What Have We Learned So Far?

Superconductors Without Any Symmetry (Besides Particle-Hole "Redundancy" Which Squares to 1 - the Usual Case) Have Z2 Classification in 1-d and 2-d

Edge States - Mirror of Topologically Nontrivial Bulk

Topological Indices Can (and should) Be Understood Through Both Bulk Topology and Edge Stability Arguments

Majorana Zero Modes Can Appear as Edge Modes in a Topologically Nontrivial Phase of the Kitaev P-Wave Wire and Can Form Non-Local Hilbert Spaces

Last Thing About 1-d (ideas will help later)

Cartan label	T	C	S	$d=1$	$d=2$	$d=3$
A (unitary)	0	0	0	-	$\mathbb{Z}$	-
AI (orthogonal)	$+1$	0	0	-	-	-
AII (symplectic)	-1	0	0	-	$\mathbb{Z}_2$	$\mathbb{Z}_2$
AIII (ch. unit.)	0	0	1	$\mathbb{Z}$	-	$\mathbb{Z}$
BDI (ch. orth.)	$+1$	$+1$	1	$\mathbb{Z}$	-	-
CII (ch. sympl.)	-1	-1	1	$\mathbb{Z}$	-	$\mathbb{Z}_2$
D (BdG)	0	$+1$	0	$\mathbb{Z}_2$	$\mathbb{Z}$	-
C (BdG)	0	-1	0	-	$\mathbb{Z}$	-
DIII (BdG)	-1	$+1$	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
CI (BdG)	$+1$	-1	1	-	-	$\mathbb{Z}$

Why is there no C (BdG) class in d=1?

(hint: because C^2=-1, we need two flavors at least. The symmetry cannot keep the two flavor Majorana edge modes from gapping - just like C^2=1 cant either, but with C^2=1 we do not need two flavors)

Work it out during break.

Two Dimensions: First the Easy Classes

Lets analyze first the classes with no chiral symmetry S

Cartan label	T	C	S	$d=1$	$d=2$	$d=3$
A (unitary)	0	0	0	-	$\mathbb{Z}$	-
AI (orthogonal)	$+1$	0	0	-	-	-
AII (symplectic)	-1	0	0	-	$\mathbb{Z}_2$	$\mathbb{Z}_2$
AIII (ch. unit.)	0	0	1	$\mathbb{Z}$	-	$\mathbb{Z}$
BDI (ch. orth.)	$+1$	$+1$	1	$\mathbb{Z}$	-	-
CII (ch. sympl.)	-1	-1	1	$\mathbb{Z}$	-	$\mathbb{Z}_2$
D (BdG)	0	$+1$	0	$\mathbb{Z}_2$	$\mathbb{Z}$	-
C (BdG)	0	-1	0	-	$\mathbb{Z}$	-
DIII (BdG)	-1	$+1$	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
CI (BdG)	$+1$	-1	1	-	-	$\mathbb{Z}$

Look at class D in 2d. We can understand the classification by thinking what protected edge states could we have if we cut the system.

If noninteracting, the k=0, Pi (along the edge) Hamiltonians are charge conjugation invariant 1-d Hamiltonians themselves.

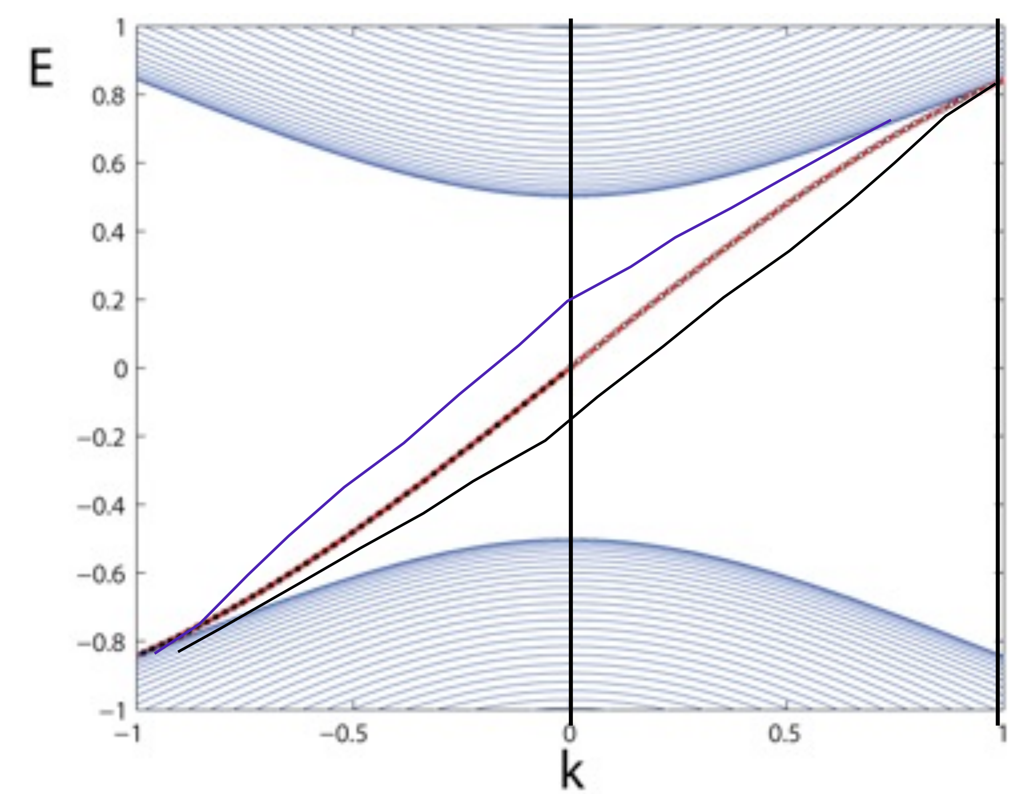
We can then clearly have a Chiral edge state - Chiral Majorana mode on one edge (say right) of the cut superconductor.

Therefore I could have a nontrivial 1-d class at k=0 and a trivial class at k =Pi

Can we have counter-propagating modes on one edge? NOT in D class. There would be nothing to protect the gap at edge from opening.

(nontrivial at both k=0,Pi is a weak topological superconductor, just decoupled layers of 1-d)

However, we can have as many Chiral modes as we want. Cannot open a gap (chiral)



General Theory for the D and C classes in 2 Dimensions

These are classes with Chiral Majorana edge modes.

They have a Z classification, which is the number of chiral edges.

They are described by a projector (spectral) Chern number, identical to the case of the Chern insulator of IQH or TKNN formula.

The only difference is that unlike in the insulator, the Chern number is not related to the Hall conductance.

P_G is the spectral projector onto the lower Bogoliubov bands $\epsilon_{ij} Tr[(\partial_i P_G)(\partial_j P_G) P_G]$

We can alternatively define the Berry potential and curvature of the Bogoliubov occupied bands and express the Chern number just as TKNN did

Thouless-Kohmoto-Nightingale-den Nijs (1982)

$$\boxed{C = \frac{e^2}{h} \sum_{bands} \frac{i}{2\pi} \int d^2k \left(\left\langle \frac{\partial u(k)}{\partial k_x} \middle| \frac{\partial u(k)}{\partial k_y} \right\rangle - \left\langle \frac{\partial u(k)}{\partial k_y} \middle| \frac{\partial u(k)}{\partial k_x} \right\rangle \right)}$$

topological invariant ! "Chern number"

$$\vec{\ }, \quad \sum \langle \ , \quad \vec{\nabla} \quad , \ \rangle$$

$$C_{,} = \frac{1}{2\pi} \int d^2k F_{xy}$$

P+iP Superconductor

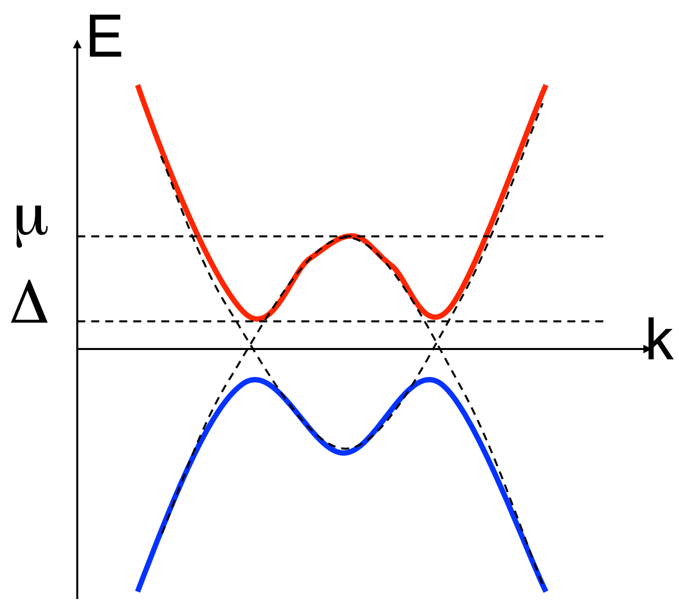
Simplest example of a chiral superconductor is the p+ip in 2d

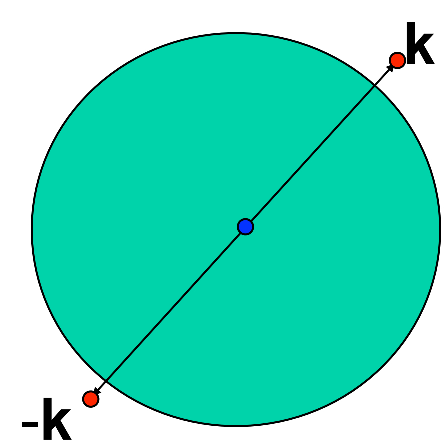
There is even a candidate material SrRu - see Prof Raghu's talk right after this one.

px+i py needed. Only px would be gapless in 2 dimensions.

$$\Psi_{\mathbf{p}} = (c_{\mathbf{p}} \quad c^{\dagger}_{-\mathbf{p}})^{T}$$

$$H_{\mathrm{BdG}} = \frac{1}{2}\sum_{\mathbf{p}} \Psi^{\dagger}_{\mathbf{p}} \begin{pmatrix} -2t(\cos p_x + \cos p_y) - (\mu - 4t) & 2i\Delta(\sin p_x + i \sin p_y) \\ -2i\Delta^{*}(\sin p_x - i \sin p_y) & +2t(\cos p_x + \cos p_y) + (\mu - 4t) \end{pmatrix} \Psi_{\mathbf{p}}$$





$$\Delta_{\mathbf{k}} = \Delta(k_x + ik_y)/k_F$$

Phases of the P+iP Superconductor

A simple way of seeing that this Hamiltonian has nontrivial edges for $0 \quad \mu \quad 2$ is to linearize it

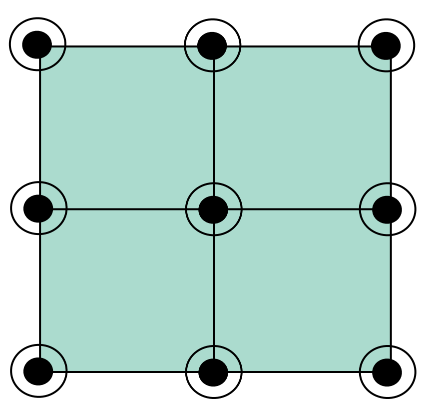
$$H_{\mathrm{BdG}} \quad \frac{1}{2}\sum_{\mathbf{p}} \Psi^{\dagger}_{\mathbf{p}} d_a(\mathbf{p} \quad \mu)\tau^a \Psi_{\mathbf{p}}$$

$$d_a(\mathbf{p} \quad \mu) \quad (-2\,\Delta\, p_y \quad -2\,\Delta\, p_x \quad p^2 \;\; 2m - \mu)$$

This now looks like a Dirac Hamiltonian.

μ<0

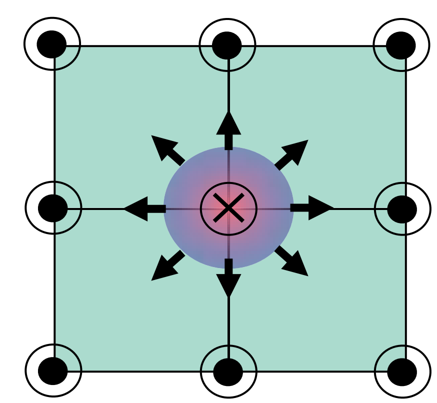
Trivial (strong pairing Fermi level below band)

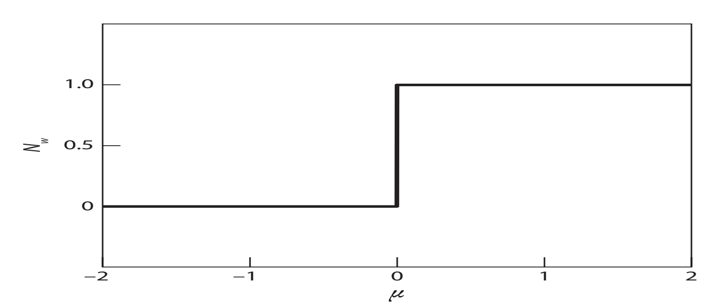


$$\frac{1}{8\pi}\int d^2p\, \epsilon^{ij}\, d \quad \left(\partial_{p_i} d \times \partial_{p_j} d\right)$$

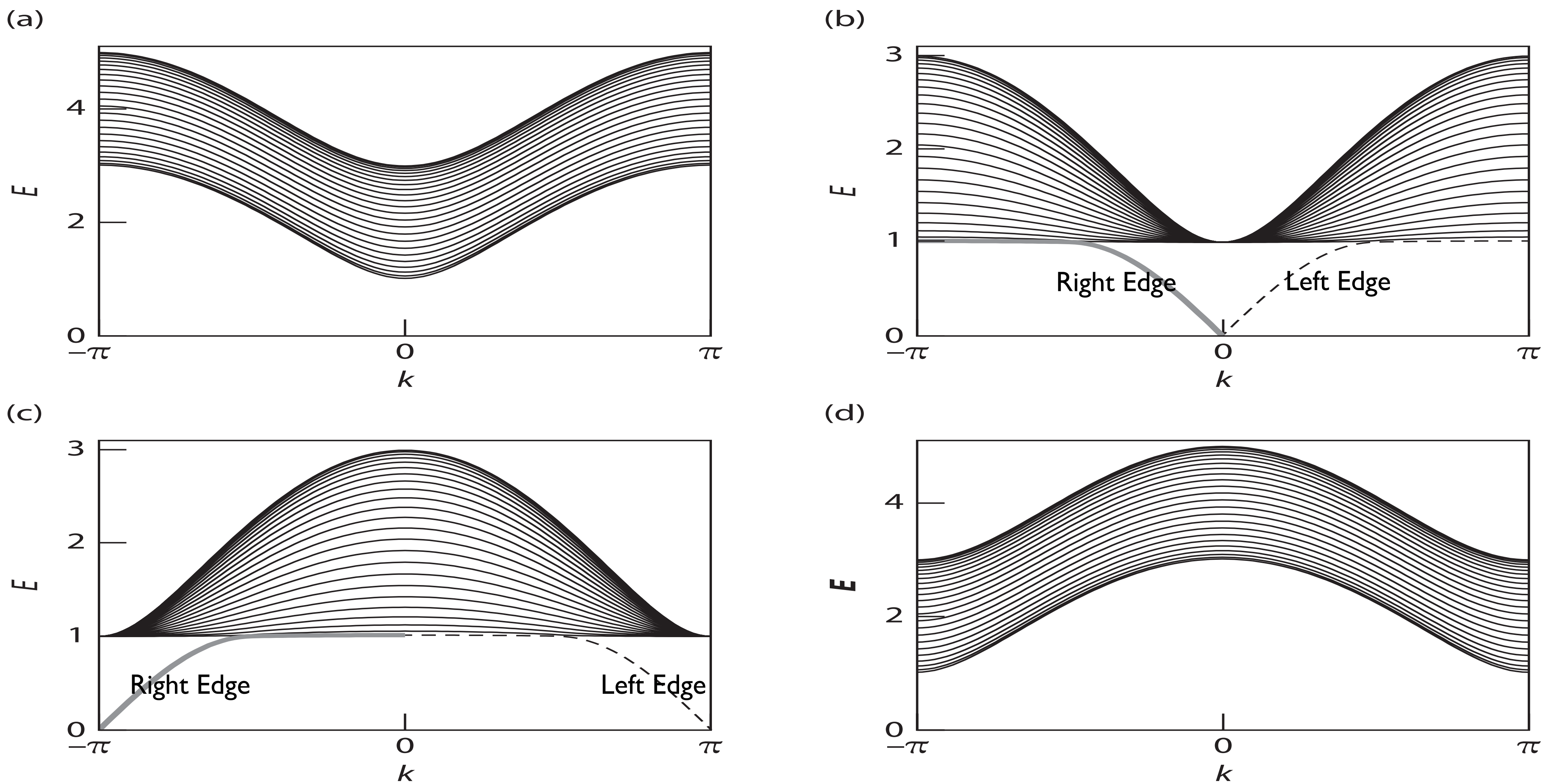
μ>0

Non-trivial
(weak pairing Fermi level in band)

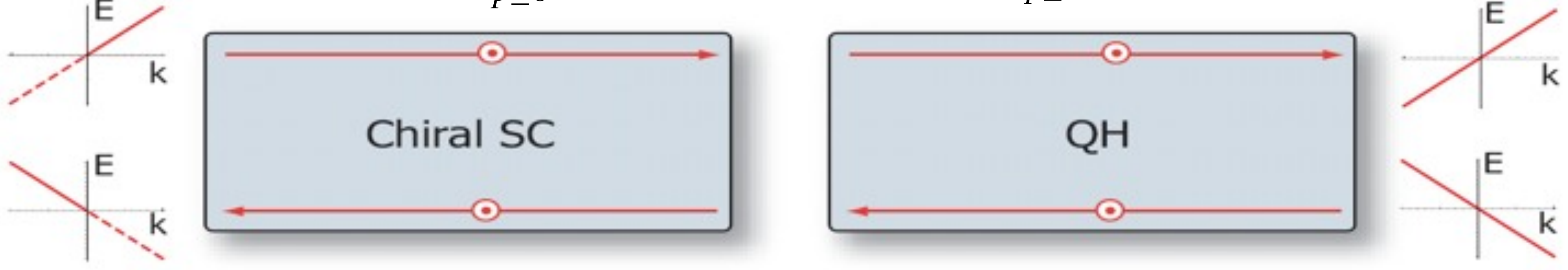




Open Boundary Spectrum of P+IP Superconductor



$$H_{\text{edge}}^{(p\text{-wave})} = \frac{\hbar v}{2} \sum_{p \geq 0} p \gamma_{-p} \gamma_p \qquad H_{\text{edge}}^{(\text{QH})} = \frac{\hbar v}{2} \sum_{p \geq 0} p \left(\gamma_{1,-p} \gamma_{1,p} + \gamma_{2,-p} \gamma_{2,p} \right)$$



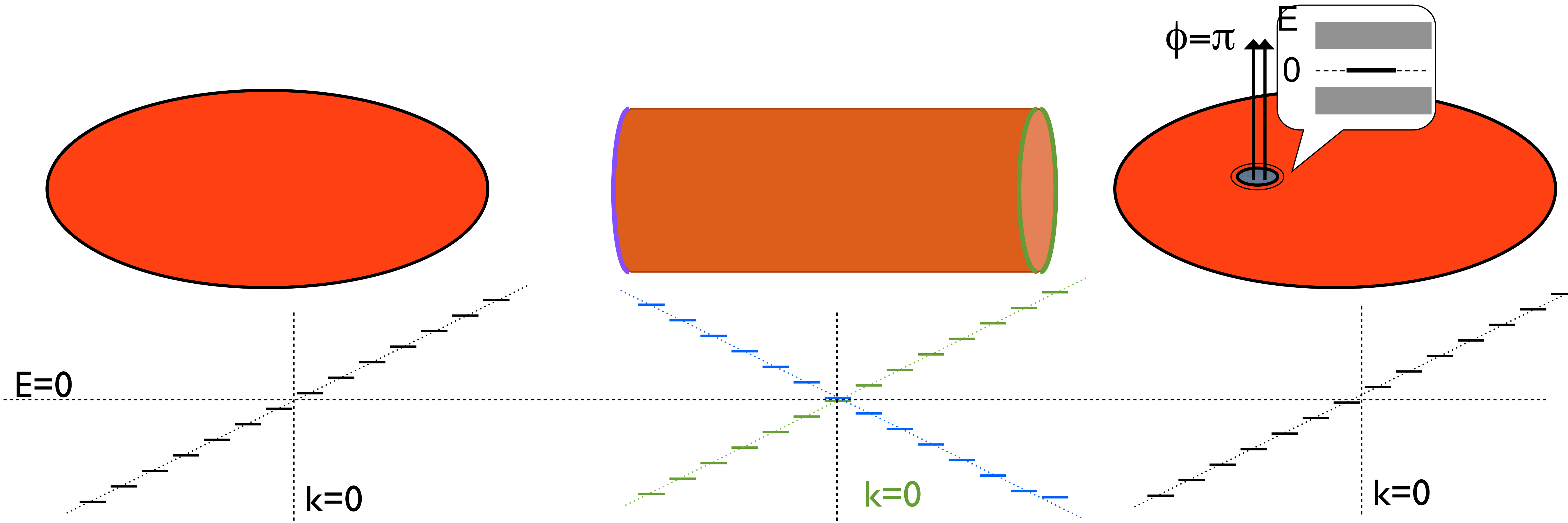
Majorana Modes on Vortices of Chiral Superconductors

One can do a proper theory of defects in topological phases and the existence/stability of modes on defects.

Defects can have a different than their host medium because they have different dimensionality

Vortices in 2D Chiral Superconductors have a Z2 classification, even though the chiral superconductor has a Z classification.

On general grounds we expect a zero mode stuck on the vortex in a p+ip superconductor. This is because in a disk geometry, the boundary conditions on the gapless edge are antiperiodic (the only way to be because otherwise PH symm would require 2 modes, with only 1 edge)



Explicit Form of A Vortex

We pick a vortex located at the origin r=0:

$$\Delta(r,\,\theta) \;=\; |\Delta(r)|e^{i\alpha(r)} \qquad\qquad |\Delta(0)| \;=\; 0$$

We take the phase $\alpha(\mathbf{r})$ to be equal to the polar angle at $\mathbf{r}$. $\quad \Psi(r) \;\rightarrow\; e^{i\alpha(\mathbf{r})/2}\Psi(r)$

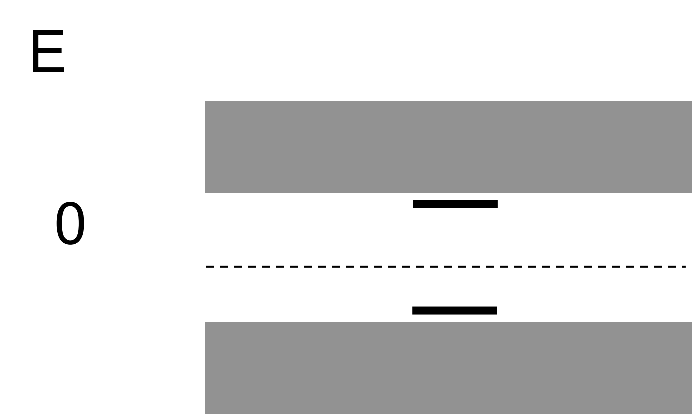
2N vortices with zero modes on them lead do a degeneracy of 2^N (N complex fermions). Then 2^(N-1) even and 2^(N-1) odd fermion parity.

$$H_{\mathrm{BdG}} = \frac{1}{2}\begin{pmatrix} -\mu & 2|\Delta(r)|e^{i\theta}\left(\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial\theta}\right) \\ -2|\Delta(r)|e^{-i\theta}\left(\frac{\partial}{\partial r} - \frac{i}{r}\frac{\partial}{\partial\theta}\right) & \mu \end{pmatrix}$$

$$\Psi_0(r,\,\theta) = \frac{i}{\sqrt{r}\mathcal{N}}\exp\left[-\frac{1}{2}\int_0^r \frac{\mu(r')}{|\Delta(r')|}dr'\right]\begin{pmatrix} -e^{i\theta/2} \\ e^{-i\theta/2} \end{pmatrix} \equiv ig(r)\begin{pmatrix} -e^{i\theta/2} \\ e^{-i\theta/2} \end{pmatrix}$$

$$\gamma = \int r\,dr\,d\theta\; ig(r)\;\left(-e^{i\theta/2}c(r,\,\theta) + e^{-i\theta/2}c^{\dagger}(r,\,\theta)\right)$$

Vortex classification is Z_2. Two majorana modes on the same vortex (as would come from a Chern number 2 topological superconductor) would split up and move into the vortex core continuum of states.



Time-Reversal Topological Superconductors

Cartan label	T	C	S	$d=1$	$d=2$	$d=3$
A (unitary)	0	0	0	-	$\mathbb{Z}$	-
AI (orthogonal)	+1	0	0	-	-	-
AII (symplectic)	−1	0	0	-	$\mathbb{Z}_2$	$\mathbb{Z}_2$
AIII (ch. unit.)	0	0	1	$\mathbb{Z}$	-	$\mathbb{Z}$
BDI (ch. orth.)	+1	+1	1	$\mathbb{Z}$	-	-
CII (ch. sympl.)	−1	−1	1	$\mathbb{Z}$	-	$\mathbb{Z}_2$
D (BdG)	0	+1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	-
C (BdG)	0	−1	0	-	$\mathbb{Z}$	-
DIII (BdG)	−1	+1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
CI (BdG)	+1	−1	1	-	-	$\mathbb{Z}$

Again we can understand everything from bulk or from edge.

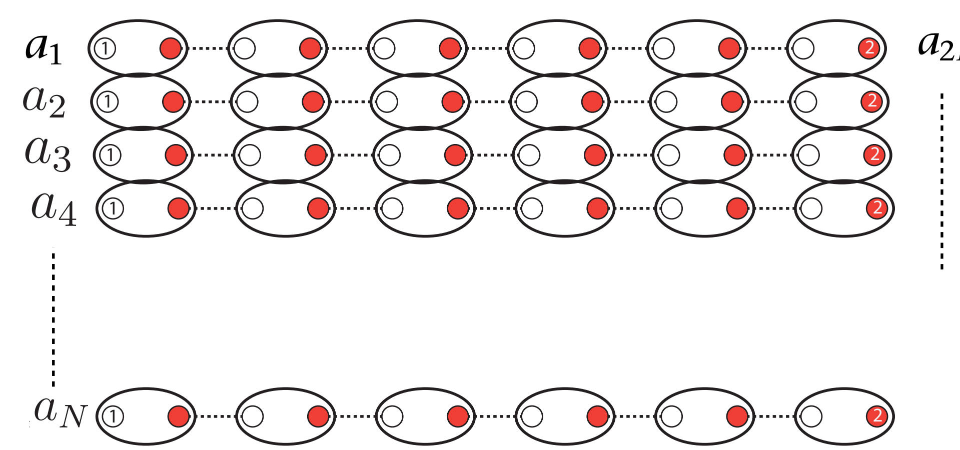
Simplest example: why is the class BD1 Z in 1D?

BD1 is class D plus added spinless time-reversal T=K (complex conjugation)

We can then add an integer number N (flavor) of D-class 1-dimensional open chains and ask what happens to their edges.

In the absence of any other symmetry, for the D class, N Majorana fermions they would have a local hilbert space and gap (mod 2) by a one-body term:

$$i\sum_{m<n=1}^{N}\gamma_m\gamma_n$$



With spinless TR added (class BD1), this term is not allowed and the classification is Z

$$Ti\sum_{m<n=1}^{N}\gamma_m\gamma_n T^{-1} = TiT^{-1}\sum_{m<n=1}^{N}\gamma_m\gamma_n = -i\sum_{m<n=1}^{N}\gamma_m\gamma_n$$

Time-Reversal Topological Superconductors With Spin

Cartan label	T	C	S	$d=1$	$d=2$	$d=3$
A (unitary)	0	0	0	-	$\mathbb{Z}$	-
AI (orthogonal)	+1	0	0	-	-	-
AII (symplectic)	−1	0	0	-	$\mathbb{Z}_2$	$\mathbb{Z}_2$
AIII (ch. unit.)	0	0	1	$\mathbb{Z}$	-	$\mathbb{Z}$
BDI (ch. orth.)	+1	+1	1	$\mathbb{Z}$	-	-
CII (ch. sympl.)	−1	−1	1	$\mathbb{Z}$	-	$\mathbb{Z}_2$
D (BdG)	0	+1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	-
C (BdG)	0	−1	0	-	$\mathbb{Z}$	-
DIII (BdG)	−1	+1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
CI (BdG)	+1	−1	1	-	-	$\mathbb{Z}$

Generic form for the pairing: $\bar{c}_{\mathbf{p}\sigma} \equiv i\sigma^{y}_{\sigma\sigma'} c_{\mathbf{p}\sigma'}$

$$H_\Delta = \sum_{\mathbf{p}} \frac{1}{2} \left[c^\dagger_{\mathbf{p}\sigma} \Delta_{\sigma\sigma'}(\mathbf{p}) \bar{c}^\dagger_{-\mathbf{p}\sigma'} + \bar{c}_{-\mathbf{p}\sigma} \left(\Delta^\dagger\right)_{\sigma\sigma'} (\mathbf{p}) c_{\mathbf{p}\sigma'} \right]$$

In this form singlet proportional to identity

Gap can be expanded (as learned in this school)

$$\Delta_{\sigma\sigma'}(\mathbf{p}) = d_0(\mathbf{p}) \mathbb{I}_{\sigma\sigma'} + d_a(\mathbf{p}) \sigma^a_{\sigma\sigma'}$$

Identity term is singled, pauli terms are triplet. $d_0(\mathbf{p}) = d_0(-\mathbf{p})$ $d_a(\mathbf{p}) = -d_a(-\mathbf{p})$

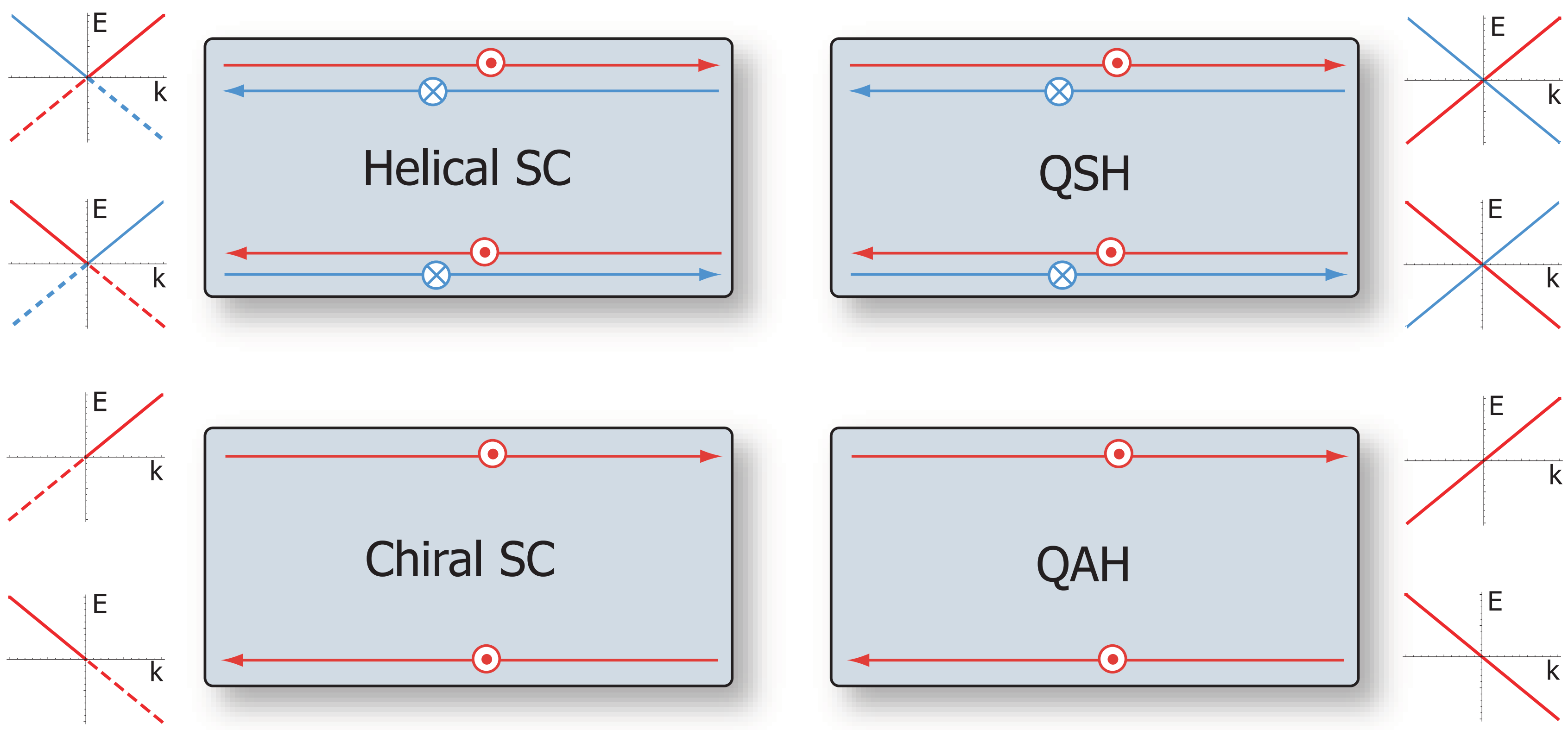
Because time-reversal is spinful, and hence squares to -1, states come in Kramers pairs. If translational invariance exists the states at k and -k are related by time-reversal, while Kramers doublets exist at k=0, Pi

Can we build a time-reversal topological superconductor in 2 dimensions?

Time-Reversal Topological Superconductors With Spin in 2-dimensions

From our experience, we know the quantum spin hall insulator is two copies of the integer quantum Hall state, plus a Z_2 mod due to time-reversal.

We expect to be able to build a "helical" superconductor by doubling the Chiral superconductors learned before.



Time-Reversal Topological Superconductors With Spin in 2-dimensions

Put together a C=1 and C=-1 Chiral topological superconductors and relate them by time-reversal. Simplest way P+iP and P-IP

$$H \quad \frac{1}{2}\sum_{\mathbf{p}} \Psi^\dagger_{\mathbf{p}} \begin{pmatrix} \frac{p^2}{2m}-\mu & 0 & 0 & -\Delta(p_x \quad ip_y) \\ 0 & \frac{p^2}{2m}-\mu & \Delta(p_x - ip_y) & 0 \\ 0 & \Delta^*(p_x \quad ip_y) & -\frac{p^2}{2m} \quad \mu & 0 \\ -\Delta^*(p_x - ip_y) & 0 & 0 & -\frac{p^2}{2m} \quad \mu \end{pmatrix} \Psi_{\mathbf{p}}$$

$$\Psi_{\mathbf{p}} \quad \left(c_{\mathbf{p}\uparrow} \;\; c_{\mathbf{p}\downarrow} \;\; -c^\dagger_{-\mathbf{p}\downarrow} \;\; c^\dagger_{-\mathbf{p}\uparrow}\right)^T$$

$$T \quad \mathbb{I} \otimes i\sigma^y K$$

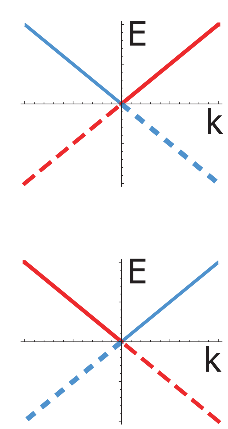
C= $\tau^y \otimes \sigma^y K$

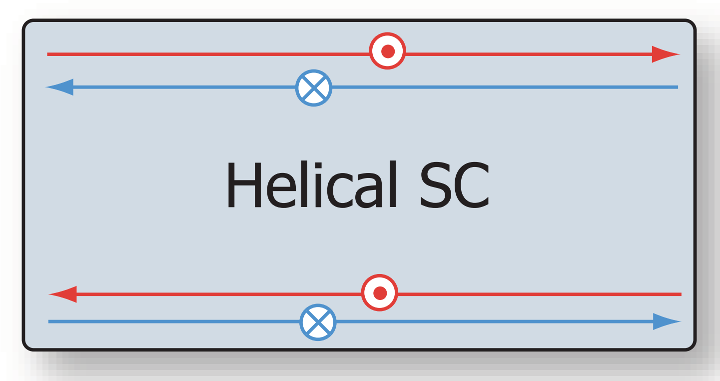
In d-vector notation of the gap: $\mathbf{d} \quad -i\Delta(p_x y \quad p_y x)$

If each spin had same chirality: $\mathbf{d} \quad -i\Delta(p_x \quad ip_y)y$; $\mathrm{Sr_2RuO_4}$ Would break TR

The P+iP half of the superconductor has chiral Majorana, the P-iP has anti-chiral Majorana edge states

$$\Phi_p = (\gamma_{p\uparrow} \quad \gamma_{p\downarrow}) \qquad H_{edge} = \sum_p \Phi_p^\dagger \begin{pmatrix} p & 0 \\ 0 & -p \end{pmatrix} \Phi_p = \sum_p \Phi_p^\dagger p \sigma^z \Phi_p$$





Stability of Edge States

The decoupled edge Hamiltonian is invariant under TR and C: $T = i\sigma^y K. \quad C = \mathbb{I}K$

$$H_{edge} = \sum_p \Phi_p^\dagger \begin{pmatrix} p & 0 \\ 0 & -p \end{pmatrix} \Phi_p = \sum_p \Phi_p^\dagger p \sigma^z \Phi_p$$

$$CH_{edge}(p)C^{-1} = p\sigma^z = -(-p)\sigma^z = -H_{edge}(-p). \qquad TH_{edge}(p)T^{-1} = p\sigma^y\sigma^z\sigma^y = -p\sigma^z = H_{edge}(-p)$$

Now try to open a one-body gap without breaking TR or C. Add a generic mass:

$$H'_{edge} = H_{edge} + m_0\mathbb{I} + m_a\sigma^a$$

not allowed by C

x, y, z not allowed by TR.
x,z break C.
If we relax TR, we can add a y mass which is allowed.
So our superconductor is protected by time-reversal

Is the classification Z2 or Z? Add 2 copies of surface $H^{(2)}_{edge}(p) = p(\mathbb{I} \otimes \sigma^z)$.

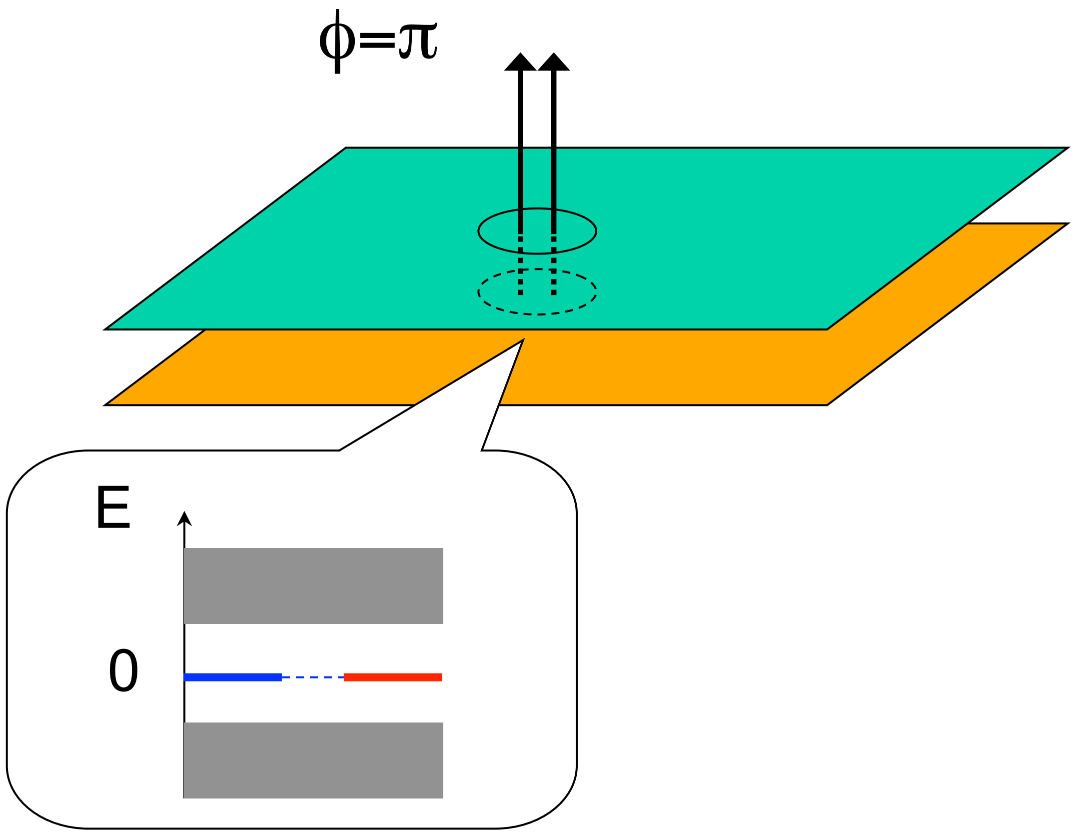
Can we find a TR, C invariant mass term that anticommutes with the edge Hamiltonian? YES! Not Z

$$H^{(2)}_{edge}(p) + m\mathcal{M} \qquad \{\mathcal{M}, \mathbb{I}\otimes\sigma^z\} = 0 \qquad \mathcal{M} = \tau^y \otimes \sigma^x$$

Vortices in Time-Reversal Invariant Superconductors

$$\Delta_{vortex} = \begin{pmatrix} 0 & -\Delta e^{i\phi_\uparrow}(p_x + ip_y) \\ \Delta e^{i\phi_\downarrow}(p_x - ip_y) & 0 \end{pmatrix}$$

$$\phi_\uparrow = -\phi_\downarrow = \theta(r)$$



$$\begin{aligned} \phi_1 &= (1/2)(\gamma_\uparrow + i\gamma_\downarrow) \\ \phi_1^\dagger &= (1/2)(\gamma_\uparrow - i\gamma_\downarrow) \\ \phi_2 &= (1/2)(\alpha_\uparrow + i\alpha_\downarrow) \\ \phi_2^\dagger &= (1/2)(\alpha_\uparrow - i\alpha_\downarrow) \end{aligned}$$

Time-Reversal Invariant Superconductors in 3 Dimensions

				+	+	+ :
	6 =					
+	− =					
: :			=			
: :	6 =	6 =	=			
: + :	− =	− =	=			
		6 =				
		− =				
	− =	6 =	=			
	6 =	− =	=			

$$\sum_{\mathbf{p}} \Psi^\dagger_{\mathbf{p}} \begin{pmatrix} \frac{p^2}{2m} - \mu & 0 & |\Delta| p_z & |\Delta| p_- \\ 0 & \frac{p^2}{2m} - \mu & |\Delta| p_+ & -|\Delta| p_z \\ |\Delta| p_z & |\Delta| p_- & -\frac{p^2}{2m} + \mu & 0 \\ |\Delta| p_+ & -|\Delta| p_z & 0 & -\frac{p^2}{2m} + \mu \end{pmatrix} \Psi_{\mathbf{p}}$$

$$_{\mathbf{p}} = \left(e^{-i\theta/2} c_{\mathbf{p}\uparrow} \;\; e^{-i\theta/2} c_{\mathbf{p}\downarrow} \;\; -e^{i\theta/2} c^\dagger_{-\mathbf{p}\downarrow} \;\; e^{i\theta/2} c^\dagger_{-\mathbf{p}\uparrow} \right)^T$$

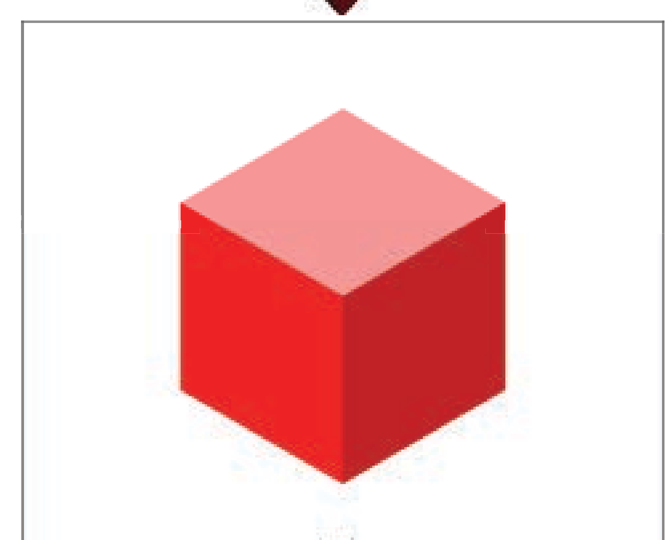
The B-Phase of He-3 exhibits this Hamiltonian

If we stare *intensely*, this is a 3-D Dirac gapped Hamiltonian. In fact, its identical to the topological insulator "effective" bulk Hamiltonian say for Bi2 Se3. For $\mu > 0$ the kinetic term winds and we have a topological superconductor (or insulator)

We expect surface states on any cut surface. Propagating Majorana fermions (non-chiral). Just like Dirac Fermion on the surface of a bulk 3D topological insulator

$$H_{surf} = \sum_{\mathbf{p}} \Phi^\dagger_{\mathbf{p}} \left(p_x \sigma^x + p_y \sigma^y \right) \Phi_{\mathbf{p}}$$

$$\Phi_{\mathbf{p}} = \left(\gamma_{\mathbf{p}\uparrow} \;\; \gamma_{\mathbf{p}\downarrow} \right)^T$$



Time-Reversal Invariant Superconductors in 3 Dimensions

So: form of the simplest 3D topological superconductor Hamiltonian same as that of the 3D topological insulator.

Also time-reversal symmetry has Kramers pairs. So very similar. Is the topological superconductor Z2? NO - it is Z!!!

Charge conjugation symmetry has the effect that, when coupled to spinful time-reversal, enhances the symmetry class to Z

$$\Phi_{\mathbf{p}} = (\gamma_{\mathbf{p}\uparrow} \ \gamma_{\mathbf{p}\downarrow})^T$$

$$T = i\sigma^y K \qquad C = i\sigma^x K \qquad H_{surf} = \sum_{\mathbf{p}} \Phi^\dagger_{\mathbf{p}} \left(p_x\sigma^x + p_y\sigma^y\right) \Phi_{\mathbf{p}}$$

These are "symmetries" of the Hamiltonian and they commute with each other. They relate p to -p points. However, crucially different from the TR-only case, we can now take their product and have a "chiral" symmetry (has nothing to do with chiral edge states in 2D)

$$\chi = CT = i\sigma^z$$

Both the edge and the bulk have this extra, chiral symmetry

$$\chi H(\mathbf{p})\chi^{-1} = -H(\mathbf{p})$$

Stability of Surface States of 3-d Topological Superconductors

The Chiral symmetry enhances the classification to Z. How?

Stability of **one** Majorana cone is guaranteed by TR, but can also be seen through Chiral symmetry

$\Phi_{\mathbf{p}} = (\gamma_{\mathbf{p}\uparrow} \ \gamma_{\mathbf{p}\downarrow})^T$

$$H_{surf} = \sum_{\mathbf{p}} \Phi^\dagger_{\mathbf{p}} \left(p_x \sigma^x + p_y \sigma^y\right) \Phi_{\mathbf{p}}$$

Try to open a gap by a mass $H' = m_z \sigma^z$ not allowed!:

$$\chi m_z \sigma^z \chi^{-1} = m_z \sigma^z \neq -m_z \sigma^z = -H(\mathbf{p})$$

So one Dirac node is stable towards opening a gap, known because of Kramers pairs

We now consider two cones, first decoupled: $H^{(2)}_{surf}(\mathbf{p}) = p_x \mathbb{I} \otimes \sigma^x + p_y \mathbb{I} \otimes \sigma^y$

We must see if we can add a mass term which anticommutes with the Hamiltonian but preserves all the symmetries. To anticommute, we need:

$$\mathcal{M} \doteq X \otimes \sigma^z$$

However, this does not anticommute with the Chiral symmetry, so it is not allowed! We have added 2 Majorana Cones together! Classification is Z!

$$\chi = CT = i\sigma^z$$

$$\chi H(\mathbf{p}) \chi^{-1} = -H(\mathbf{p})$$

Stability of Surface Majorana Cones on the Surface of a Topological Superconductor

We can add as many Majorana cones as we want, as long as their Berry phase (plus minus Pi) is of the same sign, hence the classification is Z. If we add a Berry phase Pi with a Berry phase - Pi then we can gap the system:

$T = \mathbb{I} \otimes i\sigma^y K,\ C = \mathbb{I} \otimes \mathbf{i}\sigma^x K$ $\qquad$ $H^{(2')}_{surf}(\mathbf{p}) = p_x \tau^z \otimes \sigma^x + p_y \mathbb{I} \otimes \sigma^y$ $\qquad$ $\mathcal{M} = \tau^y \otimes \sigma^x$

Conclusion: with Chiral symmetry, we can add any number of Majorana cones as long as they have the same Berry phase, hence classification is Z. Majorana modes with opposite Berry phases can still annihilate each other.

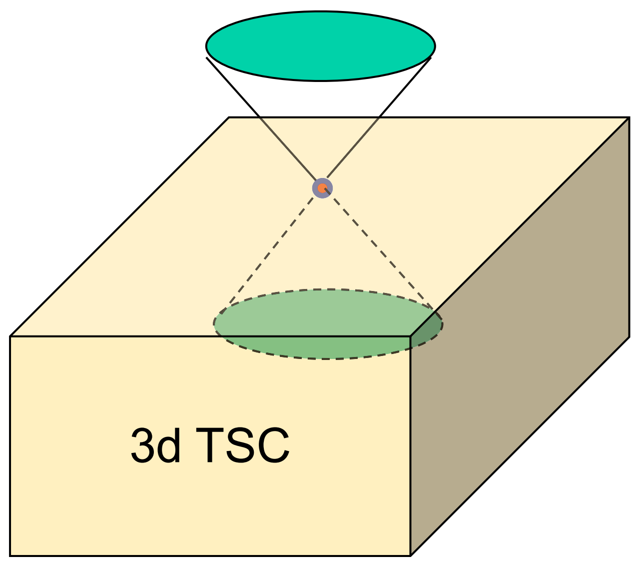
Why cant we add two Dirac nodes with just time-reversal symmetry, even if their Berry phases are both Pi?

$$H^{(2)}_{surf}(\mathbf{p}) = p_x \mathbb{I} \otimes \sigma^x + p_y \mathbb{I} \otimes \sigma^y$$

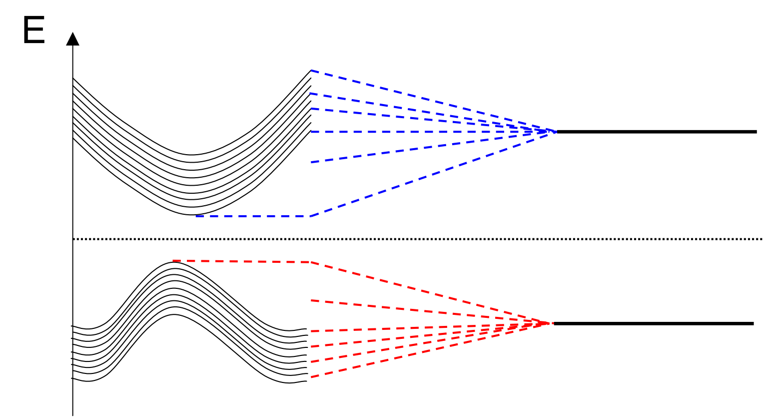
Because here I can add a mass that anticommutes with the Hamiltonian $\mathcal{M} \doteq X \otimes \sigma^z$ that also is TR invariant if I pick X= \tau_y

Another moral of the story: for Berry phases to act like monopoles and give Z classification, one needs extra symmetries (like C and T here, or Inversion and Time-Reversal in Graphene)

Bulk Index of 3D Time-Reversal Topological Superconductors



Flat-band your Bdg Hamiltonian:



Build the projector onto occupied Bogoliubov bands: $P(\mathbf{p}) \;=\; |u_1(\mathbf{p})\rangle\langle u_1(\mathbf{p})| + |u_2(\mathbf{p})\rangle\langle u_2(\mathbf{p})|$

Build the matrix: $Q(\mathbf{p}) \equiv 2P(\mathbf{p}) - 1$. With Chiral symmetry, it is easy to show that we can ALWAYS put this matrix into the following form:

$$Q(\mathbf{p}) \equiv \begin{pmatrix} 0 & q(\mathbf{p}) \\ q^\dagger(\mathbf{p}) & 0 \end{pmatrix}$$

$$\boxed{q^\dagger(\mathbf{p})q(\mathbf{p}) = q(\mathbf{p})q^\dagger(\mathbf{p}) = 1}$$

Bulk Index of 3D Time-Reversal Topological Superconductors

Importantly $q(\mathbf{p})$ is unitary U(N) matrix.

For the Dirac model of B-phase of Helium (previous slides), we have that

$$q(\mathbf{p}) = -\frac{1}{E(p)}\left(|\Delta|\mathbf{p}\cdot\sigma - iM(p)\mathbb{I}\right)$$

The matrix $q(\mathbf{p})$ is a map from the d-dimensional torus of lattice momenta ***p*** to the unitary group of U(N) where N is number of bands

These mappings have winding numbers if the d'th homotopy group of U(N) is nonzero. This is true for d=3, but not for d=1, 2.

Hence a Z winding invariant can be defined for 3-d. Lower dimensions have other invariants, which are NOT winding numbers of the q(p) matrix(but could be winding numbers of other matrices such as the Chern numbers

$$N_w = \frac{1}{24\pi^2}\int d^3p\,\epsilon^{\mu\nu\rho}\mathrm{Tr}\left[\left(q^{-1}(\mathbf{p})\partial_\mu q(\mathbf{p})\right)\left(q^{-1}(\mathbf{p})\partial_\nu q(\mathbf{p})\right)\left(q^{-1}(\mathbf{p})\partial_\rho q(\mathbf{p})\right)\right]$$

Cartan label	T	C	S	$d=1$	$d=2$	$d=3$
A (unitary)	0	0	0	-	$\mathbb{Z}$	-
AI (orthogonal)	+1	0	0	-	-	-
AII (symplectic)	−1	0	0	-	$\mathbb{Z}_2$	$\mathbb{Z}_2$
AIII (ch. unit.)	0	0	1	$\mathbb{Z}$	-	$\mathbb{Z}$
BDI (ch. orth.)	+1	+1	1	$\mathbb{Z}$	-	-
CII (ch. sympl.)	−1	−1	1	$\mathbb{Z}$	-	$\mathbb{Z}_2$
D (BdG)	0	+1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	-
C (BdG)	0	−1	0	-	$\mathbb{Z}$	-
DIII (BdG)	−1	+1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
CI (BdG)	+1	−1	1	-	-	$\mathbb{Z}$

Can We Go Beyond The Periodic Table?

To date, no real experimental confirmation of topological superconductors.

Spectroscopy (finding in-gap surface states) likely to be easiest way to see edge/surface modes, but most superconductors with spin-orbit coupling have small gaps, no arpes possible.

Can we find some topological superconductor without spin-orbit coupling but with (spinless) time-reversal? Periodic table says NO.

Cartan label	T	C	S	$d=1$	$d=2$	$d=3$
A (unitary)	0	0	0	-	$\mathbb{Z}$	-
AI (orthogonal)	$+1$	0	0	-	-	-
AII (symplectic)	-1	0	0	-	$\mathbb{Z}_2$	$\mathbb{Z}_2$
AIII (ch. unit.)	0	0	1	$\mathbb{Z}$	-	$\mathbb{Z}$
BDI (ch. orth.)	$+1$	$+1$	1	$\mathbb{Z}$	-	-
CII (ch. sympl.)	-1	-1	1	$\mathbb{Z}$	-	$\mathbb{Z}_2$
D (BdG)	0	$+1$	0	$\mathbb{Z}_2$	$\mathbb{Z}$	-
C (BdG)	0	-1	0	-	$\mathbb{Z}$	-
DIII (BdG)	-1	$+1$	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
CI (BdG)	$+1$	-1	1	-	-	$\mathbb{Z}$

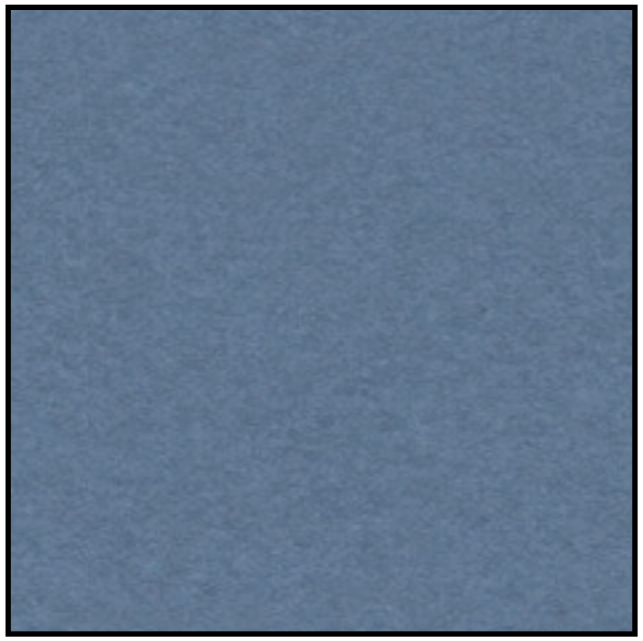
However, if we introduce point-group symmetry, we now suddenly can!

This leads to extension of the periodic table, and to the discovery of new classes, potentially found in experiments (since materials have point-group symmetries).

There are MANY crystallographic groups in 1,2,3 dimensions, large number of possible insulators

A Flavor of What Point Group Symmetry Can Do For U:

An easy example is a spinless time-reversal invariant 2 dimensional superconductor (which the classification table says is trivial, due to T^2=1)



Say now the system also has C4 symmetry (UNITARY)

$$C_4 H(k_x, k_y) C_4^{-1} = H(k_y, -k_x)$$

At Gamma=(0,0) and M= (Pi, Pi) points, the Hamiltonian commutes with the C4 matrix.

Since C4 is also unitary, the bands at Gamma=(0,0) and M= (Pi, Pi) can be described by C4 eigenvalues.

Since C4^4=1, these eigenvalues can be +1, -1, i, -i

Now suppose i have in my supercondctor the 2 occupied Bogoliubov bands which have eigenvalues i,-i under C4. These can and do exist in real materials. (C4= i \sigma_y)

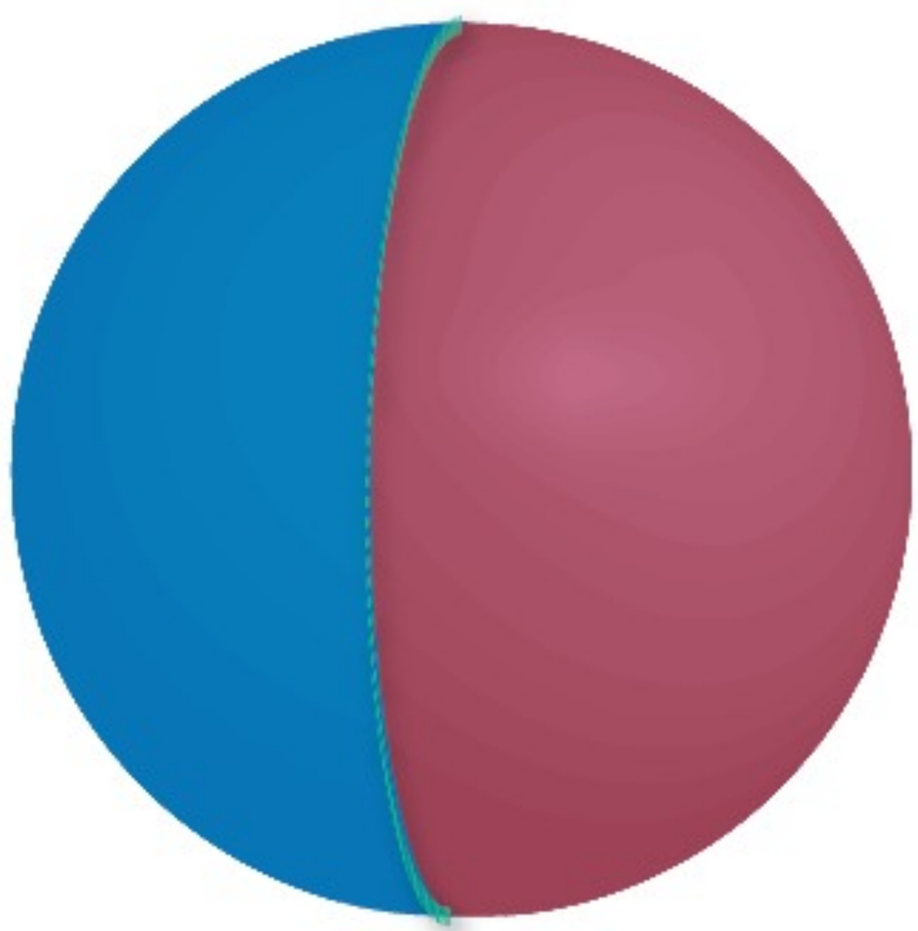
Because of spinless time-reversal T=K, now the band with C4 eigenvalue +i is degenerate (at Gamma and M) with the band of C4 eigenvalues -i. At these points only we have effectively built Kramers pairs! C4*T= i sigma_y K is now a composite operator that acts like SPINFUL time-reversal but ONLY at Gamma and M. Hence we can change the symmetry, get nontrivial

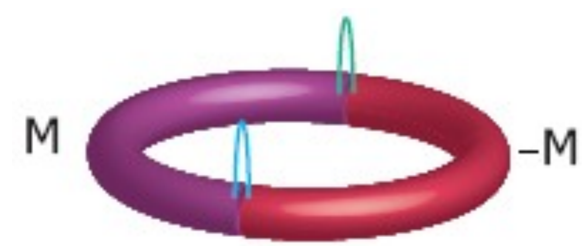
Other Ways of Building Majorana Modes and other Interesting states

We could just find the materials which exhibit each of these properties.

Or take a hint from Kane and Fu and use parent materials to generate the other interesting states.

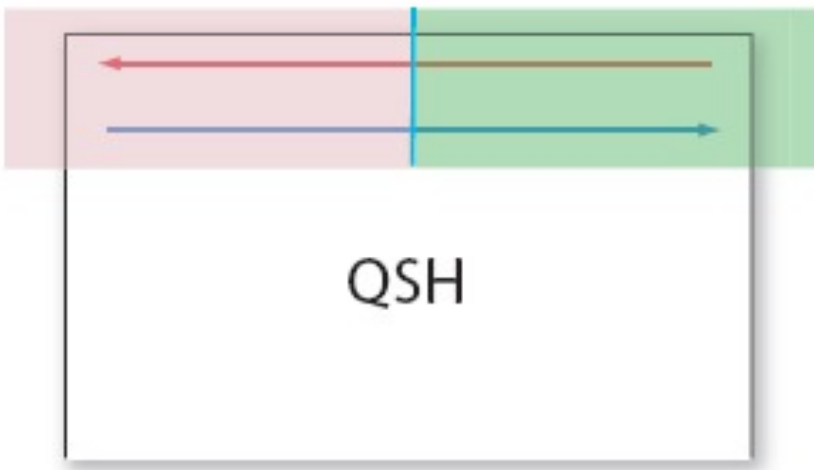
Surface of TI (e.g. BiTe,BiSe)





Blue Mass	Red Mass	Boundary State	
M	-M		IQHE
Δ	$-\Delta$		(p+ip/p-ip)
M	Δ		p+ip

Boundary Modes

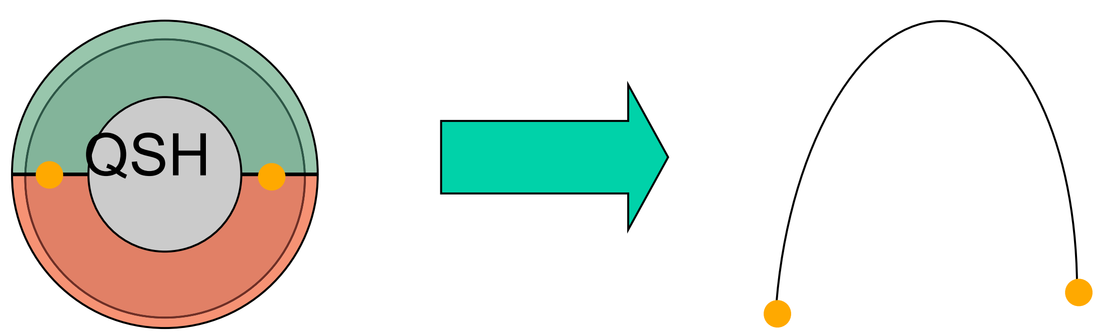


Red	Green	Bound State
M	-M	1 fermion state
Δ	-Δ	2 Majorana states
M	Δ	1 Majorana state

SSH 1d chain

Doubled p-wave wire

p-wave wire



What Next?

Find and propose materials! Find these

Look for systems without spin-orbit coupling but with point-group symmetry - what new phases.

Interacting classification.

Fractional interacting superconductors - is there any way of getting a superconductor with a fractional index? Similar to the Fractional Topological insulators effect but for superconductors