\dagger (\zeta \partial_\tau - i \mathbf{v}_2 \cdot \nabla_r) \psi_{2\alpha} \]

Order parameter: \[ \mathcal{L}_\varphi = \frac{1}{2} (\nabla_r \varphi')^2 + \frac{\zeta}{2} (\partial_\tau \varphi')^2 + \frac{s}{2} \varphi'^2 + \frac{u}{4} \varphi'^4 \]

“Yukawa” coupling: \[ \mathcal{L}_c = -\lambda \varphi' \cdot \left( \psi_{1\alpha}^\dagger \bar{\sigma}_{\alpha\beta} \psi_{2\beta} + \psi_{2\alpha}^\dagger \bar{\sigma}_{\alpha\beta} \psi_{1\beta} \right) \]

Under the rescaling \( x' = x e^{-\ell}, \tau' = \tau e^{-z\ell} \), the spatial gradients are fixed if the fields transform as

\[ \varphi' = e^{(d+z-2)\ell/2} \varphi; \quad \psi' = e^{(d+z-1)\ell/2} \psi. \]

Then the Yukawa coupling transforms as

\[ \lambda' = e^{(4-d-z)\ell/2} \lambda \]

For \( d = 2 \), with \( z = 2 \) the bare time-derivative terms \( \zeta, \tilde{\zeta} \) are irrelevant, but the Yukawa coupling is invariant. Thus we have to work at fixed \( \lambda = 1 \), and cannot expand in powers of \( \lambda \): critical theory is strongly coupled.
Critical point theory is strongly coupled in $d = 2$
Results are independent of coupling $\lambda$

\[ G_{\text{fermion}} \sim \frac{1}{i \sqrt{\omega} - v \cdot k} \]


Tuesday, January 8, 13
Critical point theory is strongly coupled in $d = 2$
Results are *independent* of coupling $\lambda$

$$G_{\text{fermion}} = \frac{Z(k_{\parallel})}{i\omega - v_F(k_{\parallel})k_{\perp}}, \quad Z(k_{\parallel}) \sim v_F(k_{\parallel}) \sim k_{\parallel}$$
Outline

1. Weak coupling theory of SDW ordering, and d-wave superconductivity

2. Universal critical theory of SDW ordering

3. Emergent pseudospin symmetry, and quadrupolar density wave

4. Quantum Monte Carlo without the sign problem
Outline

1. Weak coupling theory of SDW ordering, and d-wave superconductivity

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Emergent $[SU(2)]^4$ pseudospin symmetry

\[ \mathcal{L}_f = \psi_{1\alpha}^\dagger (\zeta \partial_\tau - i \mathbf{v}_1 \cdot \nabla_r) \psi_{1\alpha} + \psi_{2\alpha}^\dagger (\zeta \partial_\tau - i \mathbf{v}_2 \cdot \nabla_r) \psi_{2\alpha} \]

Order parameter: \[ \mathcal{L}_\varphi = \frac{1}{2} (\nabla_r \varphi) ^2 + \frac{\zeta}{2} (\partial_\tau \varphi) ^2 + \frac{s}{2} \varphi^2 + \frac{u}{4} \varphi^4 \]

“Yukawa” coupling: \[ \mathcal{L}_c = -\lambda \varphi \cdot (\psi_{1\alpha}^\dagger \tilde{\sigma}_{\alpha\beta} \psi_{2\beta} + \psi_{2\alpha}^\dagger \tilde{\sigma}_{\alpha\beta} \psi_{1\beta}) \]
**Emergent $[SU(2)]^4$ pseudospin symmetry**

\[ \mathcal{L}_f = \psi_{1\alpha}^\dagger (\zeta \partial_\tau - iv_1 \cdot \nabla_r) \psi_{1\alpha} + \psi_{2\alpha}^\dagger (\zeta \partial_\tau - iv_2 \cdot \nabla_r) \psi_{2\alpha} \]

Order parameter: \[ \mathcal{L}_\varphi = \frac{1}{2} (\nabla_r \varphi)^2 + \frac{\bar{\zeta}}{2} (\partial_\tau \varphi)^2 + \frac{s}{2} \varphi^2 + \frac{u}{4} \varphi^4 \]

“Yukawa” coupling: \[ \mathcal{L}_c = -\lambda \bar{\varphi} \cdot \left( \psi_{1\alpha}^\dagger \bar{\sigma}_{\alpha\beta} \psi_{2\beta} + \psi_{2\alpha}^\dagger \bar{\sigma}_{\alpha\beta} \psi_{1\beta} \right) \]

Introduce the spinors

\[ \Psi_{1\alpha} = \begin{pmatrix} \psi_{1\alpha} \\ \epsilon_{\alpha\beta} \psi_{1\beta}^\dagger \end{pmatrix}, \quad \Psi_{2\alpha} = \begin{pmatrix} \psi_{2\alpha} \\ \epsilon_{\alpha\beta} \psi_{2\beta}^\dagger \end{pmatrix} \]

Then the Lagrangian is invariant under the SU(2) transformation $U$ with

\[ \Psi_1 \rightarrow U \Psi_1, \quad \Psi_2 \rightarrow U \Psi_2 \]

Note that $U$ can be chosen independently at the 4 pairs of hotspots.

This symmetry relies on the linearization of the fermion dispersion about the hot spots.
Unconventional pairing at and near hot spots

\[ \left\langle c_{\mathbf{k}\alpha}^{\dagger} c_{-\mathbf{k}\beta}^{\dagger} \right\rangle = \varepsilon_{\alpha\beta} \Delta \left( \cos k_x - \cos k_y \right) \]
Unconventional particle-hole pairing at \( \Phi \) and near \( \Phi \).

\[
\left\langle c_{k-Q/2, \alpha}^\dagger c_{k+Q/2, \alpha} \right\rangle = \Phi (\cos k_x - \cos k_y)
\]

\( Q \) is \('2k_F'\) wavevector.

After pseudospin rotation


Unconventional particle-hole pairing at and near hot spots

\[ \langle c_{k - Q/2, \alpha}^\dagger c_{k + Q/2, \alpha} \rangle = \Phi(\cos k_x - \cos k_y) \]

\( Q \) is ‘2\( k_F \)’ wavevector

After pseudospin rotation


\( \Phi \) corresponds to a 2\( k_F \) bond-nematic or a quadrupole density wave

Unconventional particle-hole pairing at and near hot spots
Quadrupole density wave
Quadrupole density wave
Quadrupole density wave
Quadrupole density wave

\[ \left\langle c_{\mathbf{k}-Q/2,\alpha}^\dagger c_{\mathbf{k}+Q/2,\alpha} \right\rangle = \Phi(\cos k_x - \cos k_y) \]
No modulations on sites, $\langle c^\dagger_{r\alpha} c_{s\alpha} \rangle$ is modulated only for $r \neq s$.

$$\langle c^\dagger_{k-Q/2,\alpha} c_{k+Q/2,\alpha} \rangle = \Phi(\cos k_x - \cos k_y)$$
No modulations on sites, $\langle c_{\mathbf{r} \alpha}^\dagger c_{\mathbf{s} \alpha} \rangle$ is modulated only for $\mathbf{r} \neq \mathbf{s}$.

$$\langle c_{\mathbf{k} - \mathbf{Q}/2, \alpha}^\dagger c_{\mathbf{k} + \mathbf{Q}/2, \alpha} \rangle = \Phi(\cos k_x - \cos k_y)$$
Quadrupole density wave

No modulations on sites, $\langle c^\dagger_{r\alpha} c_{s\alpha} \rangle$ is modulated only for $r \neq s$.

$$\left\langle c^\dagger_{k-Q/2,\alpha} c_{k+Q/2,\alpha} \right\rangle = \Phi(\cos k_x - \cos k_y)$$

Local Ising nematic order with an envelope which oscillates.
Quadrupole density wave

No modulations on sites, $\langle c_{r\alpha}^{\dagger} c_{s\alpha} \rangle$ is modulated only for $r \neq s$.

$$\langle c_{k-Q/2,\alpha}^{\dagger} c_{k+Q/2,\alpha} \rangle = \Phi(\cos k_x - \cos k_y)$$
Strength of instability at quantum criticality

BCS theory

\[ 1 + \lambda_{e-ph} \log \left( \frac{\omega_D}{\omega} \right) \]
Strength of instability at quantum criticality

BCS theory

\[ 1 + \lambda_{e-ph} \log \left( \frac{\omega_D}{\omega} \right) \]

Electron-phonon coupling

Debye frequency

Implies

\[ T_c \sim \omega_D \exp \left( -\frac{1}{\lambda} \right) \]
Spin density wave quantum critical point

\[ 1 + \frac{\alpha}{\pi(1 + \alpha^2)} \log^2 \left( \frac{E_F}{\omega} \right) \]

Y. Wang and A. Chubukov, arXiv:1210.2408
Strength of instability at quantum criticality

Spin density wave quantum critical point

\[ 1 + \frac{\alpha}{\pi(1 + \alpha^2)} \log^2 \left( \frac{E_F}{\omega} \right) \]

\( \alpha = \tan \theta \), where \( 2\theta \) is the angle between Fermi lines. Independent of interaction strength \( U \) in 2 dimensions.

Y. Wang and A. Chubukov, arXiv:1210.2408
\[ G_{\text{fermion}} = \frac{Z(k_\parallel)}{i\omega - v_F(k_\parallel)k_\perp}, \quad Z(k_\parallel) \sim v_F(k_\parallel) \sim k_\parallel \]

\[ \int dk_\parallel \frac{1}{k_\parallel^2} \left( \frac{Z^2(k_\parallel)}{v_F(k_\parallel)} \right) \log \frac{k_\parallel^2}{\omega} \]
\[ G_{\text{fermion}} = \frac{Z(k_{||})}{i\omega - v_F(k_{||})k_\perp}, \quad Z(k_{||}) \sim v_F(k_{||}) \sim k_{||} \]

\[ \int dk_{||} \frac{1}{k_{||}^2} \left( \frac{Z^2(k_{||})}{v_F(k_{||})} \right) \log \frac{k_{||}^2}{\omega} \]

Cooper logarithm
\[ G_{\text{fermion}} = \frac{Z(k_{||})}{i\omega - v_F(k_{||})k_{\perp}}, \quad Z(k_{||}) \sim v_F(k_{||}) \sim k_{||} \]

\[ \int dk_{||} \frac{1}{k_{||}^2} \left( \frac{Z^2(k_{||})}{v_F(k_{||})} \right) \log \frac{k_{||}^2}{\omega} \]

Spin fluctuation propagator

Cooper logarithm
Enhancement of pairing susceptibility by interactions

Spin density wave quantum critical point

\[ 1 + \frac{\alpha}{\pi(1 + \alpha^2)} \log^2 \left( \frac{E_F}{\omega} \right) \]

- \( \log^2 \) singularity arises from Fermi lines; singularity at hot spots is weaker.
- Interference between BCS and quantum-critical logs.
- Momentum dependence of self-energy is crucial.
- Not suppressed by \( 1/N \) factor in \( 1/N \) expansion.
Enhancement of $\Phi$ susceptibility by interactions

Spin density wave quantum critical point

$$1 + \frac{\alpha}{3\pi(1 + \alpha^2)} \log^2 \left( \frac{E_F}{\omega} \right)$$

- Emergent pseudospin symmetry of low energy theory also induces $\log^2$ in a single “$d$-wave” particle-hole channel. Fermi-surface curvature reduces prefactor by $1/3$.
- $\Phi$ corresponds to a $2k_F$ bond-nematic or a quadrupole density wave

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1. Weak coupling theory of SDW ordering, and d-wave superconductivity

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Low energy theory for critical point near hot spots
QMC for the onset of antiferromagnetism

Hot spots in a single band model
Hot spots in a two band model

Faithful realization of the \textit{generic} universal low energy theory for the onset of antiferromagnetism.

\textbf{QMC for the onset of antiferromagnetism}

Hot spots in a two band model

QMC for the onset of antiferromagnetism

Hot spots in a two band model

QMC for the onset of antiferromagnetism

Sign problem is absent as $K$ connects hotspots in distinct bands

Hot spots in a two band model

QMC for the onset of antiferromagnetism

Sign problem is absent as \( K \) connects hotspots in distinct bands.

Particle-hole or point-group symmetries or commensurate densities \textit{not} required!


Hot spots in a two band model
QMC for the onset of antiferromagnetism

Electrons with dispersion $\varepsilon_\mathbf{k}$
interacting with fluctuations of the
antiferromagnetic order parameter $\varphi$.

$$\mathcal{Z} = \int \mathcal{D}c_\alpha \mathcal{D}\varphi \exp (-S)$$

$$S = \int d\tau \sum_\mathbf{k} c^\dagger_\mathbf{k}\alpha \left( \frac{\partial}{\partial \tau} - \varepsilon_\mathbf{k} \right) c_\mathbf{k}\alpha$$

$$+ \int d\tau d^2x \left[ \frac{1}{2} (\nabla_x \varphi)^2 + \frac{r}{2} \varphi^2 + \ldots \right]$$

$$- \lambda \int d\tau \sum_i \varphi_i \cdot (-1)^x_i c^\dagger_{i\alpha} \bar{\sigma}_{\alpha\beta} c_{i\beta}$$
Electrons with dispersions $\varepsilon^{(x)}_{k}$ and $\varepsilon^{(y)}_{k}$ interacting with fluctuations of the antiferromagnetic order parameter $\vec{\varphi}$.

$$Z = \int \mathcal{D}c^{(x)}_{\alpha} \mathcal{D}c^{(y)}_{\alpha} \mathcal{D}\varphi \exp (-S)$$

$$S = \int d\tau \sum_{k} c^{(x)\dagger}_{k\alpha} \left( \frac{\partial}{\partial \tau} - \varepsilon^{(x)}_{k} \right) c^{(x)}_{k\alpha}$$

$$+ \int d\tau \sum_{k} c^{(y)\dagger}_{k\alpha} \left( \frac{\partial}{\partial \tau} - \varepsilon^{(y)}_{k} \right) c^{(y)}_{k\alpha}$$

$$+ \int d\tau d^{2}x \left[ \frac{1}{2} (\nabla_{x} \varphi)^{2} + \frac{r}{2} \varphi^{2} + \ldots \right]$$

$$- \lambda \int d\tau \sum_{i} \vec{\varphi}_{i} \cdot (-1)^{x_{i}} c^{(x)\dagger}_{i\alpha} \vec{\sigma}_{\alpha\beta} c^{(y)}_{i\beta} + \text{H.c.}$$

QMC for the onset of antiferromagnetism

Electrons with dispersions $\varepsilon^{(x)}_k$ and $\varepsilon^{(y)}_k$
interacting with fluctuations of the antiferromagnetic order parameter $\vec{\varphi}$.

$$
\mathcal{Z} = \int \mathcal{D}c^{(x)}_\alpha \mathcal{D}c^{(y)}_\alpha \mathcal{D}\vec{\varphi} \exp (-S)
$$

$$
S = \int d\tau \sum_k c^{(x)}_k \left( \frac{\partial}{\partial \tau} - \varepsilon^{(x)}_k \right) c^{(x)}_k
$$

$$
+ \int d\tau \sum_k c^{(y)}_k \left( \frac{\partial}{\partial \tau} - \varepsilon^{(y)}_k \right) c^{(y)}_k
$$

$$
+ \int d\tau d^2x \left[ \frac{1}{2} (\nabla_x \vec{\varphi})^2 + \frac{r}{2} \vec{\varphi}^2 + \ldots \right]
$$

$$
- \lambda \int d\tau \sum_i \vec{\varphi}_i \cdot (-1)^{x_i} c^{(x)}_{i\alpha} \vec{\sigma}_{\alpha\beta} c^{(y)}_{i\beta} + \text{H.c.}
$$


No sign problem!
QMC for the onset of antiferromagnetism

Electrons with dispersions $\varepsilon^{(x)}_k$ and $\varepsilon^{(y)}_k$
interacting with fluctuations of the
antiferromagnetic order parameter $\bar{\varphi}$.

$$Z = \int Dc^{(x)}_\alpha Dc^{(y)}_\alpha D\bar{\varphi} \exp(-S)$$

$$S = \int d\tau \sum_k c^{(x)}_k \left( \frac{\partial}{\partial \tau} - \varepsilon^{(x)}_k \right) c^{(x)}_k$$

$$+ \int d\tau \sum_k c^{(y)}_k \left( \frac{\partial}{\partial \tau} - \varepsilon^{(y)}_k \right) c^{(y)}_k$$

$$+ \int d\tau d^2x \left[ \frac{1}{2} (\nabla x \bar{\varphi})^2 + \frac{r}{2} \bar{\varphi}^2 + \ldots \right]$$

$$- \lambda \int d\tau \sum_i \bar{\varphi}_i \cdot (-1)^{x_i} c^{(x)}_{i\alpha} \bar{\sigma}_{\alpha\beta} c^{(y)}_{i\beta} + \text{H.c.}$$

E. Berg, M. Metlitski, and S. Sachdev,

Applies without changes to the microscopic band
structure in the iron-based superconductors
QMC for the onset of antiferromagnetism

Electrons with dispersions $\varepsilon^{(x)}_k$ and $\varepsilon^{(y)}_k$ interacting with fluctuations of the antiferromagnetic order parameter $\varphi$.

$$Z = \int Dc_\alpha^{(x)} Dc_\alpha^{(y)} D\varphi \exp (-S)$$

$$S = \int d\tau \sum_k c_{k\alpha}^{(x)} \left( \frac{\partial}{\partial \tau} - \varepsilon^{(x)}_k \right) c_{k\alpha}^{(x)}$$

$$+ \int d\tau \sum_k c_{k\alpha}^{(y)} \left( \frac{\partial}{\partial \tau} - \varepsilon^{(y)}_k \right) c_{k\alpha}^{(y)}$$

$$+ \int d\tau d^2 x \left[ \frac{1}{2} (\nabla_x \varphi)^2 + \frac{r}{2} \varphi^2 + \ldots \right]$$

$$- \lambda \int d\tau \sum_i \varphi_i \cdot (-1)^{x_i} c_{i\alpha}^{(x)} \bar{\sigma}_\alpha \beta c_{i\beta}^{(y)} + H.c.$$
Hot spots in a two band model

QMC for the onset of antiferromagnetism

E. Berg, M. Metlitski, and S. Sachdev,
QMC for the onset of antiferromagnetism

Move one of the Fermi surface by \((\pi, \pi, \pi)\)

QMC for the onset of antiferromagnetism

Now hot spots are at Fermi surface intersections

QMC for the onset of antiferromagnetism


Expected Fermi surfaces in the AFM ordered phase
QMC for the onset of antiferromagnetism

Electron occupation number $n_k$ as a function of the tuning parameter $r$

QMC for the onset of antiferromagnetism

AF susceptibility, $\chi_\varphi$, and Binder cumulant as a function of the tuning parameter $r$

QMC for the onset of antiferromagnetism

\[ \bar{P} = x_{\max} \]

\[ r_c \]

\[ P_+/P_- \]

s/d pairing amplitudes as a function of the tuning parameter \( r \)

Conclusions

Solved sign-problem for generic universal theory for the onset of antiferromagnetism in two-dimensional metals.
Solved sign-problem for generic universal theory for the onset of antiferromagnetism in two-dimensional metals.

Obtained \textit{(first ?)} convincing evidence for the presence of unconventional superconductivity at strong coupling and near SDW quantum criticality.
Conclusions

Solved sign-problem for generic universal theory for the onset of antiferromagnetism in two-dimensional metals.

Obtained (first ?) convincing evidence for the presence of unconventional superconductivity at strong coupling and near SDW quantum criticality.

Good prospects for studying competing charge orders, and non-Fermi liquid physics at non-zero temperature.