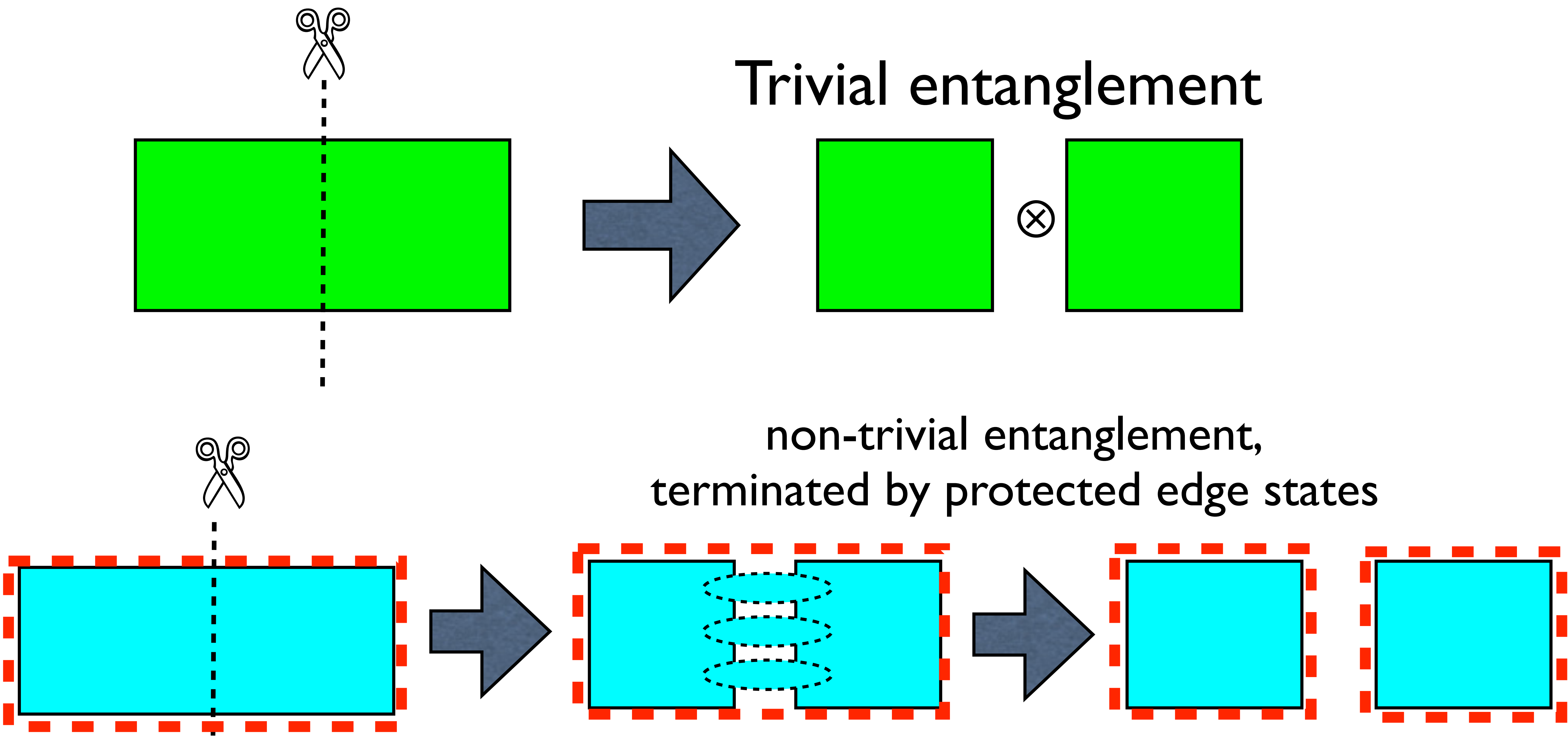
Theory Winter School
"Quantum Information Meets Many-Body-Physics, Entanglement, Thermalization and Chaos"

Bipartite Entanglement, Topological Order, and the Entanglement Spectrum

F. Duncan M. Haldane
Princeton University

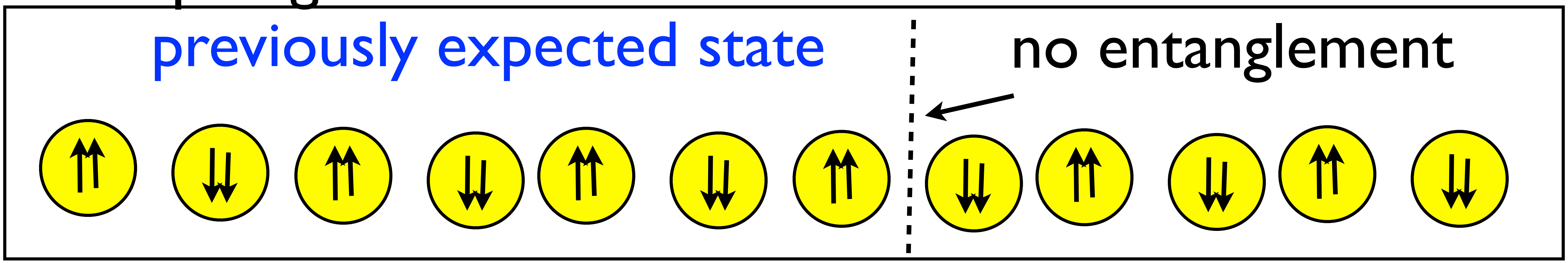
- Novel entanglement properties of "Topological Quantum States of Matter"
- Introduction using Spin Chains
- Quantum Hall effect, Laughlin states, and non-Abelian generalizations.

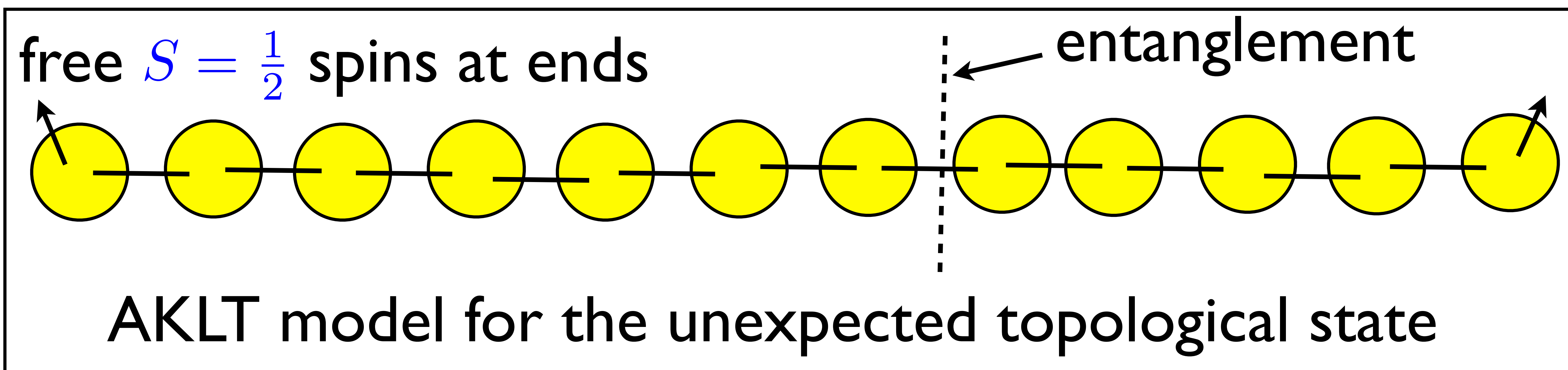
- In recent years, it has been realized that quantum condensed matter can exhibit unexpected properties associated with long range quantum entanglement



- Surprise #1: gapped spin-liquid state of spin-1 antiferromagnetic chains

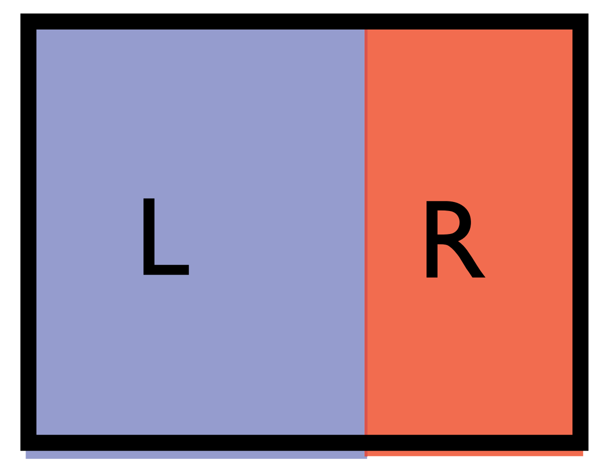
- In 1981 I unexpectedly discovered that a S=1 chain on spins could have a novel state that is now understood as the simplest example of "topological matter"





- Entanglement in its simplest form can be characterized by a bipartite (Schmidt) decomposition of a pure quantum state into products of states of two subsystems "Left" and "Right"

$$|\Psi\rangle = \sum_{\alpha\beta} M_{\alpha\beta} |\Psi^L_\alpha\rangle \otimes |\Psi^R_\beta\rangle$$



a (rectangular) complex matrix

orthonormal basis of "Left" degrees of freedom

orthonormal basis of "Right" degrees of freedom

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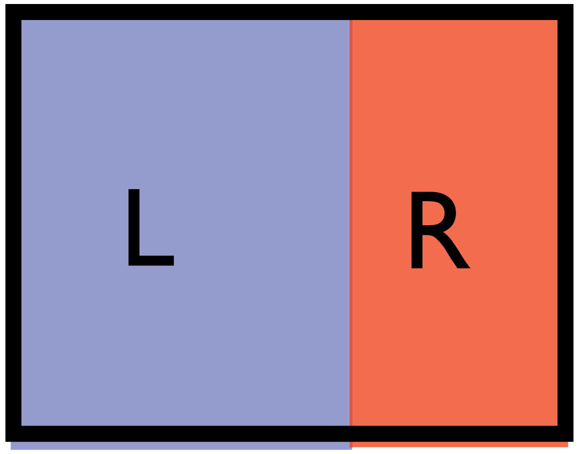
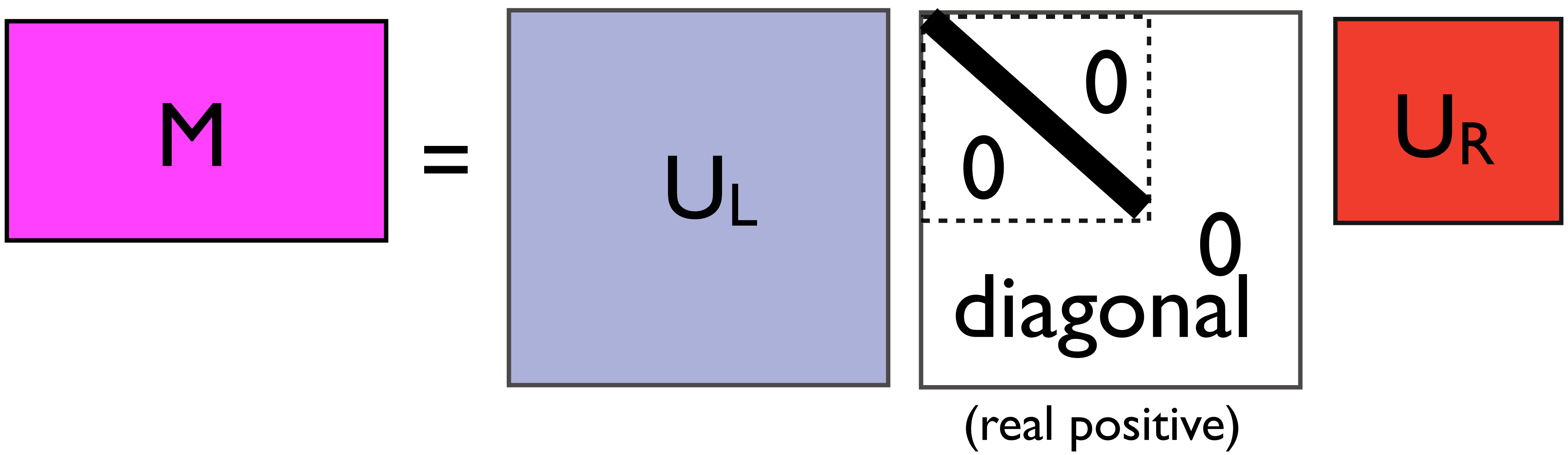


a (rectangular) complex matrix

orthonormal basis of "Left" degrees of freedom

orthonormal basis of "Right" degrees of freedom

- Any matrix has a "singular value decomposition"



$$|\Psi\rangle = \sum_\nu e^{-\frac{1}{2}\xi_\nu} |\Psi^L_\nu\rangle \otimes |\Psi^R_\nu\rangle$$

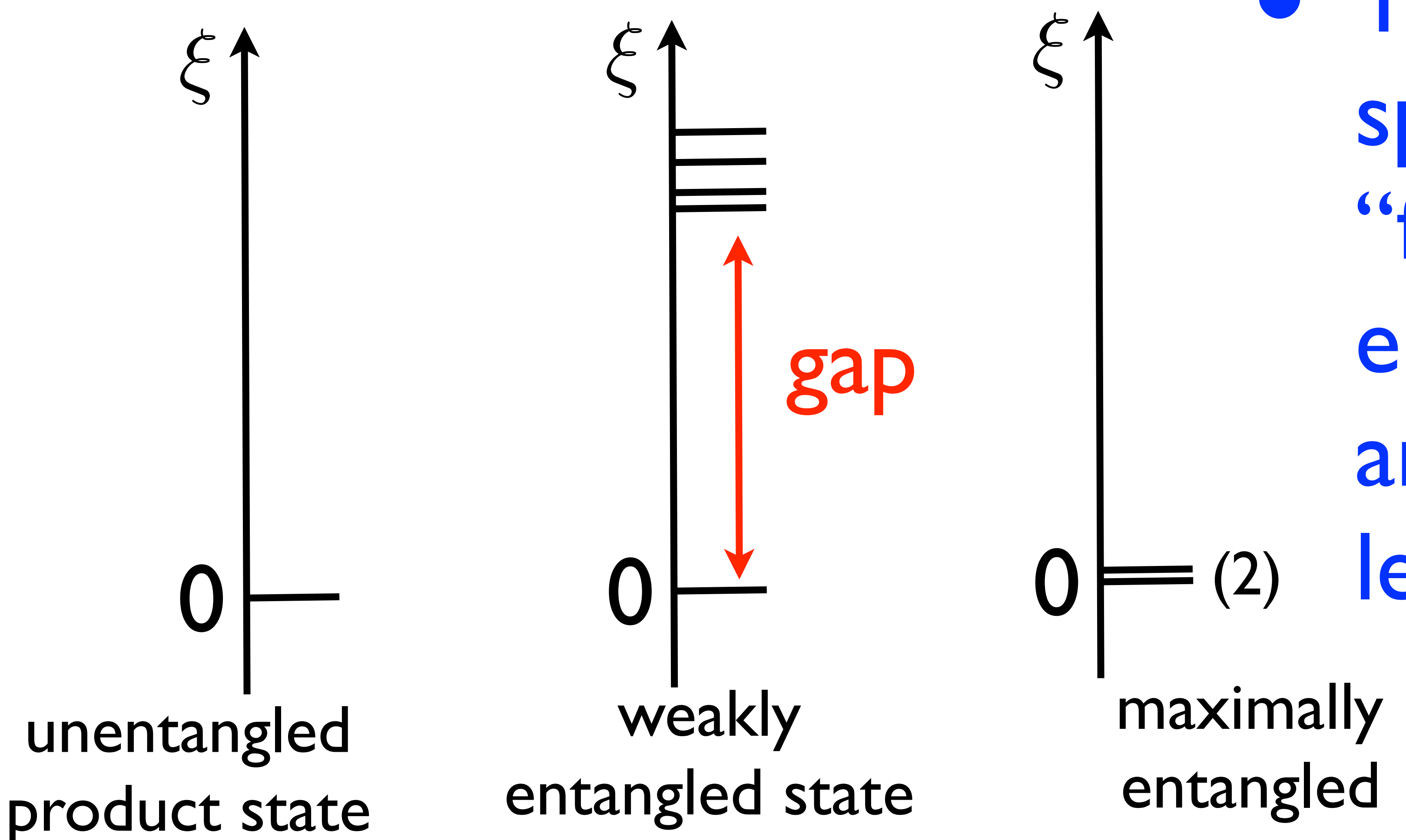
"entanglement spectrum" eigenvalues

orthonormal basis of "Left" degrees of freedom

orthonormal basis of "Right" degrees of freedom

Schmidt decomposition

$$|\Psi\rangle = \sum_{\nu} e^{-\frac{1}{2}\xi_\nu} |\Psi^L_\nu\rangle \otimes |\Psi^R_\nu\rangle$$



$$|\Psi\rangle = |\Psi^L\rangle \otimes |\Psi^R\rangle$$

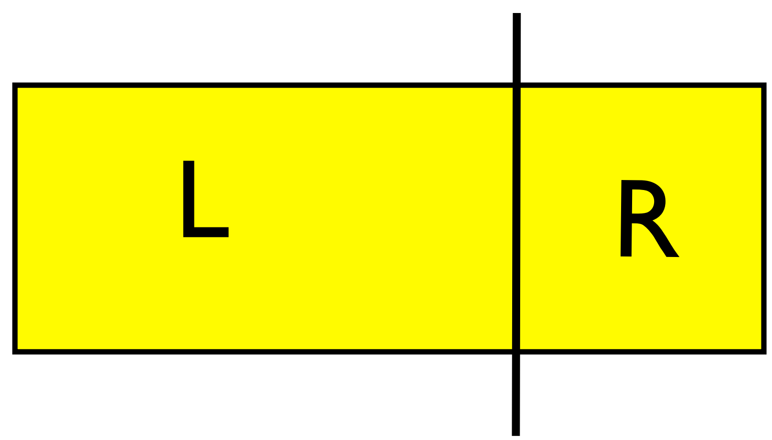
- The "entanglement spectrum" is a "fingerprint" of the entanglement, analogous to energy levels

- Bipartite Entanglement and the Schmidt Decomposition:

basis of states of L

$$|\Psi\rangle = \sum_{i=i}^{D_L} \sum_{j=1}^{D_R} \Psi(i,j) |\Phi_i^L\rangle \otimes |\Phi_j^R\rangle$$

basis of states of R



- singular value decomposition

$$\Psi(i,j) = \sum_{\alpha} e^{-\frac{1}{2}\beta_\alpha} \psi_{i\alpha}^L(i) \psi_\alpha^R(j)$$

$$\sum_i \psi_\alpha^L(i) \psi_{\alpha'}^L(i) = \delta_{\alpha\alpha'}$$

$$\sum_i \psi_\alpha^R(i) \psi_{\alpha'}^R(i) = \delta_{\alpha\alpha'}$$

$$\Psi(i,j) = \sum_\alpha e^{-\frac{1}{2}\beta_\alpha} \psi^L_{i\alpha}(i)\psi^R_\alpha(j)$$

real positive

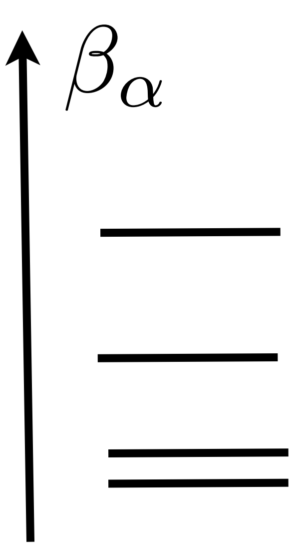
- The normalization of the state is

$$\sum_{i,j} |\Psi(i,j)|^2 = \sum_\alpha e^{-\beta_\alpha}$$

- The probability of a component is

$$p_\alpha = \frac{e^{-\beta_\alpha}}{\sum_{\alpha'} e^{-\beta_{\alpha'}}}$$

analogy with thermodynamics

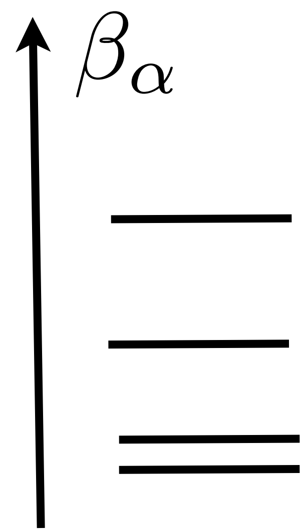


"Entanglement spectrum" (like energy levels)

The absolute value of the levels is fixed by the normalization , but only the relative values are significant

$$\Psi(i,j) = \sum_\alpha e^{-\frac{1}{2}\beta_\alpha} \psi^L_{i\alpha}(i)\psi^R_\alpha(j)$$

analogy with thermodynamics



"Entanglement spectrum"
(like energy levels)

The absolute value of the levels is fixed by the normalization , but only the relative values are significant

The von Neumann entanglement entropy coincides with the thermodynamic entropy of the set of levels at temperature $k_B T = 1$

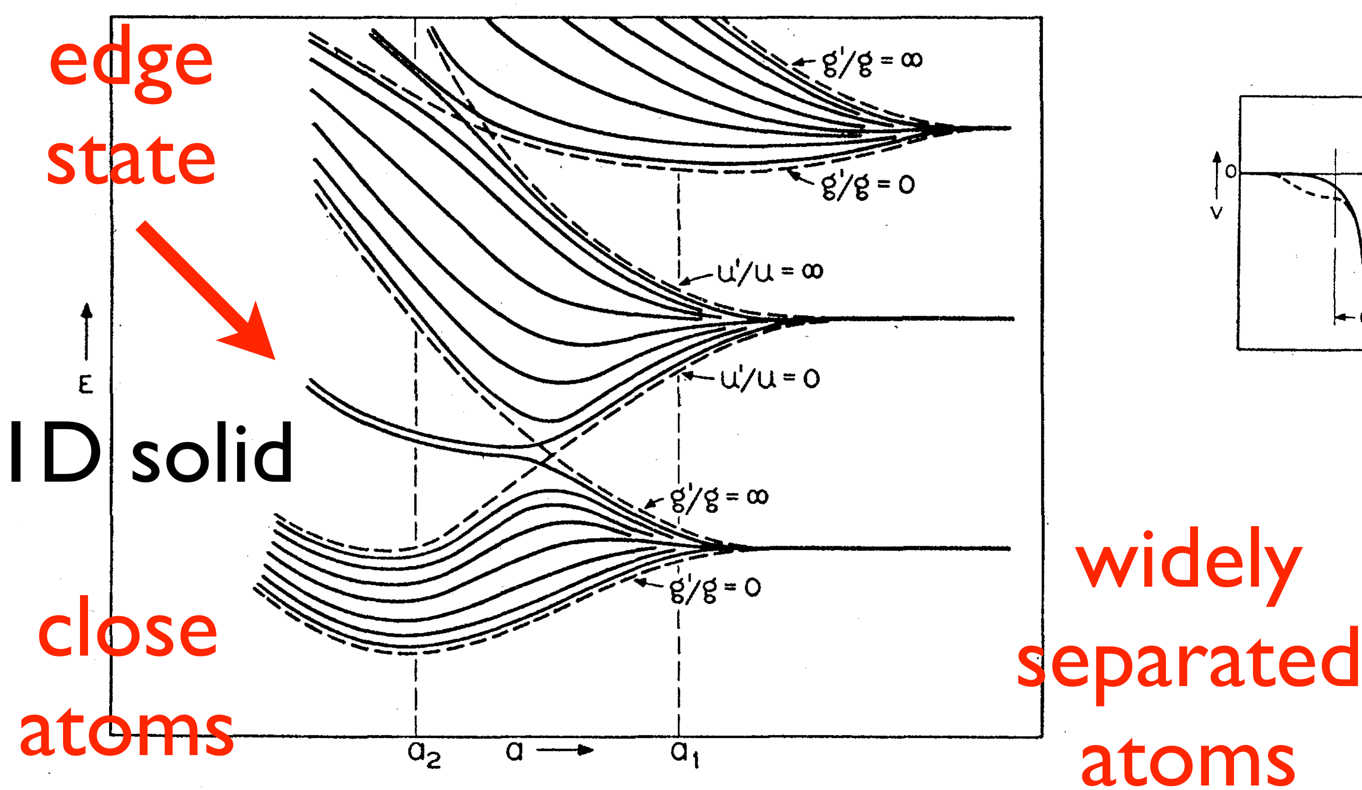
- The entanglement spectrum contains information about the entanglement between two halves of a system across a cut.
- It plays a key role in analyzing topological order
- The structure of the dominant terms in the Schmidt expansion is analogous to the low energy excitations of a many-body Hamiltonian.

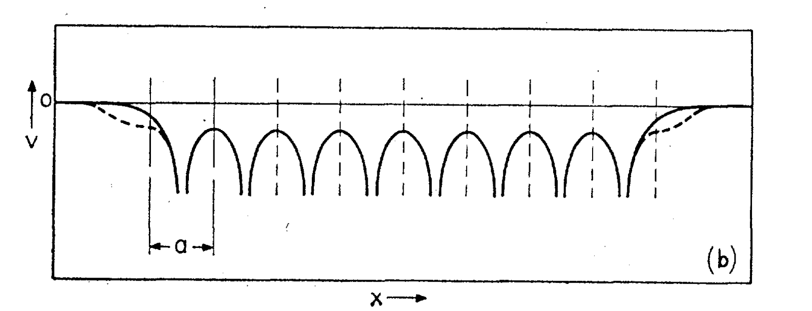
- Edge states and Entanglement.
- Topological states characteristically have protected edge states at the boundary between trivial and non-trivial regions
- They arise inevitably to terminate entanglement in the bulk

- Topologically-trivial states of insulating matter can in principal be assembled by bringing their constituent atoms together, with all electrons remaining bound during the process
- Topologically non-trivial states of matter cannot be adiabatically connected to atomic matter. At some point during their formation, bound electrons are liberated, then rebound in a state with non-trivial entanglement

- One of the striking characteristic properties of band topological insulators (or "Symmetry-Protected Topological States")is their edge states

Shockley 1939



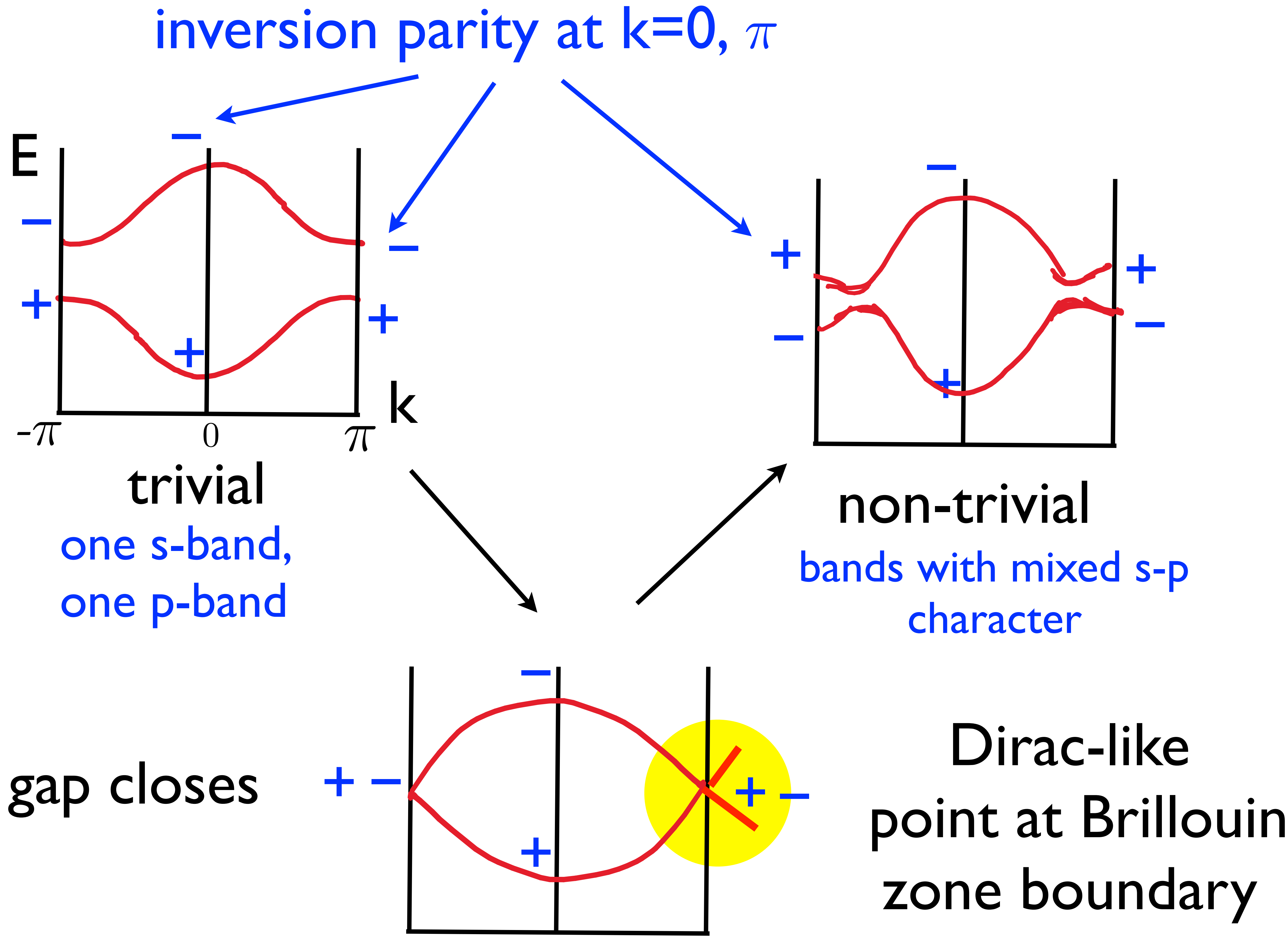


Fermi level pinned to edge state if neutral charge +1/2 electron if full , -1/2 if empty, per edge

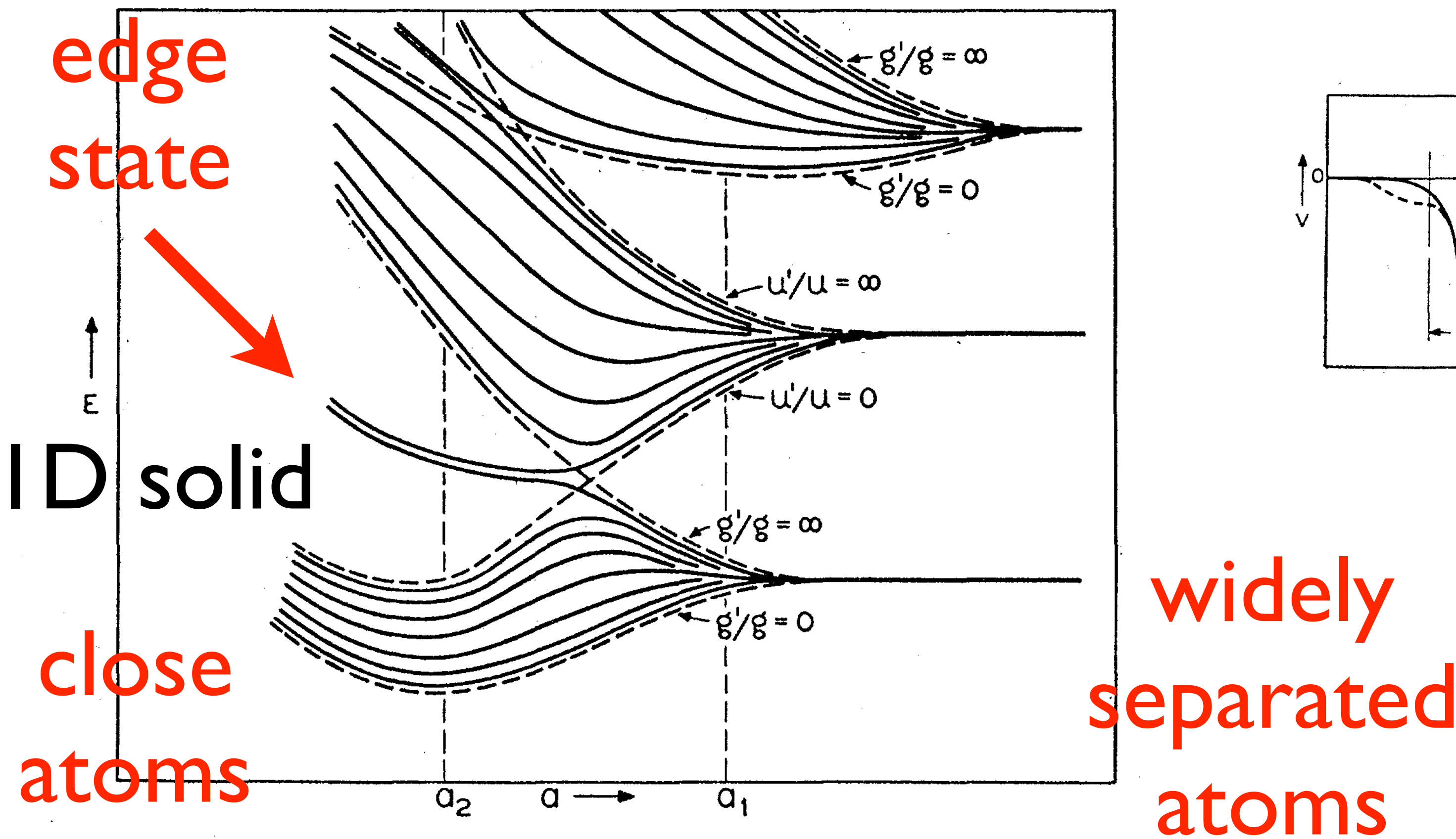
protective symmetry: spatial inversion

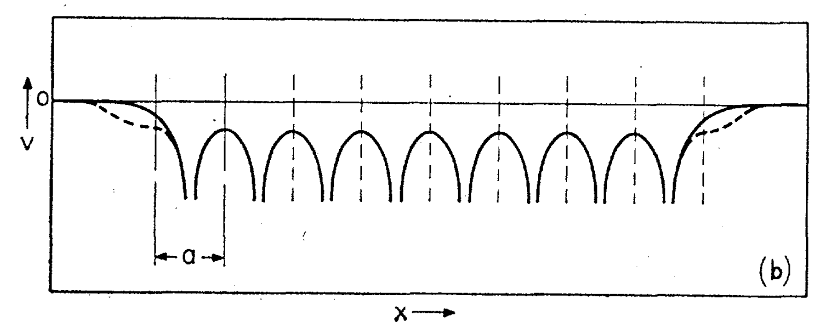
Z_2 invariant: $I_{k=0} \times I_{k=\pi} = \pm 1$

s-p band-inversion at k = π



inversion parity at k=0, π
E
k
-π
0
π
trivial
one s-band,
one p-band
non-trivial
bands with mixed s-p character
gap closes
Dirac-like point at Brillouin zone boundary





Fermi level pinned
to edge state if neutral
charge +1/2 electron if
full , -1/2 if empty, per
edge

- If both edge states are occupied, there is **<u>one</u>** extra electron, 50% at one edge, 50% at the other (half an electron at each edge)
- If both are empty there is half a hole at each edge

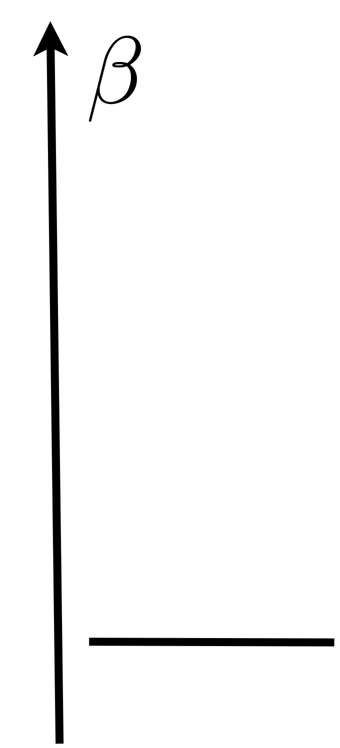
- Quantum Spin chains have been very fruitful in developing understanding of entanglement in condensed matter systems
- The controversial and unexpected "Haldane gap" in the Spin-1 chain led to the development of tensor product states and DMRG techniques, which were subsequently clarified with ideas from quantum information theory

- A simple model for an unentangled product state is the model

$$H = D \sum_i (S_i^z)^2$$

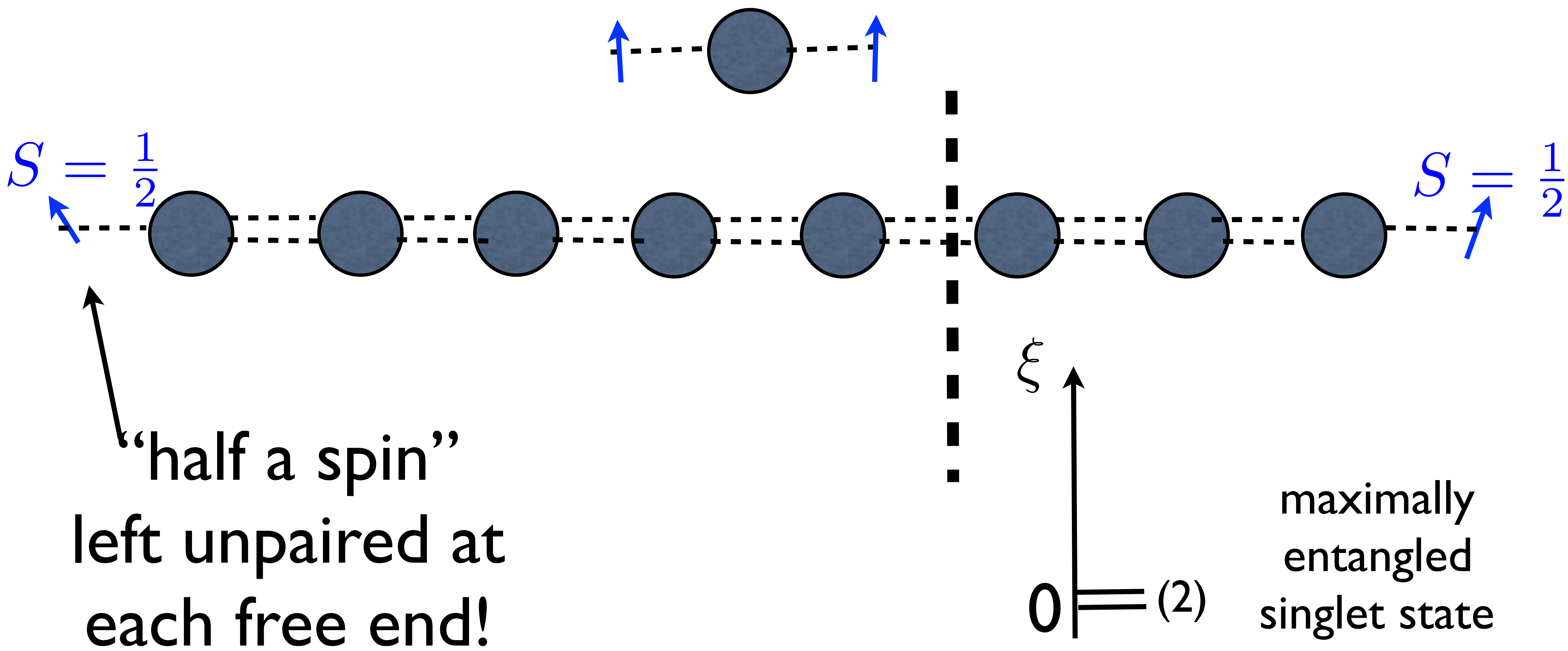
$$|\Psi_0\rangle = \ldots \otimes |0\rangle \otimes |0\rangle \otimes |0\rangle \otimes \ldots$$

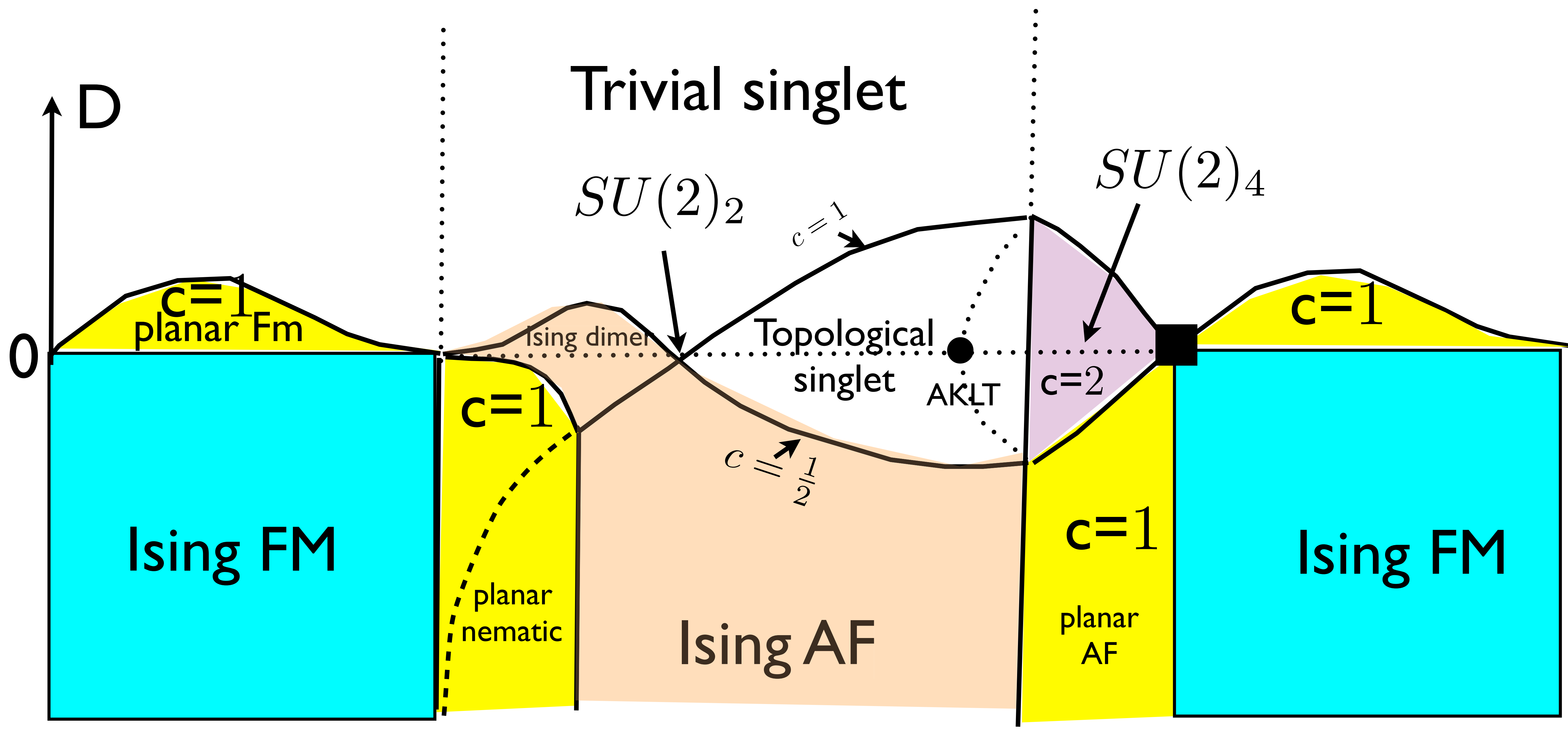
- The entanglement spectrum has a single level



- AKLT state (Affleck, Kennedy, Lieb, Tasaki)

- regard a "spin-1" object as symmetrized product of two spin-$1/2$ spins, and pair one of these in a singlet state with "half" of the neighbor to the right, half with the neighbor to the left:





$$H=\sum_i J\vec{S}_i\cdot\vec{S}_{i+1}+K(\vec{S}_i\cdot\vec{S}_{i+1})^2+D(S_i^z)^2$$

spin 1

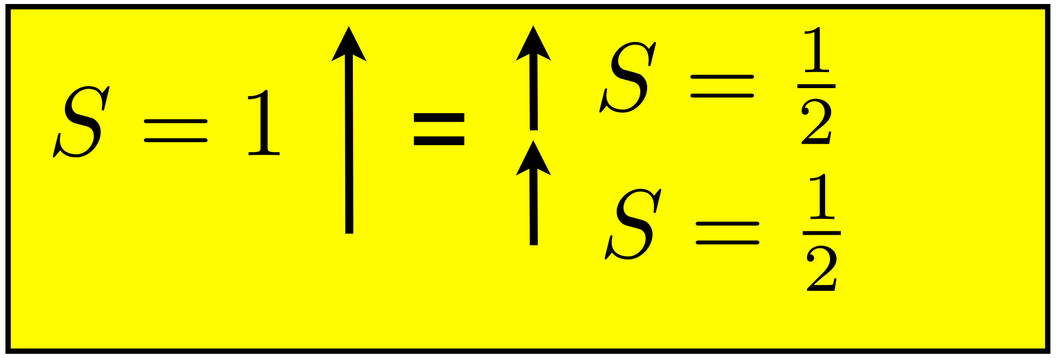
- X-G Wen and collaborators X.Chen, Z.Gu have developed a classiffication of SPT states in general (not just free fermions) using powerful mathematical tools of **cohomology theory**

- Their starting point was to identify the fundamental example as the non-trivial spin-1 chain that I identified may years ago using key ideas from Michael Berry's geometrical phase.

- They realised that the symmetry analysis needed for the 1D chain was a simple example of a cohomology argument that works in higher dimensions too!

- This instructive example of an SPT state is the spin-1 chain "Haldane gap" state,
- This exhibits fractionalization, topological order and entanglement, characterized by the <u>entanglement spectrum</u> (Li and FDMH 2008) which has become an important tool for investigating Topological Order.

A spin-1 degree of freedom can be represented as **two** spin-1/2 degrees of freedom, projected into a symmetric state.



$$H = \sum_n \boldsymbol{S}_n \cdot \boldsymbol{S}_{n+1} \qquad S = 1$$

"Physical model"

$$H^{AKLT} = \sum_n \boldsymbol{S}_n \cdot \boldsymbol{S}_{n+1} + \frac{1}{3}(\boldsymbol{S}_n \cdot \boldsymbol{S}_{n+1})^2$$

"Toy model"

$$\mathcal{L} = g^{\mu\nu}\partial_\mu \hat{\Omega} \cdot \partial_\nu \hat{\Omega} + \frac{1}{4\pi}\theta \epsilon^{\mu\nu} \hat{\Omega} \cdot \partial_\mu \hat{\Omega} \times \partial_\nu \hat{\Omega}$$

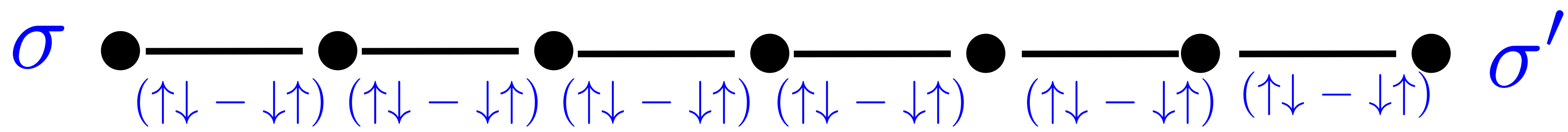
Field theory with "topological term"

Topological term

- In the presence of protective symmetries (spatial inversion and time-reversal)

$\theta = 0 (\mod 2\pi)$ integer S

$\theta = \pi (\mod 2\pi)$ half-odd-integer S



valence bond picture (AKLT) spin -1

2x2 Matrix product state)

$$\sum_{\sigma\sigma'} \psi_{\sigma}^{L*} (\boldsymbol{M}^{(1)} \dots \boldsymbol{M}^{(N)})_{\sigma\sigma'} \psi_{\sigma'}^{R}$$

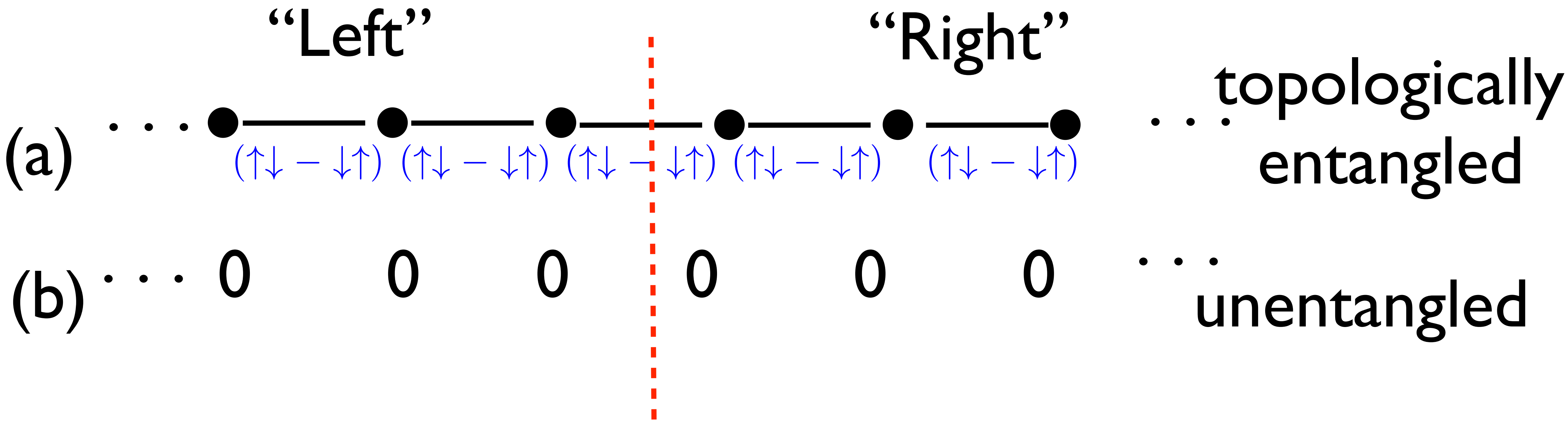
gapped (incompressible) state,unbroken symmetry

free spin-(1/2) states at free ends!

$$H = \sum_i J \boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1} + D(S_i^z)^2$$

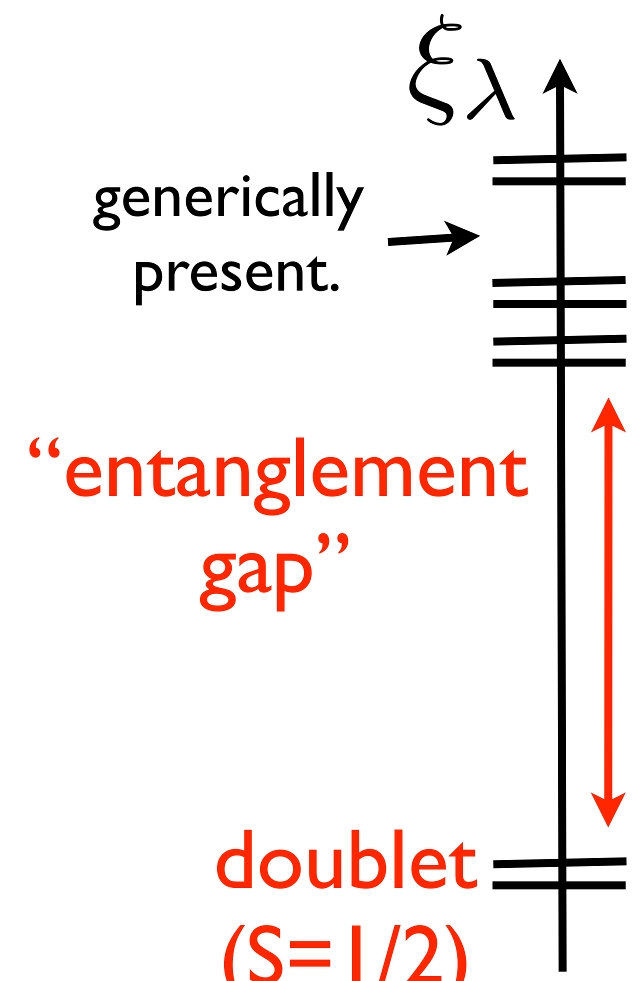
- Large D favours a state with $S^z_i = 0$, all i.

- topological order = long-range entanglement



$$|\Psi\rangle = \sum_\lambda e^{-\xi_\lambda/2} |\Psi^L_\lambda\rangle \otimes |\Psi^R_\lambda\rangle$$

Bipartite Schmidt-decomposition of ground state reveals entanglement



- a gapless "topological entanglement spectrum" separated from other Schmidt eigenvalues by an "entanglement gap" is characteristic of long-range topological order (Li + FDMH, PRL 2008)

- Topological states of matter have been a major theme in the recent developments in understanding novel quantum effects.
- key questions are: why do they occur,what features of materials favor such states, and how can we understand the energetics that drives their emergence.
- I will principally discuss the fractional quantum Hall effect, but this is a general question

- Fractional Quantum Hall effect

- thirty years after its experimental discovery and theoretical description in terms of the Laughlin state, the fractional quantum Hall effect remains a rich source of new ideas in condensed matter physics.
- The key concept is "**flux attachment**" that forms "**composite particles**" and leads to topological order.
- Recently, it has been realized that flux attachment also has interesting **geometric** properties

$$\Psi = \prod_{i<j}(z_i - z_j)^3 \prod_i e^{-\frac{1}{2}z_i^* z_i}$$

Laughlin 1983

- elegant wavefunction, describes topologically-ordered fluid with fractional charge fractional statistics excitations
- exact ground state of modified model keeping only short range part of coulomb repulsion
- Validity confirmed by numerical exact diagonalization

30 years later:
unanswered question:
we know it works, but why?

my answer:
hidden geometry

some widespread misconceptions about the Laughlin state

- "it describes particles in the lowest Landau level"

No Landau level was specified: all specifics of the Landau level are hidden in the form of $U(\boldsymbol{r}_{12})$

- "It is a Schrödinger wavefunction"

Non-commutative geometry has no Schrödinger representation (this requires classical locality); it only has a Heisenberg representation.

- "Its shape is determined by the shape of the Landau orbit"

The interaction potential $U(\boldsymbol{r}_{12})$ determines its geometry (shape)

- "It has no continuously-tunable variational parameter"

Its geometry is a continuously-variable variational parameter

- In a 2D Landau level, we apparently start from a Schrödinger picture, but end with a "quantum geometry" which is no longer correctly described by Schrödinger wavefunctions in real space because of "quantum fuzziness" (non-locality)
- It remains correctly described by the Heisenberg formalism in Hilbert space.

- Top-level model (Schrödinger):

$$p_i = -i\hbar\boldsymbol{\nabla}_r - e\boldsymbol{A}(\boldsymbol{r})$$

$$\boldsymbol{\nabla}_r \times \boldsymbol{A}(\boldsymbol{r}) = B$$

$$H = \sum_i \varepsilon(\boldsymbol{p}_i) + \sum_{i<j} V_0(\boldsymbol{r}_i - \boldsymbol{r}_j)$$

not necessarily quadratic (**no** Galilean invariance should be assumed)

bare Coulomb interaction controlled by (possibly anisotropic) dielectric tensor of medium (no rotational invariance should be assumed)

- model has inversion symmetry if $\varepsilon(\boldsymbol{p}) = \varepsilon(-\boldsymbol{p})$, but even this need not be assumed

$$\boldsymbol{r} = r^a \boldsymbol{e}_a$$

displacement (contravariant index)

$$\boldsymbol{e}_a \cdot \boldsymbol{e}_b = \delta_{ab}$$

orthonormal basis of tangent vectors of 2D plane: $a = 1,2$

Euclidean metric of 2D plane

$$p_a = \boldsymbol{e}_a \cdot \boldsymbol{p}$$

dynamical momentum (covariant index)

- Two independent Heisenberg algebras:

antisymmetric (2D Levi-Civita) symbol

$$[p_a, p_b] = i\hbar e B \epsilon_{ab}$$
$$[r^a, p_b] = i\hbar \delta^a_b$$
$$[r^a, r^b] = 0$$

organize as →

$$[\bar{R}^a, \bar{R}^b] = i\ell_B^2 \epsilon^{ab}$$
$$[R^a, \bar{R}^b] = 0$$
$$[R^a, R^b] = -i\ell_B^2 \epsilon^{ab}$$

$$\bar{R}^a = \hbar^{-1}\epsilon^{ab} p_b \ell_B^2$$

Landau orbit radius vector

$$R^a = r^a - \bar{R}^a$$

Landau orbit guiding-center displacement

$$2\pi\ell_B^2 = \frac{2\pi\hbar}{eB} > 0$$

quantum area (per h/e flux quantum)

- Note: origin of guiding-center displacement has a gauge ambiguity under $\boldsymbol{A}(\boldsymbol{r}) \mapsto \boldsymbol{A}(\boldsymbol{r})$ + constant

- Landau quantization $\varepsilon(\boldsymbol{p})|\Psi_n\rangle = E_n|\Psi_n\rangle$

discrete spectrum of macroscopically-degenerate Landau levels

- Project residual interaction in a single partially occupied "active" Landau level, all other dynamics is frozen by Pauli principle when gap between Landau levels dominates interaction potential

$$[R^a, R^b] = -i\ell_B^2\epsilon^{ab}$$

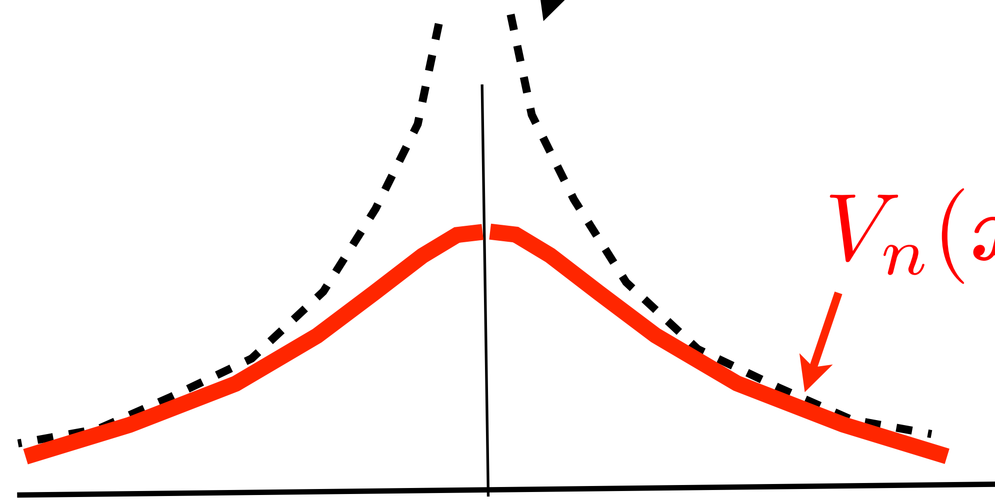
$$H = \sum_{i<j} V_n(\boldsymbol{R}_i - \boldsymbol{R}_j)$$

residual problem is non-commutative quantum geometry!

$$[R^a, R^b] = -i\ell_B^2 \epsilon^{ab}$$

$$H = \sum_{i<j} V_n(\boldsymbol{R}_i - \boldsymbol{R}_j)$$

Identical quantum particles
(fermions or bosons)



We now have the final form of the problem:

- The potential $V_n(x)$ is a **very smooth** (in fact entire) function that depends on the form- factor of the partially-occupied Landau level

- The essential clean-limit symmetries are translation and inversion: $\boldsymbol{R}_i \mapsto \boldsymbol{a} \pm \boldsymbol{R}_i$

- the essential model Hamiltonian for a partially-filled 2D Landau level

$$H_0 = \sum_{i<j} V_2(\boldsymbol{R}_i - \boldsymbol{R}_j)$$

dominant 2-particle interaction with no kinetic energy

$$H = H_0 + \sum_i V_1(\boldsymbol{R}_i)$$

1-particle term as a small perturbation

$$[R_i^x, R_i^y] = -i\ell_B^2$$

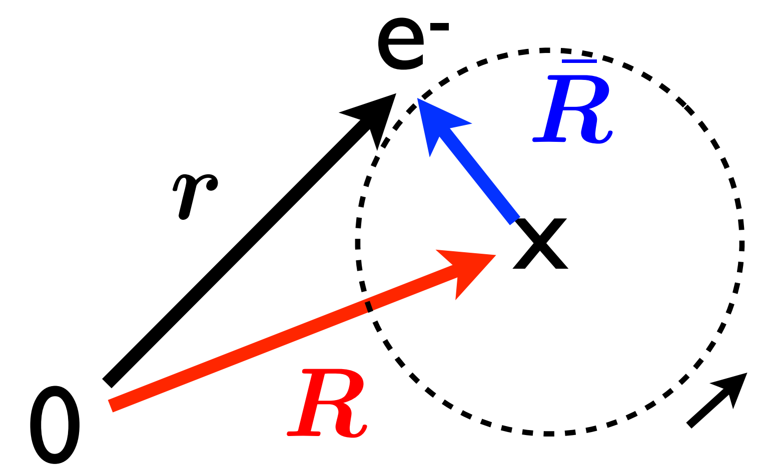
non-commutative geometry

the source of all dynamics in this problem !

- Where did this come from?

$$p_a = -i\hbar\nabla_a - eA_a(\boldsymbol{x})$$

$$[p_x, p_y] = i\hbar eB$$



- Landau orbit radius vector

$$\bar{\boldsymbol{R}} = \frac{1}{eB}(p_y, -p_x)$$

- Landau orbit guiding center

$$\boldsymbol{R} = \boldsymbol{r} - \bar{\boldsymbol{R}}$$

$$\boldsymbol{r} = \boldsymbol{R} + \bar{\boldsymbol{R}} \qquad [R^a, \bar{R}^b] = 0$$

$$[r^x, r^y] = 0$$

$$[\bar{R}^x, \bar{R}^y] = i\ell_B^2$$

$$[R^x, R^y] = -i\ell_B^2$$

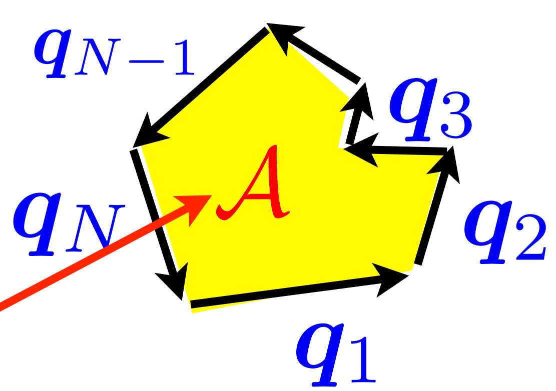
after Landau-level quantization, only the guiding centers remain as dynamical variables

- Rather than the commutation relation (here $[R^x, R^y] = -i\ell_B^2$), von Neumann pointed out that the fundamental presentation of the Heisenberg algebra was the exponentiated form which here is $U(\boldsymbol{q}) = \exp(i\, q_a R^a)$

- Let $\sum_{i=1}^{N} \boldsymbol{q}_i = 0$ be a polygonal path in q-space

$$U(\boldsymbol{q}_1)U(\boldsymbol{q}_2)\ldots U(\boldsymbol{q}_N) = e^{\frac{1}{2}i\mathcal{A}\ell_B^2}\mathbb{1}$$

closed path-ordered product



q-space area enclosed by closed path

- V_2 must be an"ultra smooth function" in $\mathbb{R}2$ (entire in $\mathbb{C}2$) (this is required for it to be well-defined when it has a non-commutative argument)

$$H_0 = \sum_{i<j} V_2(\boldsymbol{R}_i - \boldsymbol{R}_j)$$

$$V_2(\boldsymbol{x}) = \int \frac{d^2\boldsymbol{q}\ell_B}{2\pi} \tilde{V}_c(\boldsymbol{q})|f_n(\boldsymbol{q})|^2 e^{i\boldsymbol{q}\cdot\boldsymbol{x}}$$

unsmooth Coulomb interaction

$$\frac{e^2}{4\pi\epsilon_0\epsilon}\frac{1}{|q|}$$

$$\exp -\tfrac{1}{2}|q|^2\ell_B^2$$

landau level form factor

$V(\boldsymbol{R}) = V(\boldsymbol{x} + (\boldsymbol{R} - \boldsymbol{x}))$ has an absolutely convergent expansion in $\delta\boldsymbol{R} = (\boldsymbol{R} - \boldsymbol{x})$

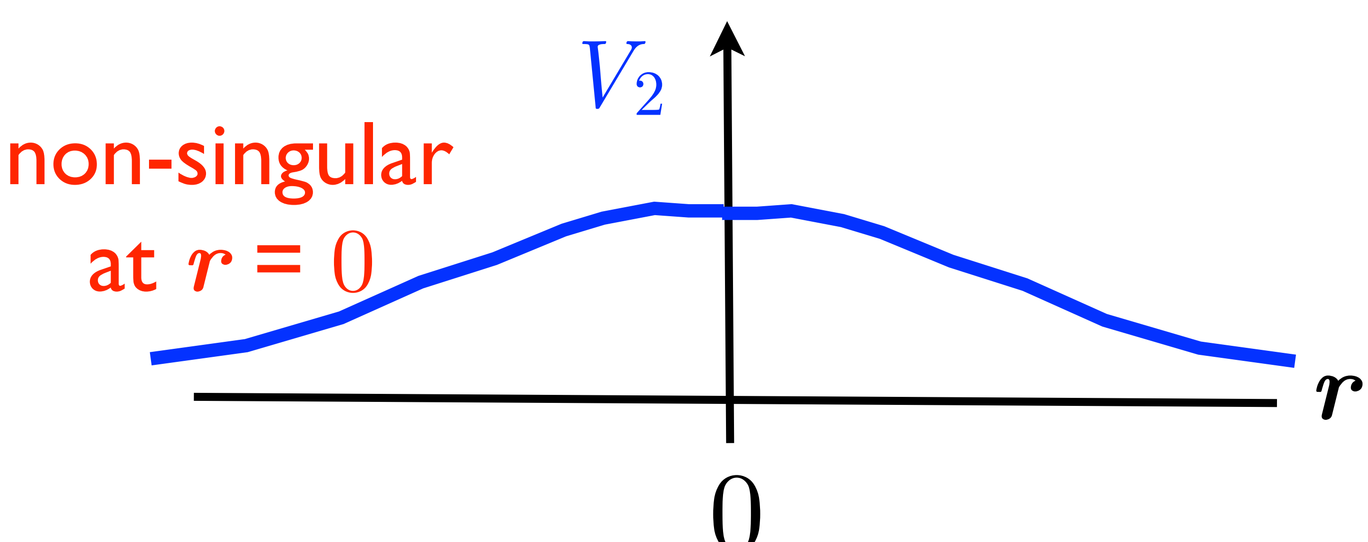
$$\delta R^a \delta R^b \ldots \delta R^c \rightarrow \{\delta R^a, \delta R^b, \ldots, \delta R^c\}$$ (symmetrized product)

- The two-body interaction potential is smooth because it is the bare unretarded Coulomb potential convoluted with the Landau-orbit form factor of the partially-filled level.

$$V_2(\boldsymbol{r}) = \int \frac{d^2\boldsymbol{q}\ell^2}{2\pi} e^{i\boldsymbol{q}\cdot\boldsymbol{r}} \tilde{V}(\boldsymbol{q})|f_n(\boldsymbol{q})|^2$$

- It depends on the structure of the Landau orbits of the partially-filled level through the form factor $f_n(\boldsymbol{q})$.

- For all $\boldsymbol{r}$, $V_2(\boldsymbol{r} + \delta\boldsymbol{r})$ has an analytic (entire) expansion in $\delta\boldsymbol{r}$, because the form-factor $f_n(\boldsymbol{q})$ is a rapidly-decreasing function.



$V_2(\boldsymbol{r}) = V_2(-\boldsymbol{r})$

real, even

$$H_2 = \sum_{i<j} V_2(\boldsymbol{R}_i - \boldsymbol{R}_j) \qquad [R_i^x, R_j^y] = -i\delta_{ij}\ell_B^2$$

The entire "clean limit" problem

Depending on the filling factor ν and the form of the interaction potential $V_2(r)$, this problem is known to have the following types of ground states:

- incompressible (gapped) translationally-invariant inversion-symmetric topologically-ordered fractional quantum Hall (FQH)states
- compressible (gapless) states with **broken translational symmetry** (stripe and bubble phases, Wigner crystal)
- gapless "Composite Fermi Liquid" (CFL) states with **unbroken translational symmetry** which can be argued to exhibit a neutral fermion Fermi surface

exhibits a gapless **anomalous Hall effect** (AHE)
(like ferromagnetic metals)

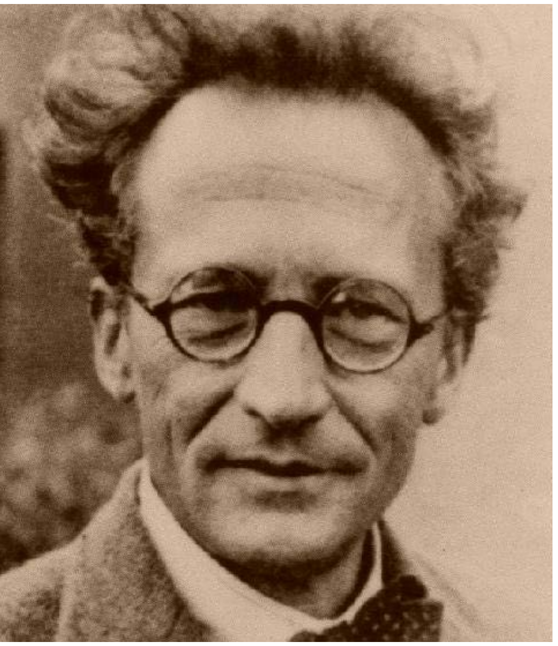
$$H_0 = \sum_{i<j} V_2(\boldsymbol{R}_i - \boldsymbol{R}_j) \quad [R_i^x, R_i^y] = -i\ell_B^2$$

- A quantum geometry does not support a Schrödinger representation

$$\Psi(\boldsymbol{x}) = \langle \boldsymbol{x} | \Psi \rangle$$

$$\langle \boldsymbol{x} | \boldsymbol{x}' \rangle = 0 \quad \boldsymbol{x} \neq \boldsymbol{x}'$$

such a basis does not exist when coordinate components do not commute



Schrödinger vs Heisenberg

- Schrödinger's picture describes the system by a **wavefunction** $\psi(r)$ in real space

- Heisenberg's picture describes the system by a **state** $|\psi\rangle$ in Hilbert space

- They are only equivalent if the basis $|\boldsymbol{r}\rangle$ of states in real-space are orthogonal:

$$\psi(\boldsymbol{r}) = \langle \boldsymbol{r} | \psi \rangle$$

requires

$$\langle \boldsymbol{r} | \boldsymbol{r}' \rangle = 0 \quad (\boldsymbol{r} \neq \boldsymbol{r}')$$

← this fails in a quantum geometry

Q: **When is a "wavefunction" NOT a wavefunction?**

A: **When it describes a "quantum geometry"**

- In this case space is "fuzzy"(non-commuting components of the coordinates), and the Schrödinger description in real space (i.e., in "classical geometry") fails, though the Heisenberg description in Hilbert space survives
- The closest description to the classical-geometry Schrödinger description is in a non-orthogonal overcomplete **coherent-state** basis of the quantum geometry.

- but the most famous result in FQHE was presented as a "lowest Landau level wavefunction"

$$\Psi(\boldsymbol{x}_1, \boldsymbol{x}_2 \ldots, x_N) = \prod_{i<j} (z_i - z_j)^m \prod_i e^{-\frac{1}{4}|x_i|^2/\ell_B^2}$$

Laughlin 1983

$$z = x + iy$$

- lowest Landau level states of $H_1 = \frac{1}{2m}(p_x^2 + p_y^2)$ have wavefunctions of the form $\psi(\boldsymbol{x}) = f(z) e^{-\frac{1}{4}|x|^2/\ell_B^2}$

holomorphic function

$$\Psi(\boldsymbol{x}_1, \boldsymbol{x}_2 \ldots, x_N) = \prod_{i<j} (z_i - z_j)^m \prod_i e^{-\frac{1}{4}|x_i|^2/\ell_B^2}$$

- For $m = 1$, this is the Slater determinant describing the uncorrelated filled lowest Landau level
- for $m > 1$ it is a highly correlated state exhibiting "flux attachment"
- It was initially proposed as a "trial wavefunction" with no continuously-adjustable variational parameter, that gave a lower energy that all other proposed states

But:

- The essential problem

$$H_0 = \sum_{i<j} V_2(\boldsymbol{R}_i - \boldsymbol{R}_j) \qquad [R_i^x, R_i^y] = -i\ell_B^2$$

is an "any Landau level" problem, not a "lowest Landau level" problem and does not reference $H_1 = \frac{1}{2m}(p_x^2 + p_y^2)$

- The Laughlin-like FQHE state occurs in the second Landau level, as well as the lowest.

- "quantum geometry" is not described by Schrodinger wavefunctions

We need to reinterpret the "Laughlin wavefunction"

- In fact, the Laughlin state does have a hidden variational parameter, its geometry

two independent Heisenberg algebras:

$$a^\dagger = \frac{R^x + iR^y}{\sqrt{2}\ell_B}$$

$$[a, a^\dagger] = 1$$

Heisenberg algebra of guiding centers

$$\bar{a}^\dagger = \frac{p_x + ip_y}{\sqrt{(2|\hbar eB|}}$$

$$[\bar{a}, \bar{a}^\dagger] = 1$$

Heisenberg algebra of Galileian-invariant Landau orbits

$$H_1 = \frac{1}{2m}(p_x^2 + p_y^2)$$

$$|\Psi\rangle = \prod_{i<j}(a_i^\dagger - a_j^\dagger)^m|0\rangle \qquad a_i|0\rangle = 0$$

Heisenberg form of Laughlin state

← no longer references any particular Landau level

$$a^\dagger = \frac{R^x + iR^y}{\sqrt{2}\ell_B}$$

$$H_1 = \frac{1}{2m}(p_x^2 + p_y^2)$$

$$|\Psi\rangle = \prod_{i<j}(a_i^\dagger - a_j^\dagger)^m|0\rangle$$

$$H_0 = \sum_{i<j} V_2(\boldsymbol{R}_i - \boldsymbol{R}_j)$$

- the original form of $a_i^\dagger$ was inherited from the shape of the Landau orbits
- Instead, it should be determined by the shape of the interaction potential

$$\frac{1}{2\ell_B^2} g_{ab} R^a R^b = \frac{1}{2}(a^\dagger a + a a^\dagger) \qquad \det g = 1$$

a metric

$$|\Psi(g)\rangle = \prod_{i<j}(a_i^\dagger - a_j^\dagger)^m|0(g)\rangle$$

$$\frac{1}{2\ell_B^2}g_{ab}R^aR^b = \frac{1}{2}(a^\dagger a + aa^\dagger)$$

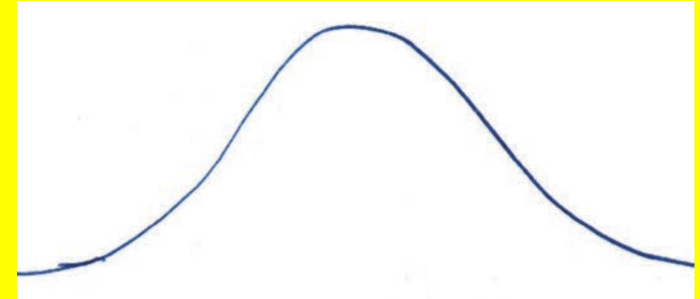
$$a_i|0(g)\rangle = 0$$

- The "Laughlin state" is a family of states continuously parametrized by a "unimodular" (unit determinant) metric
- The metric characterizes the shape of the correlation hole formed by "flux attachment" and should be chosen to minimize the correlation energy of $$H_0 = \sum_{i<j} V_2(\boldsymbol{R}_i - \boldsymbol{R}_j)$$
- The uncorrelated filled Landau level state is left invariant* by a change of metric

*when periodic boundary conditions are imposed

- As well as being a variational trial wavefunction, the Laughlin state is the true ground state of a certain short-range interaction potential

This is the **entire** problem: nothing other than this matters!



$$H = \sum_{i<j} U(\boldsymbol{R}_i - \boldsymbol{R}_j)$$

$$[R^x, R^y] = -i\ell_B^2$$

like phase-space, has Heisenberg uncertainty principle

- H has translation and inversion symmetry

$$[(R_1^x + R_2^x), (R_1^y - R_2^y)] = 0$$

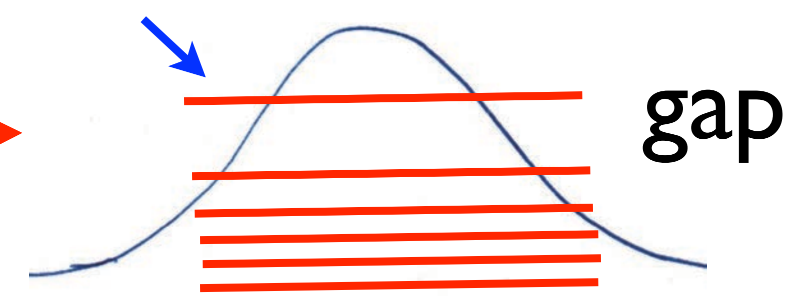
$$[H, \sum_i \boldsymbol{R}_i] = 0$$

- generator of translations and electric dipole moment!

$$[(R_1^x - R_2^x), (R_1^y - R_2^y)] = -2i\ell_B^2$$

- relative coordinate of a pair of particles behaves like a single particle

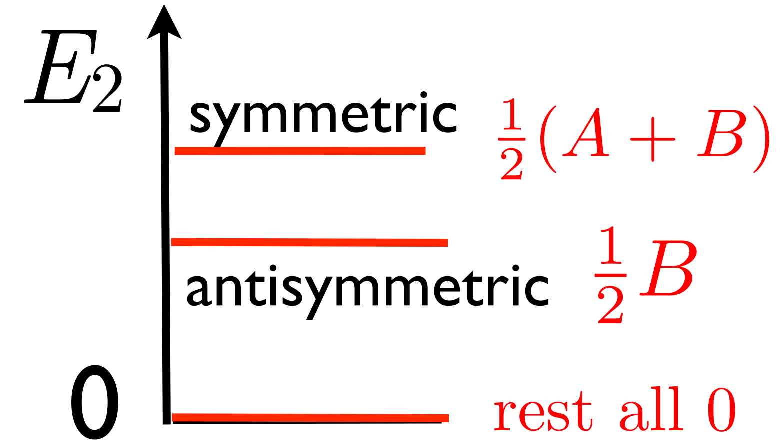
want to avoid this state



two-particle energy levels

- Solvable model! ("short-range pseudopotential")

$$U(r_{12}) = \left(A + B\left(\frac{(r_{12})^2}{\ell_B^2}\right)\right) e^{-\frac{(r_{12})^2}{2\ell_B^2}}$$



- Laughlin state

$$|\Psi_L^m\rangle = \prod_{i<j}\left(a_i^\dagger - a_j^\dagger\right)^m |0\rangle$$

$$a_i|0\rangle = 0 \qquad a_i^\dagger = \frac{R^x + iR^y}{\sqrt{2}\ell_B}$$

$$E_L = 0 \qquad [a_i, a_j^\dagger] = \delta_{ij}$$

maximum density null state

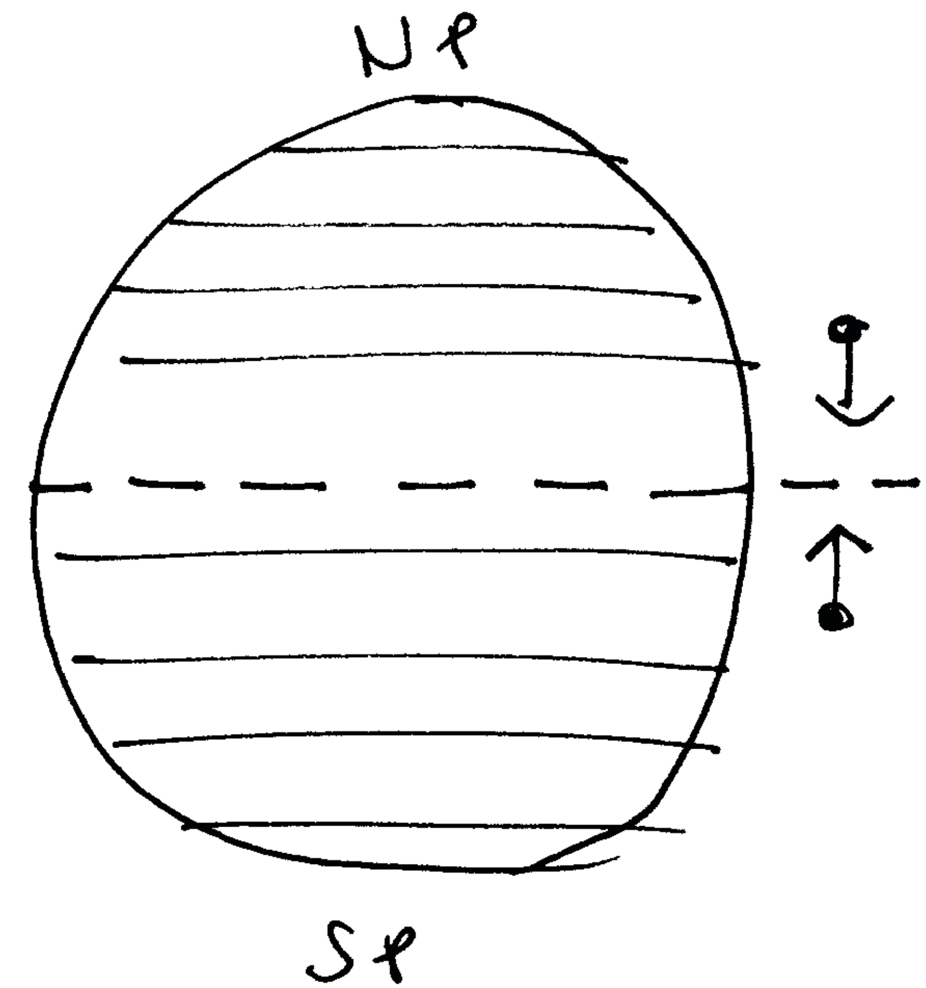
- m=2: (bosons): all pairs avoid the symmetric state $E_2 = ½(A+B)$

- m=3: (fermions): all pairs avoid the antisymmetric state $E_2 = ½B$

- The key idea for understanding both the Fractional Quantum Hall and Composite Fermi Liquid states is “Flux attachment”

Entanglement spectra and "dominance"

- Schmidt decomposition of Fock space into N and S hemispheres.
- Classify states by Lz and N in northern hemisphere, relative to dominant configuration. Lz always decreases relative to this (squeezing)



Laughlin FQHE state

$$\Psi = \Phi(z_1, z_2, \ldots, z_N) \prod_{i=1}^{N} e^{-\varphi(\boldsymbol{r}_i)}$$

lowest Landau level

N-variable (anti)symmetric polynomial

$$\nabla^2 \varphi(\boldsymbol{r}) = 2\pi B(\boldsymbol{r})/\Phi_0$$

- ν = 1/m Laughlin state

$$\Phi(z_1, z_2, \ldots, z_N) = \prod_{i<j} (z_i - z_j)^m$$

- "occupation number"-like representation in orbitals z^m, m = 0,1,..., N_Φ = m(N-1) orbitals

m=0 orbital → 1001001001001001001...1001 (m=3)

This is the "dominant" configuration of the Laughlin state

“Dominance”

- convert occupation pattern to a **partition** **λ**, “padded” with zeroes to length N:
- $1001001 \rightarrow \lambda = \{\lambda_1, \lambda_2, \lambda_3\} = \{6,3,0\}$
- λ dominates λ' if
  - $|\lambda| \equiv (\sum_i \lambda_i) = |\lambda'| = M$
  - $(\sum_{j \leq i} \lambda'_j) \leq (\sum_{j \leq i} \lambda_j)$ for all $i = 1,2,..N-1$

“dominance” and “squeezing”

- (pairwise) squeezing: move a particle from orbital m_1-1 to m_1 and another from m_2+1 to m_2 where $m_1 \leq m_2$.

$$100100100100100100\ldots1001 \quad A$$

$$100010100100101000\ldots1001 \quad B$$

A dominates B (A > B)

- dominance is a <u>partial ordering</u>: if A > B and B > C, then A > C.

Fermionic 2/4=1/2 Moore-Read state

uniform vacuum state on sphere:

1100110011001100110011001100110011

even fermion number -e/2 double quasihole (h/e vortex) at North Pole:

01100110011001100110011001100110011

odd fermion number -e/2 double quasihole (h/e vortex) at North Pole:

100110011001100110011001100110011

fractionalization: one -e/4 quasihole (h/2e vortex) at North Pole, one near equator.

101010101010101001100110011001100110011

These translate into explicit wavefunctions that can be calculated in finite-size systems

Represent bipartite Schmidt decomposition like an excitation spectrum (with Hui Li)

$$|\Psi\rangle = \sum_\alpha e^{-\beta_\alpha/2} |\Psi_{N\alpha}\rangle \otimes |\Psi_{S\alpha}\rangle$$

- like CFT of edge states.
- A lot more information than single number (entropy)

- many zero eigenvalues

$$e^{-\beta_\alpha} = 0$$

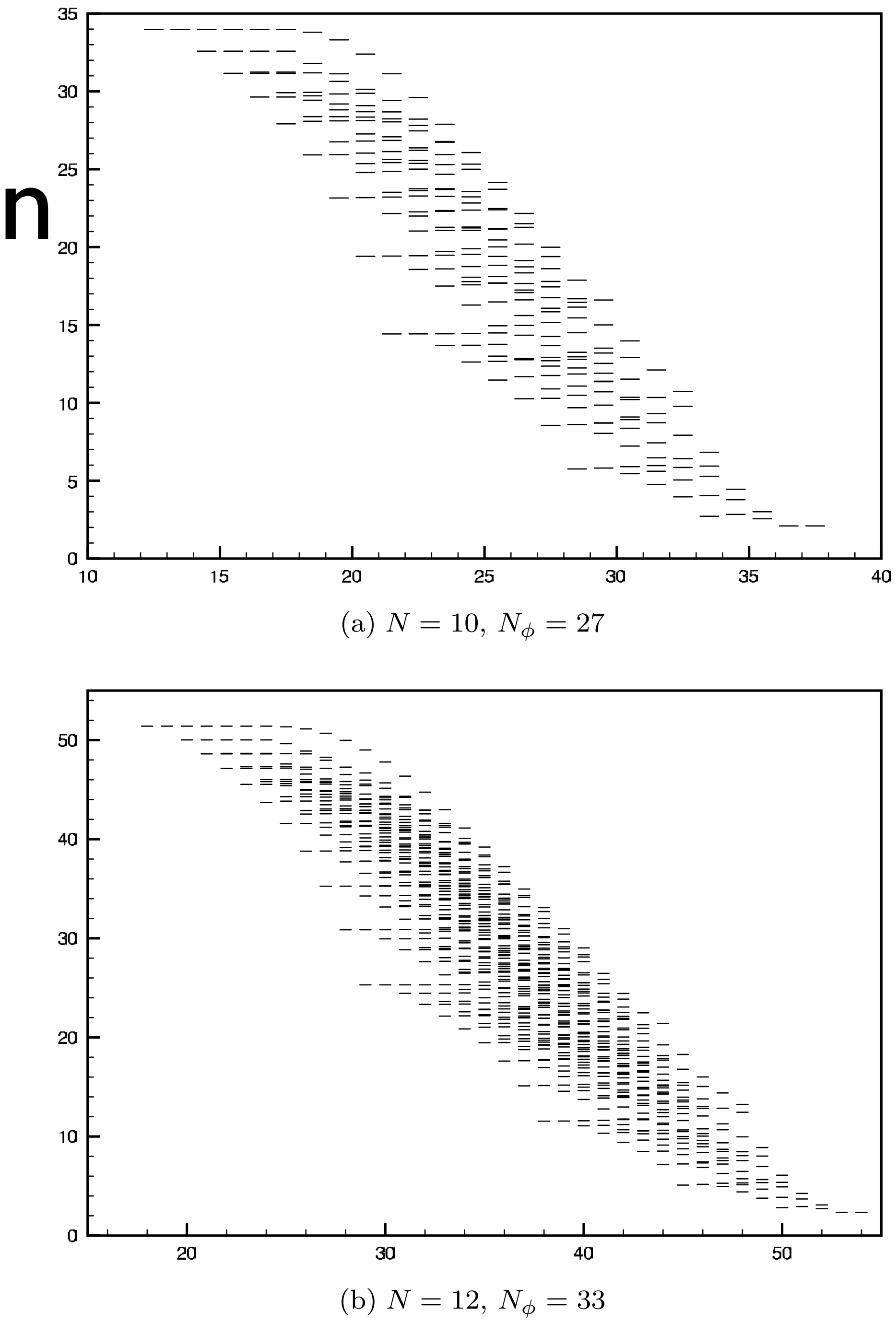
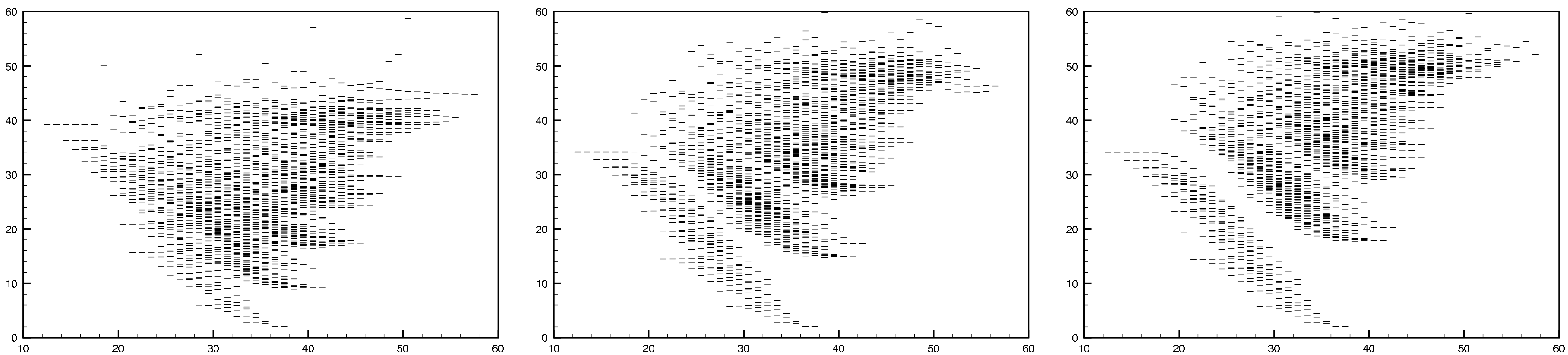


FIG. 1: Entanglement spectrum for the 1/3-filling Laughlin states, for $N = 10, m = 3, N_\phi = 27$ and $N = 12, m = 3, N_\phi = 33$. Only sectors of $N_A = N_B = N/2$ are shown.

Look at difference between Laughlin state, entanglement spectrum and state that interpolates to Coulomb ground state.

3



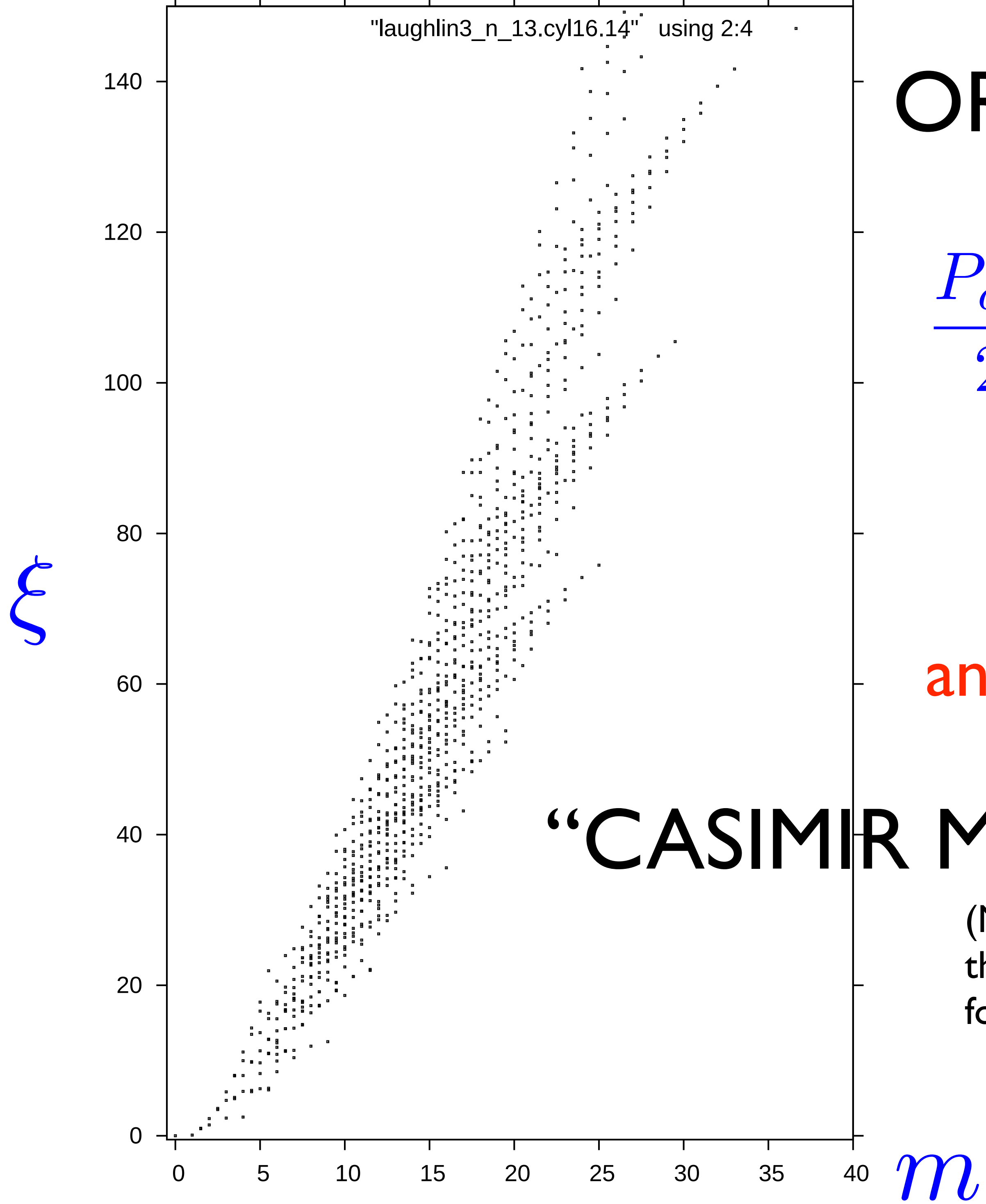
(a) $x = 1$ (b) $x = 1/3$ (c) $x = 1/10$

FIG. 2: Entanglement spectrum for the ground state, for a system of $N = 10$ electrons in the lowest Landau level on a sphere enclosing $N_\phi = 27$ flux quanta, of the Hamiltonian in Eq. (12) for various values of x.

$$H = xH_c + (1 - x)V_1$$

x=0 is pure Laughlin

Can we identify topological order in "physical as opposed to model wavefunctions from low-energy entanglement spectra?



ORBITAL CUT

$$\frac{P_a L^a}{2\pi} = \frac{\sum_\alpha m_\alpha e^{-\xi_\alpha}}{\sum_\alpha e^{-\xi_\alpha}} = \eta_H^{cd}\epsilon_{ac}\epsilon_{bd}\frac{L^a L^b}{2\pi\ell_B^2}$$

$$+\frac{1}{24}(\tilde{c}-\nu) - h$$

signed conformal anomaly (chiral stress-energy anomaly)

chiral anomaly

virasoro level of sector

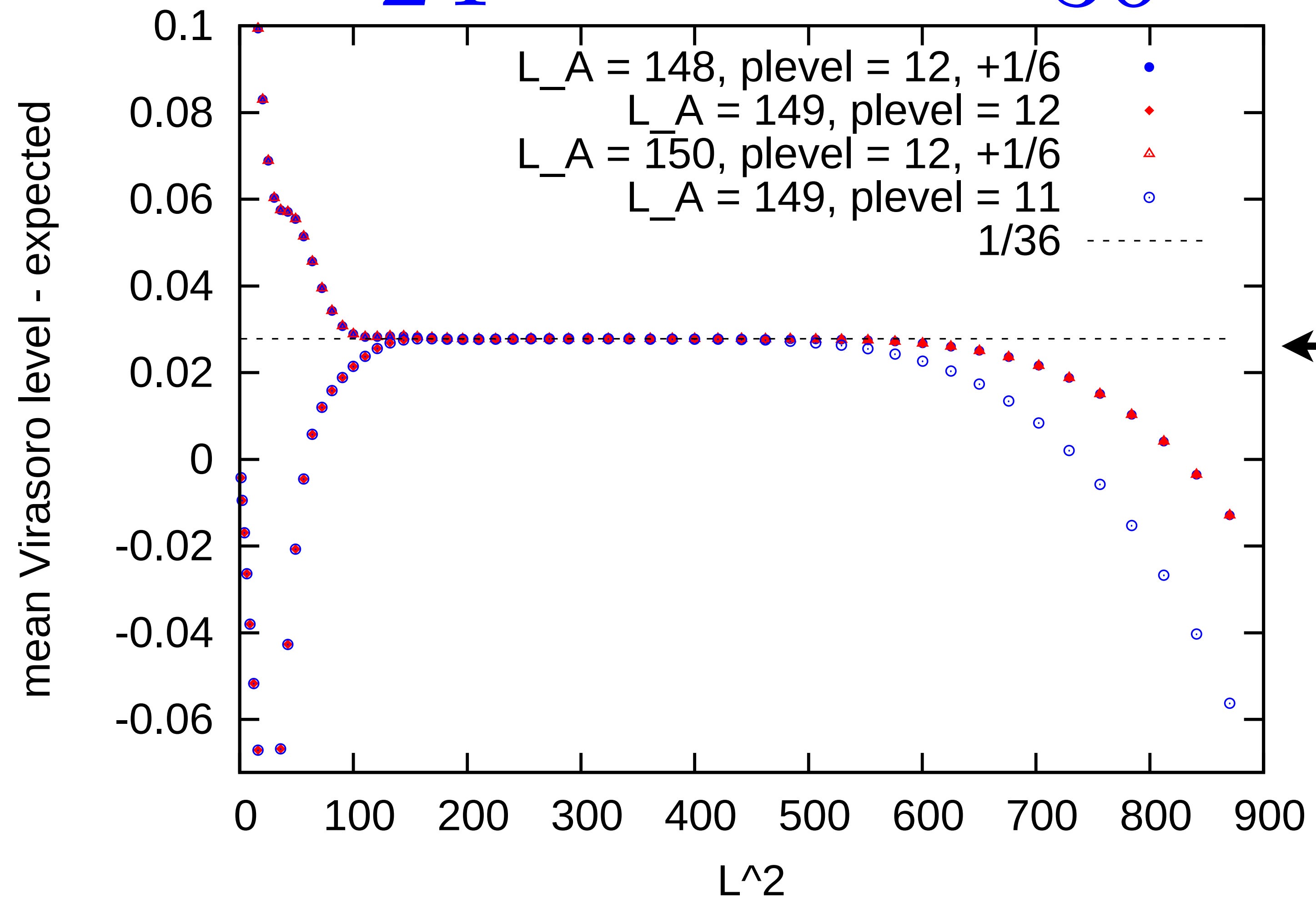
"CASIMIR MOMENTUM" term

(NOT "real-space cut" which requires the Landau orbit degrees of freedom and their form factor to be included

- Hall viscosity gives "thermally excited" momentum density on entanglement cut, relative to "vacuum", at von Neumann temperature T = 1

Yeje Park, Z Papic, N Regnault

$$\frac{1}{24}\left(\tilde{c}-\nu\right)=\frac{1}{24}\left(1-\tfrac{1}{3}\right)=\frac{1}{36}$$



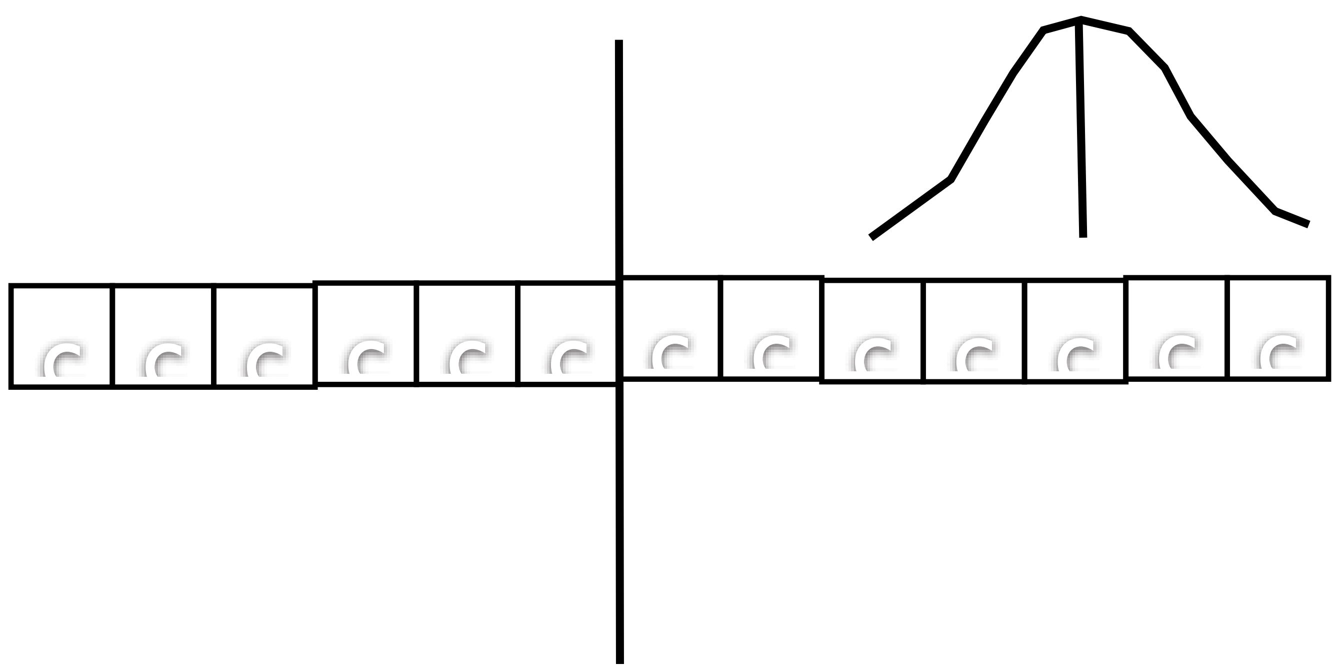
$\frac{1}{36}$

Matrix-product state calculation on cylinder with circumference L
("plevel" is Virasoro level at which the auxilliary space is truncated)

“fuzzy continuum” vs Lattice

- orbital vs “real space” cut

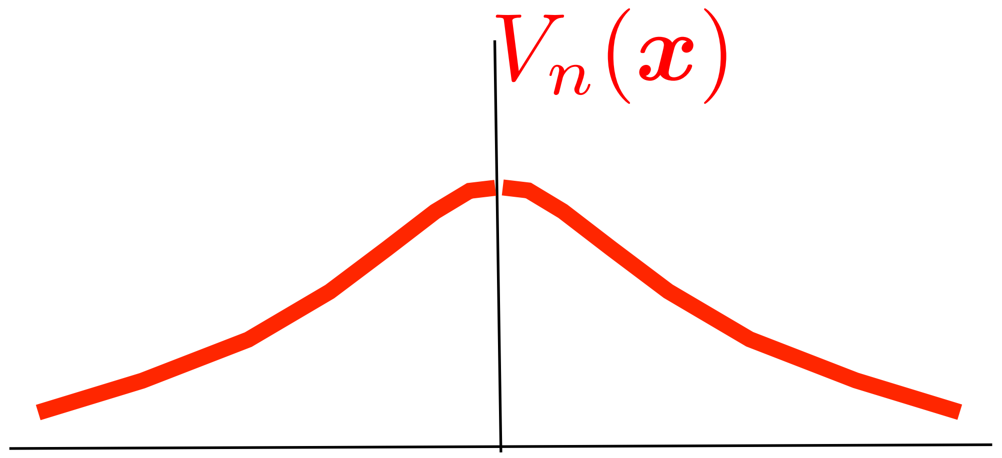
wavefunction of
Landau level orbital



- The fundamental problem is in the projected space, not its extension to “real space”

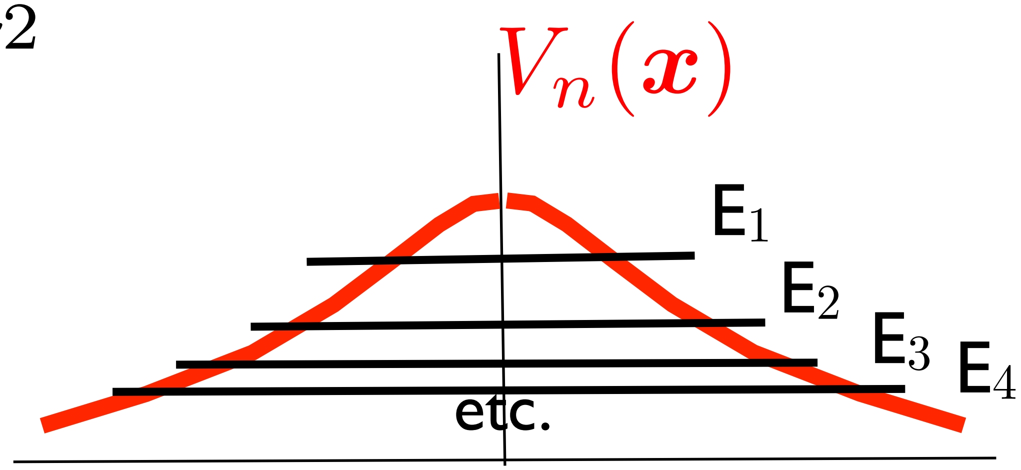
$$[R^a, R^b] = -i\ell_B^2 \epsilon^{ab}$$

$$H = \sum_{i<j} V_n(\boldsymbol{R}_i - \boldsymbol{R}_j)$$



- The quadratic expansion of this even function around the origin defines a natural "interaction metric"
- The problem is often simplified by giving it a continuous rotation symmetry that respects this metric, but this is non-generic, and not necessary.
- This metric and a rotation symmetry are important in model FQH wavefunctions based on cft, which have a stronger conformal invariance property.

- It is straightforward to solve the two-body Hamiltonian: $\boldsymbol{R}_{12} = \boldsymbol{R}_1 - \boldsymbol{R}_2$



$$[R^a_{12}, R^b_{12}] = 2i\ell_B^2\epsilon^{ab}$$

equivalent to a one-particle problem

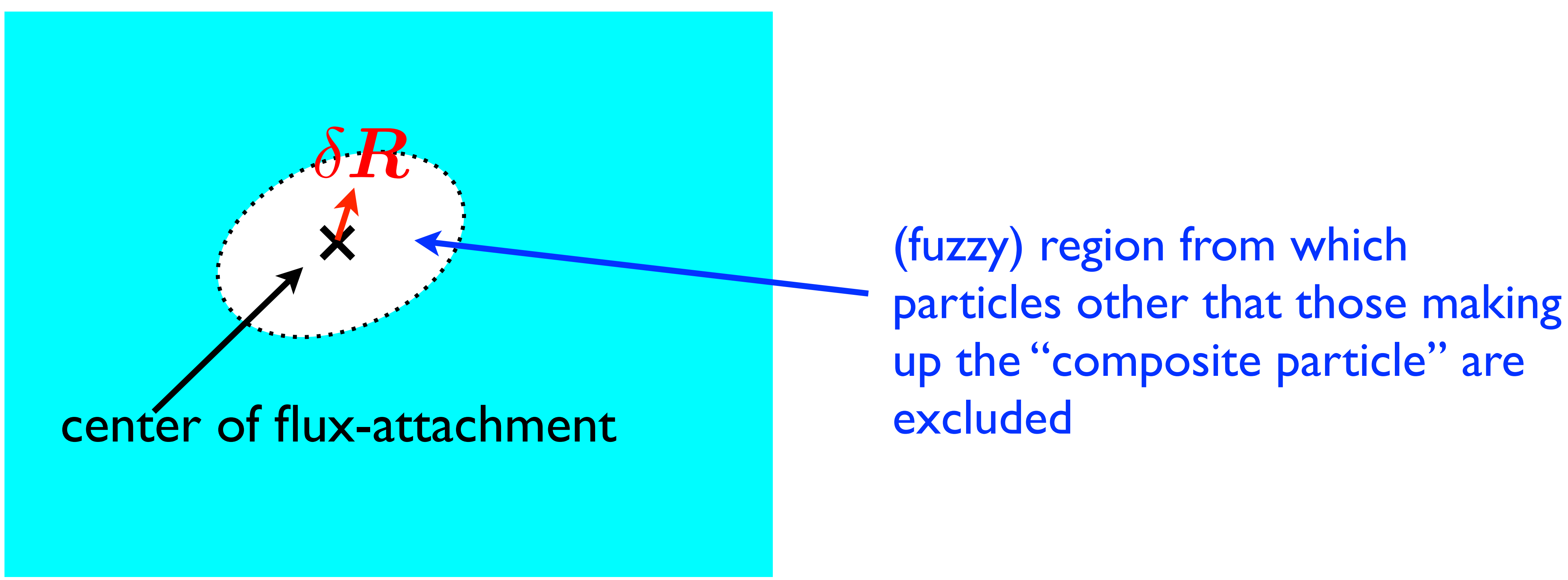
$$H = V_n(\boldsymbol{R}_{12})$$

- If there is a rotational symmetry, the energy levels (called "pseudopotentials") completely characterize the interaction potential.

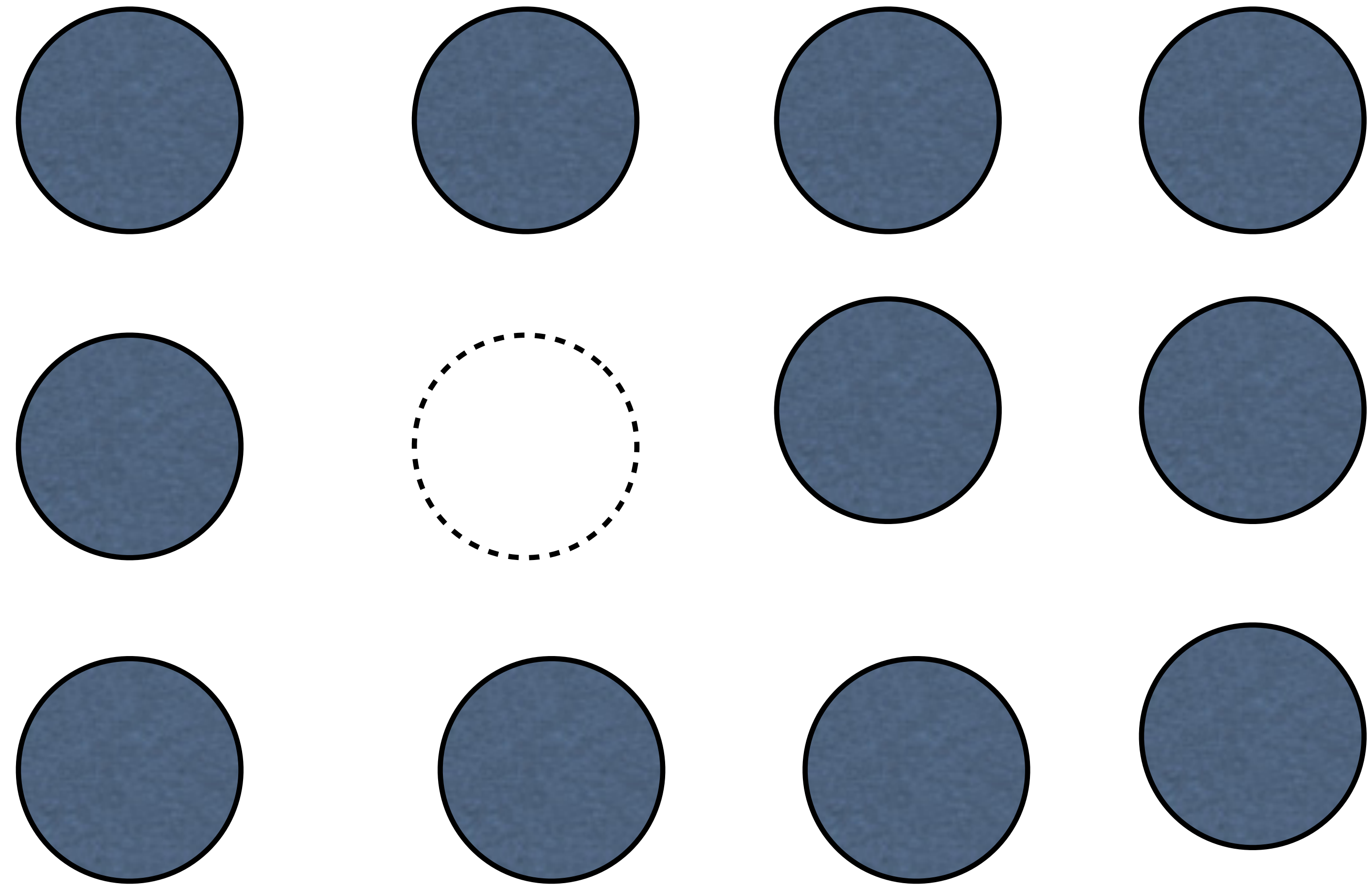
- a large gap between energy levels favors **flux attachment** with a shape close to that of the "interaction metric"

- Flux attachment is a gauge condensation that removes the gauge ambiguity of the guiding centers, giving each one a "natural" origin, so they define a physical electric dipole moment of the "composite particle" in which they are bound by the "attached flux".

- This is analogous to how the "the vector potential becomes an observable" (in a hand-waving way) in the London equations for a superconductor.



- quantum solid
- unit cell is correlation hole
- defines geometry



- repulsion of other particles make an attractive potential well strong enough to bind particle

solid melts if well is not strong enough to contain zero-point motion (Helium liquids)

- In Maxwell's equations, the momentum density is

$$\pi_i = \epsilon_{ijk} D^j B_k \qquad D^i = \epsilon_0 \delta^{ij} E_j + P^i$$

- The momentum of the condensed matter is

$$\boldsymbol{p} = \boldsymbol{d} \times \boldsymbol{B}$$

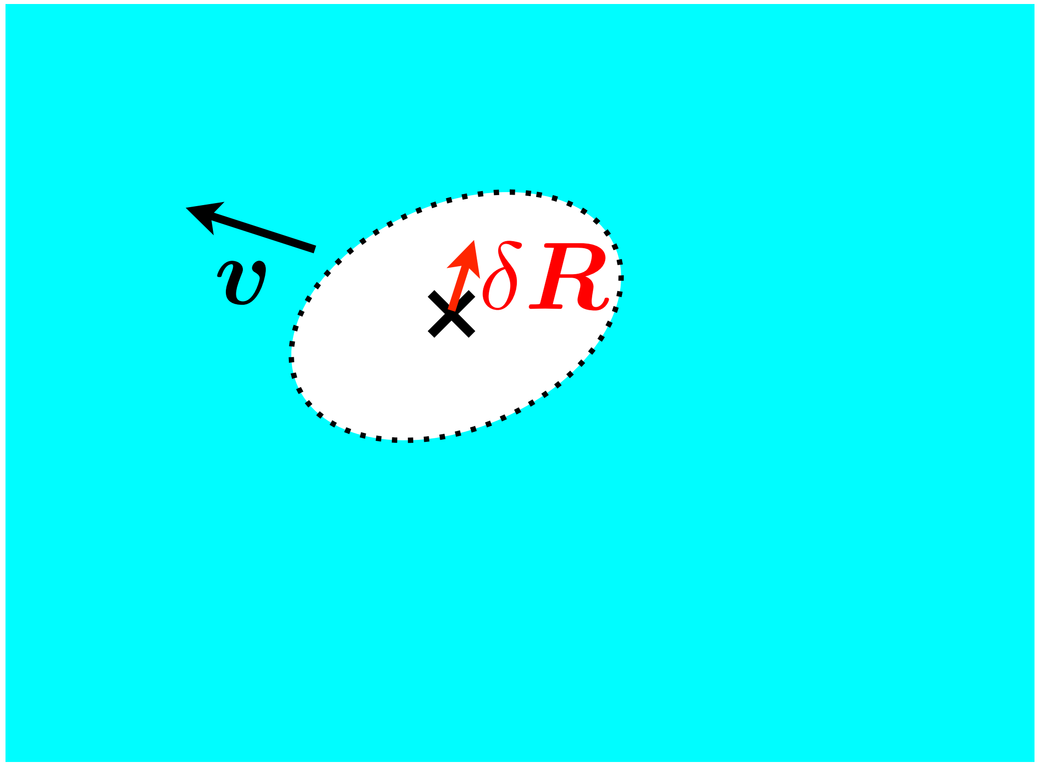
↖ electric dipole moment

- in 2D the guiding-center momentum then is

$$p_a = eB\epsilon_{ab}\delta R^b$$

- The electrical polarization energy of the dielectric composite particle then gives its energy-momentum dispersion relation, with no involvement of any "Newtonian inertia" involving an effective mass

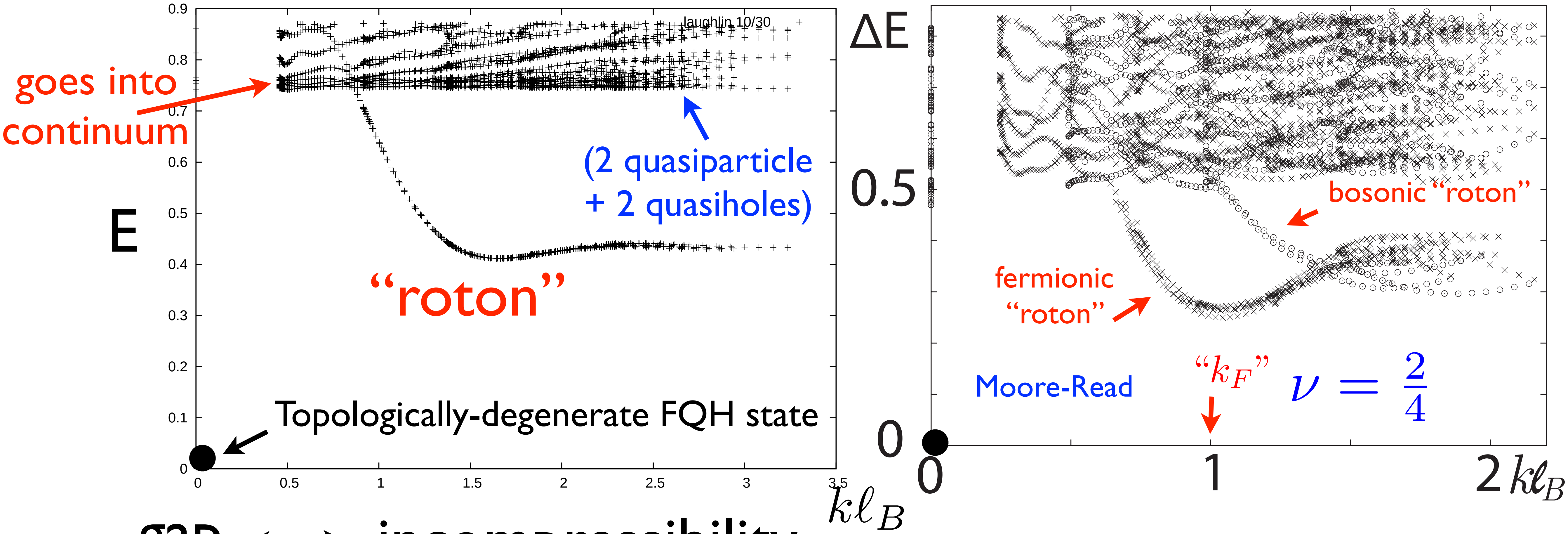
- The Berry phase generated by motion of the "other particles" that "get out of the way" as the vortex-like "flux-attachment" moves with the particle(s) it encloses can be formally-described as a Chern-Simons gauge field that cancels the Bohm-Aharonov phase, so that the composite object propagates like a neutral particle.



- If the composite particle is a boson, it condenses into the zero-momentum **(zero electric dipole-moment)** inversion-symmetric state, giving an incompressible-fluid Fractional Quantum Hall state, with an energy gap for excitations that carry momentum or electric dipole moment ("quantum incompressibility", no transmission of pressure through the bulk).

- All FQH states have an elementary unit (analogous to the unit cell of a crystal) that is a composite boson under exchange.

- It may be sometimes be useful to describe this boson as a a bound state of composite fermions (with their own preexisting flux attachment) bound by extra flux (Jain's picture)



gap ⟷ incompressibility

Collective mode with short-range V_1 pseudopotential, 1/3 filling (Laughlin state is exact ground state in that case)

Collective mode with short-range three-body pseudopotential, 1/2 filling (Moore-Read state is exact ground state in that case)

- momentum $\hbar k$ of a quasiparticle-quasihole pair is proportional to its **electric dipole moment p_e**

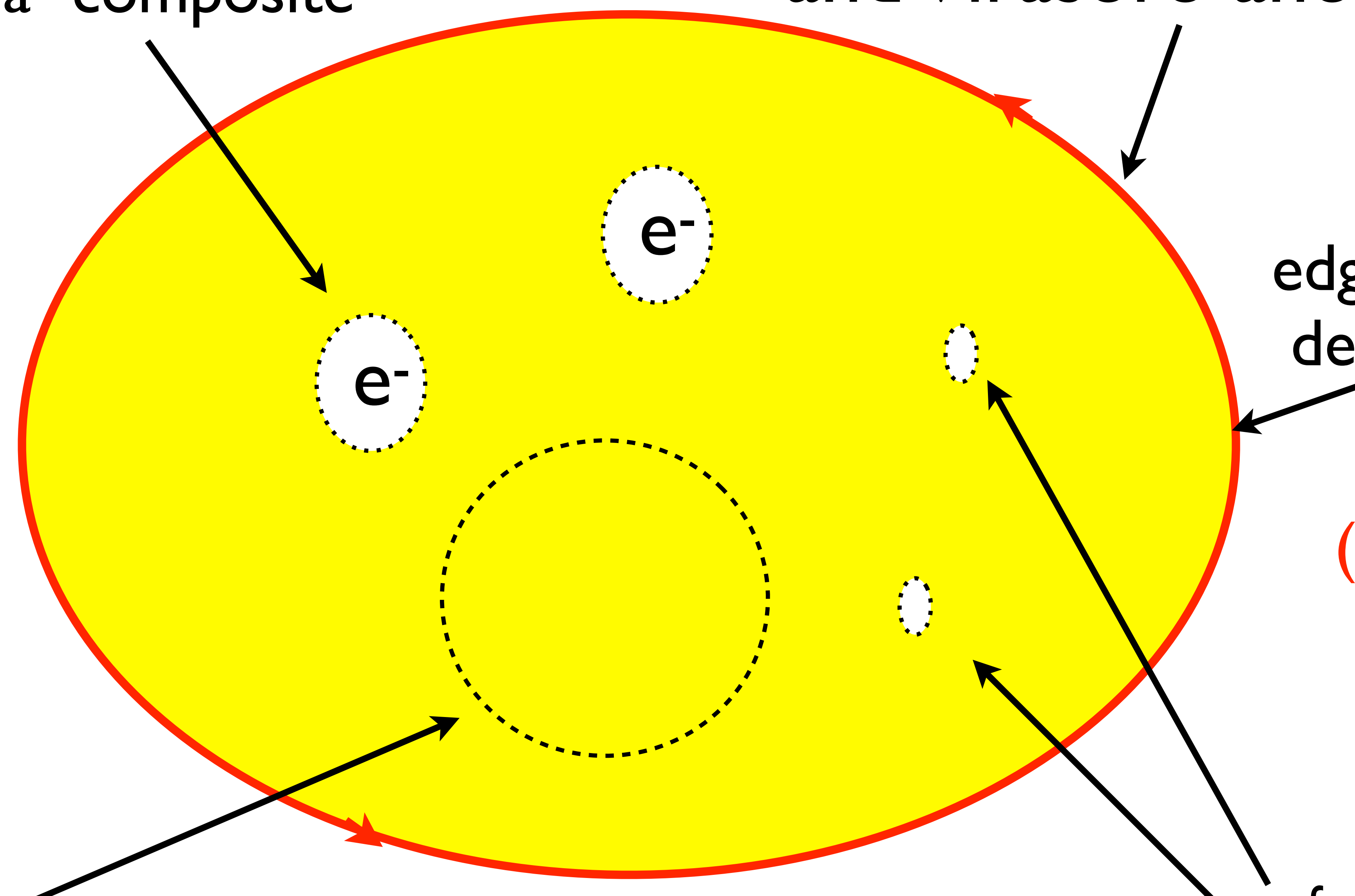
$$\hbar k_a = \epsilon_{ab} B p_e^b$$

gap for electric dipole excitations is a MUCH stronger condition than charge gap: fluid **does not transmit pressure through bulk!**

• Anatomy of Laughlin state

electron with "flux attachment" to form a "composite boson"

Chiral edge mode with chiral anomaly and Virasoro anomaly



geometric edge dipole moment determined by Hall viscosity

(Wen-Zee term)

Topological and geometric bulk properties revealed by entanglement spectrum of cut

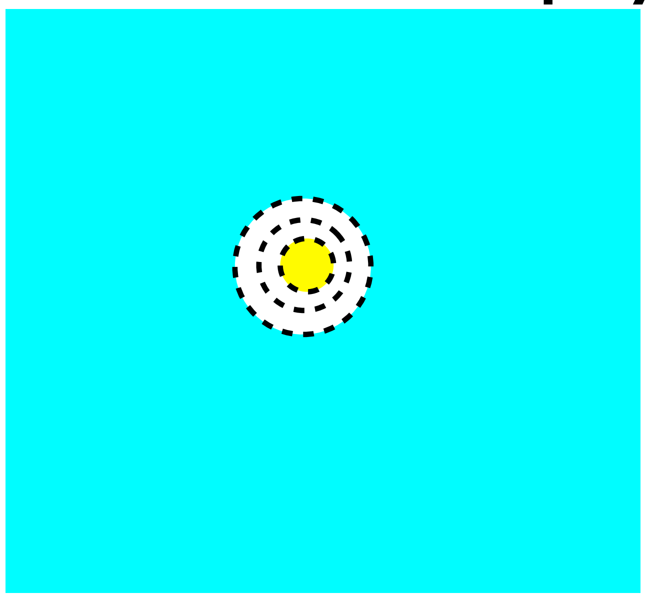
fractionally-charged e/3 quasiholes obeying (Abelian) fractional statistics

- the essential unit of the 1/3 Laughlin state is the electron bound to a correlation hole corresponding to "units of flux", or three of the available single-particle states which are exclusively occupied by the particle to which they are "attached"

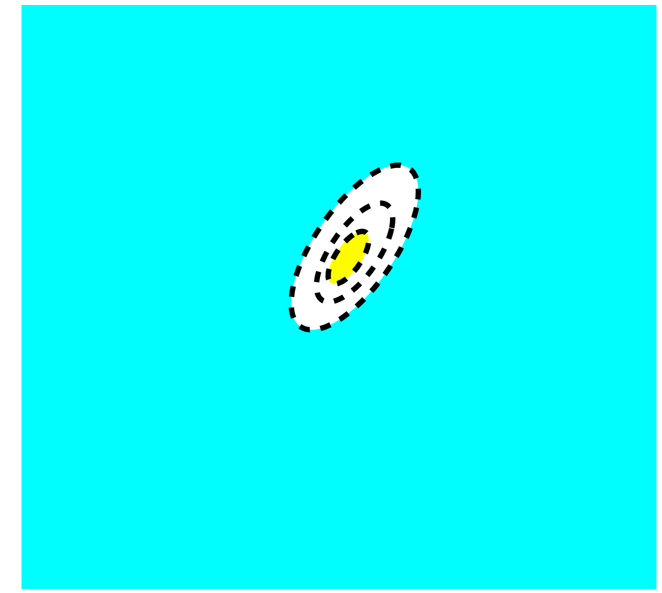
- In general, the elementary unit of the FQHE fluid is a "composite boson" of p particles with q "attached flux quanta"

- This is the analog of a unit cell in a solid....

- The Laughlin state is parametrized by a unimodular metric: what is its physical meaning?



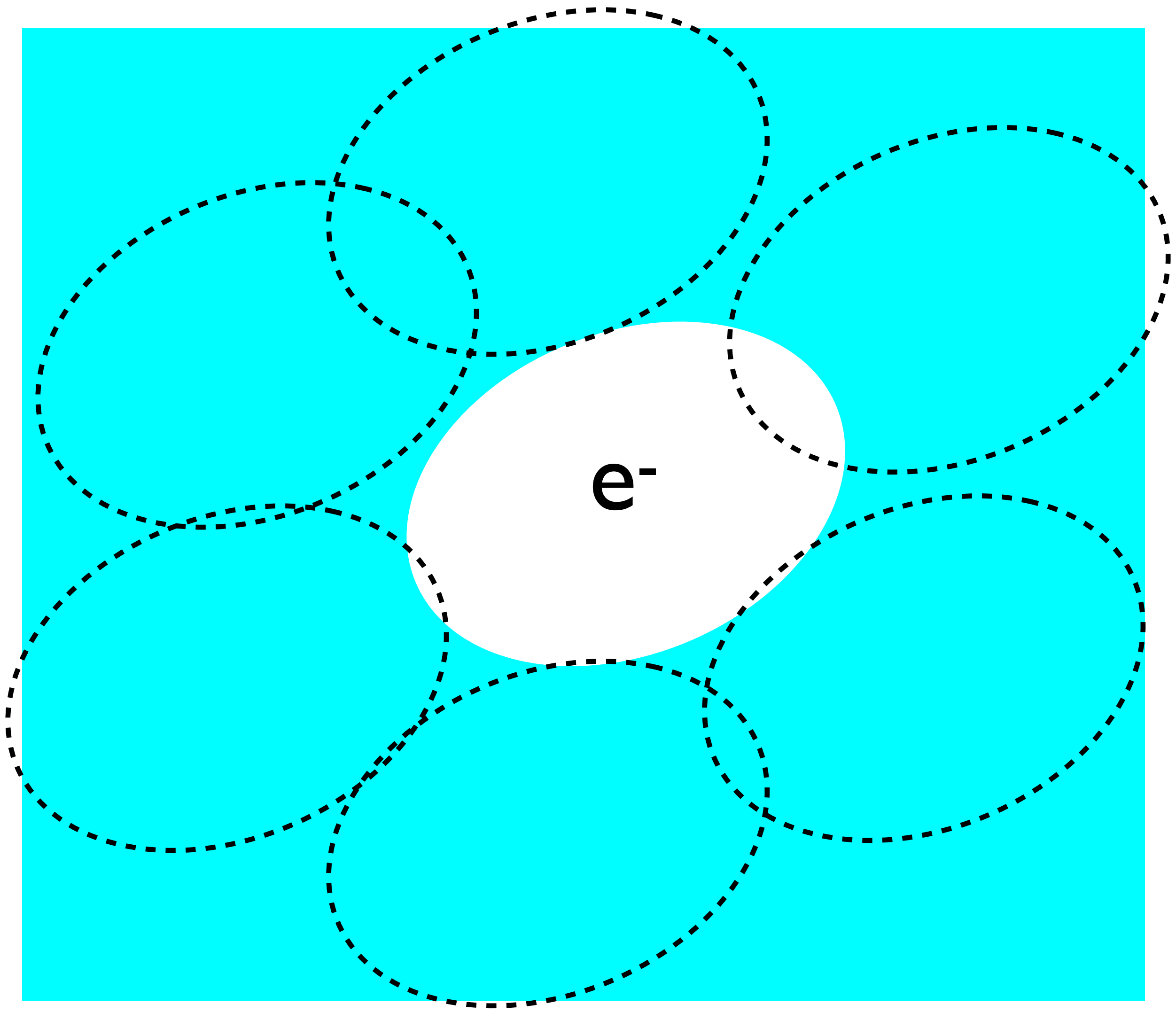
correlation holes
in two states with
different metrics



- In the $\nu = 1/3$ Laughlin state, each electron sits in a correlation hole with an area containing 3 flux quanta. The metric controls the *shape* of the correlation hole.

- In the $\nu = 1$ filled LL Slater-determinant state, there is <u>no</u> correlation hole (just an exchange hole), and this state does <u>**not**</u> depend on a metric but no broken symmetry

- similar story in FQHE:



- "flux attachment" creates correlation hole
- defines an emergent geometry
- potential well must be strong enough to bind electron
- continuum model, but similar physics to Hubbard model
- new physics: Hall viscosity, geometry...........

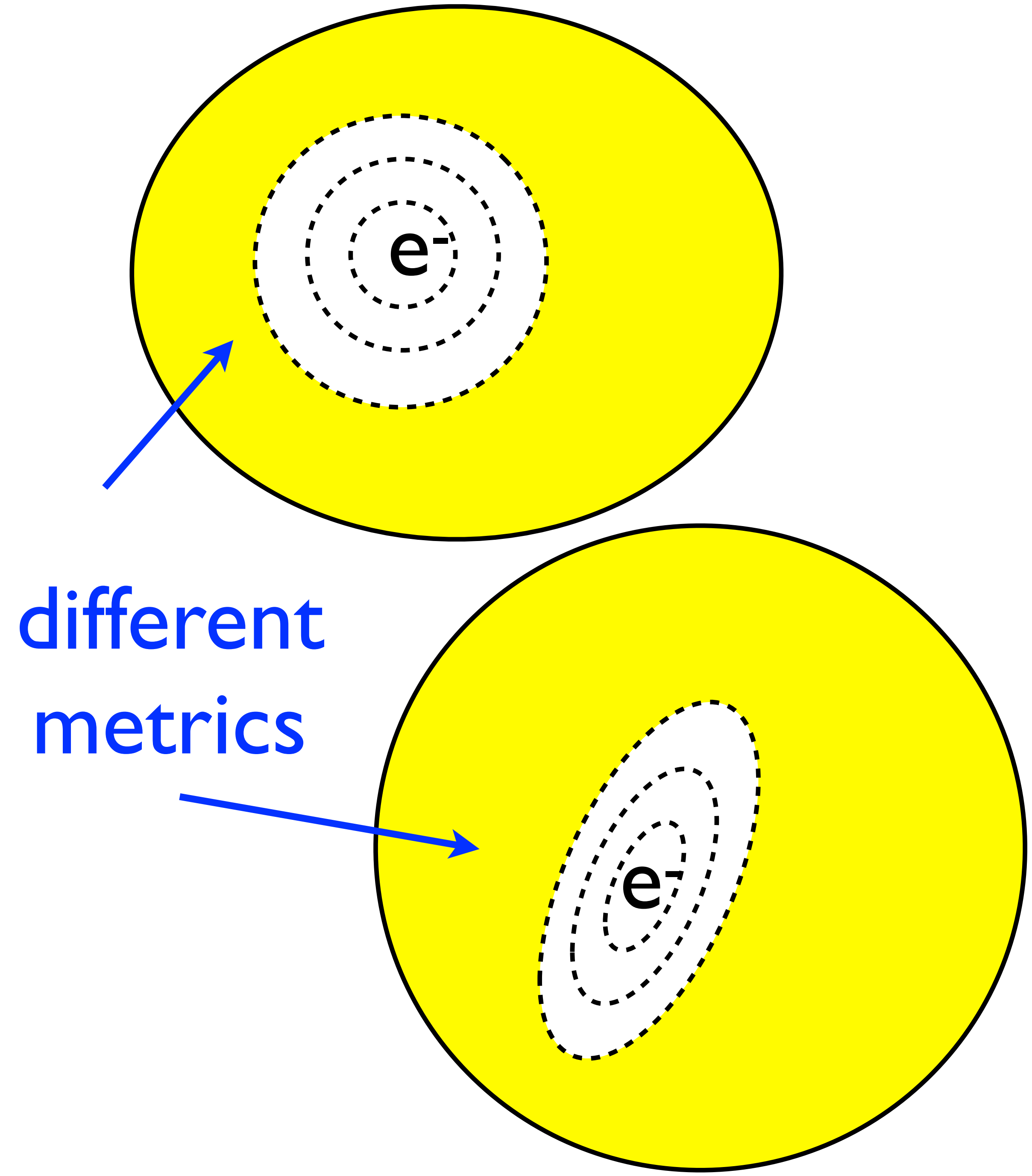
- composite boson: if the central orbital of a basis of eigenstates of L(g) is filled, the next two are empty

$$|\Psi_L^3\rangle = \prod_{i<j} \left(a_i^\dagger - a_j^\dagger\right)^3 |0\rangle$$

$$L(g)|\psi_m\rangle = (m + \tfrac{1}{2})|\psi_m\rangle$$

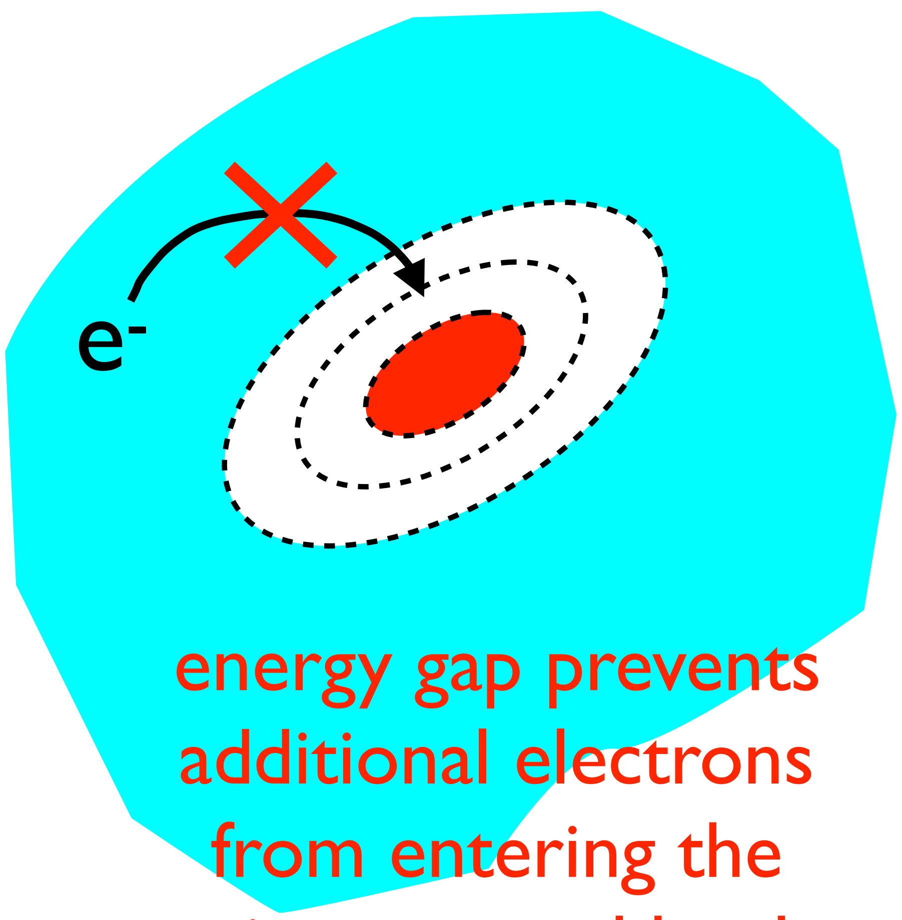
- this correlation hole is equivalent to "attachment of three flux quanta" or vortices that travel with the particle, generating a Berry phase that cancels the Bohm-Aharonov phase and transmutes Fermi to Bose exchange statistics.

- this shape of the corelation hole - and hence its correlation energy - varies with the metric g_{ab}



- Origin of FQHE incompressibility is analogous to origin of **Mott-Hubbard gap** in lattice systems.
- There is an energy gap for putting an **extra particle** in a quantized region that is **already occupied**

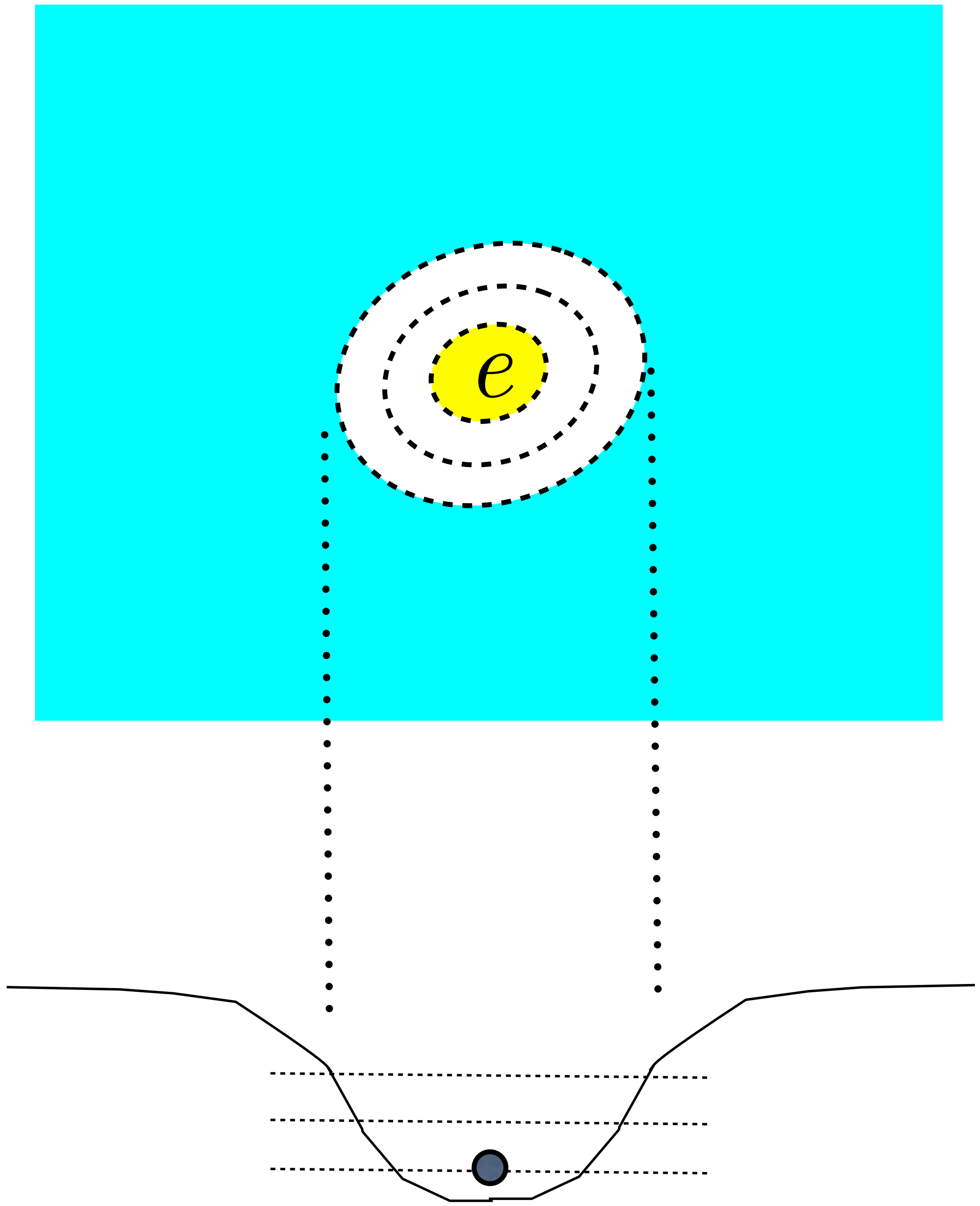
- **On the lattice** the "quantized region" is an atomic orbital with a fixed shape
- **In the FQHE** only the **<u>area</u>** of the "quantized region" is fixed. The **<u>shape</u>** must adjust to minimize the correlation energy.



energy gap prevents additional electrons from entering the region covered by the composite boson

$1/3$ Laughlin state

If the central orbital is filled, the next two are empty



The composite boson has inversion symmetry about its center

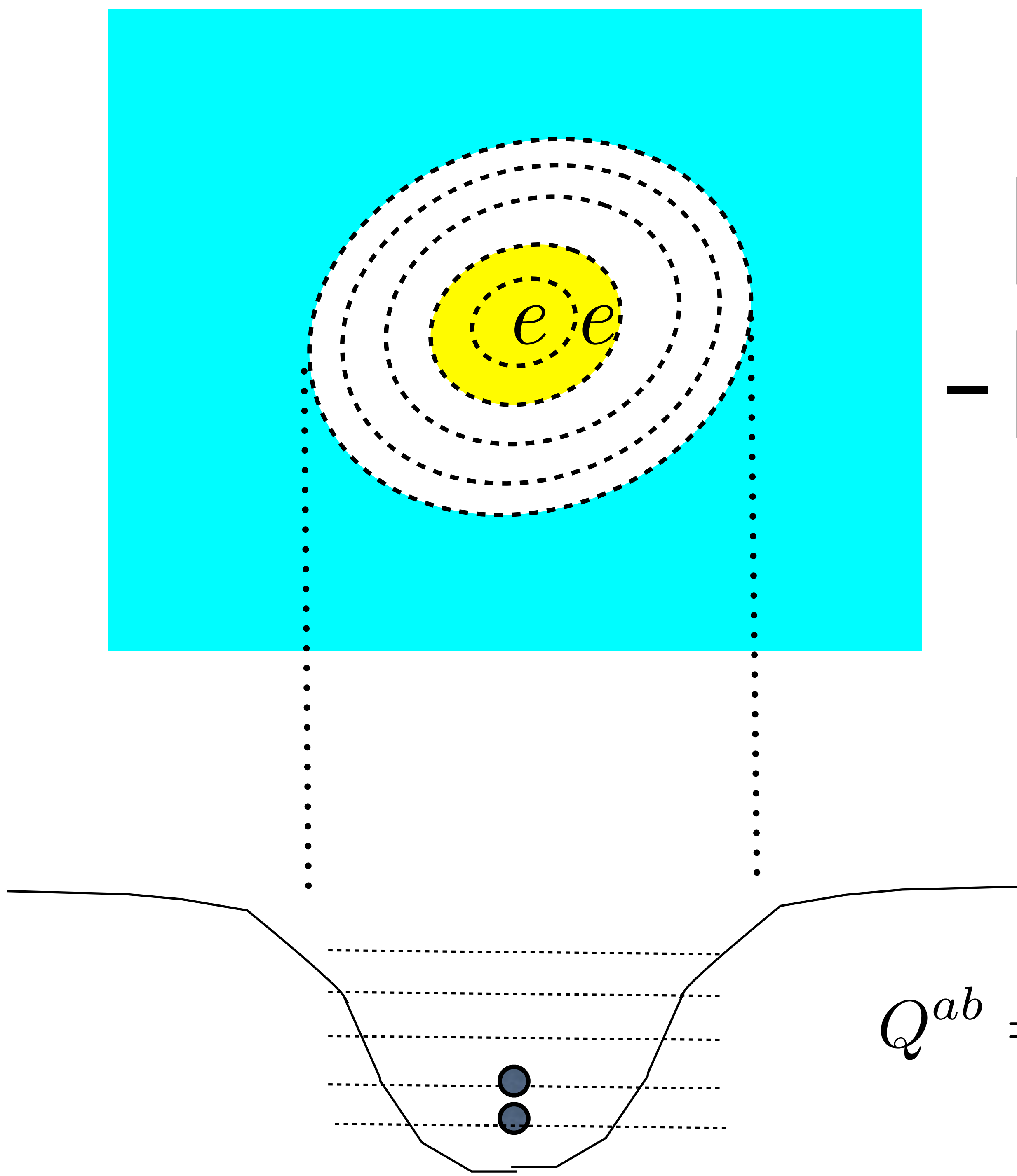
It has a "spin"

	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$		
	1	0	0	.....	$L = \frac{1}{2}$
−	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	.....	$- L = \frac{3}{2}$

$$s = -1$$

the electron excludes other particles from a region containing 3 flux quanta, creating a potential well in which it is bound

$2/5$ state



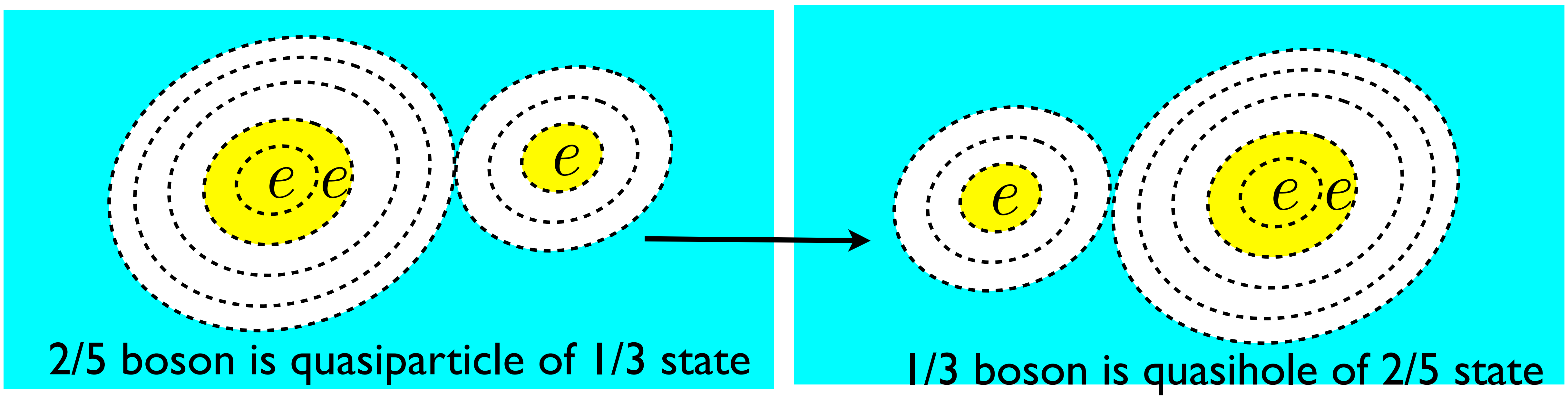
$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$				
1	1	0	0	0	.....	$L = 2$
$-$ $\frac{2}{5}$	$\frac{2}{5}$	$\frac{2}{5}$	$\frac{2}{5}$	$\frac{2}{5}$	.....	$- L = 5$
						$s = -3$

$$L = \frac{g_{ab}}{2\ell_B^2} \sum_i R_i^a R_i^b$$

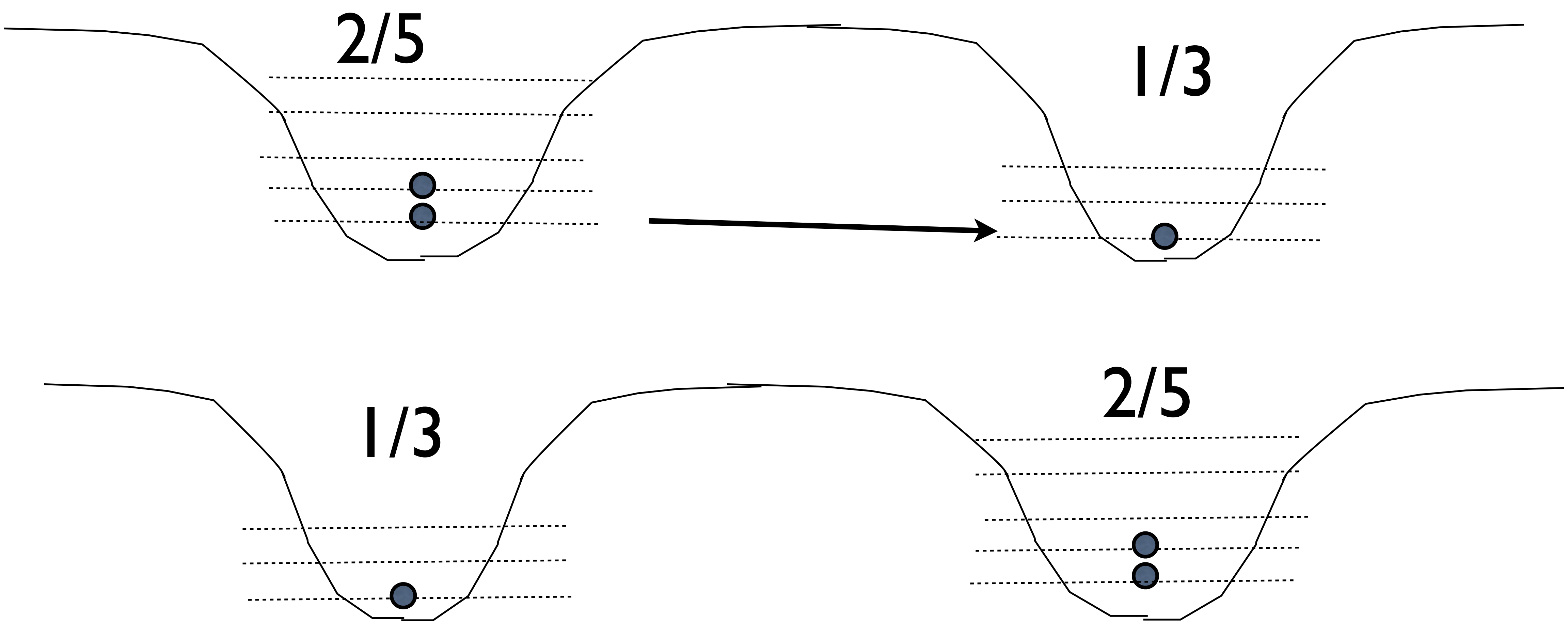
$$Q^{ab} = \int d^2r \, r^a r^b \delta\rho(r) = s\ell_B^2 g^{ab}$$

second moment of neutral
composite boson
charge distribution

hopping of a “composite fermion” (electron + 2 flux quanta)



Jain’s “pseudo Landau levels”

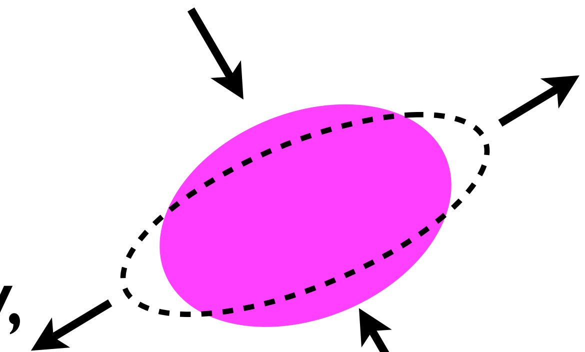


- The composite boson behaves as a neutral particle because the Berry phase (from the disturbance of the the other particles as its "exclusion zone" moves with it) cancels the Bohm-Aharonov phase

- It behaves as a boson provided its statistical spin cancels the particle exchange factor when two composite bosons are exchanged

p particles	$(-1)^{pq} = (-1)^p$	fermions
q orbitals	$(-1)^{pq} = 1$	bosons

- The metric (shape of the composite boson) has a preferred shape that minimizes the correlation energy, but fluctuates around that shape



$\delta E \propto (\text{distortion})^2$

- The zero-point fluctuations of the metric are seen as the O(q^4) behavior of the "guiding-center structure factor" (Girvin et al, (GMP), 1985)

- long-wavelength limit of GMP collective mode is fluctuations of (spatial) metric (analog of "graviton")

FDMH, Phys. Rev. Lett. **107**, 116801 (2011)

- Furthermore, the local electric charge density of the fluid with $\nu = p/q$ is determined by a combination of the magnetic flux density and the Gaussian curvature of the intrinsic metric

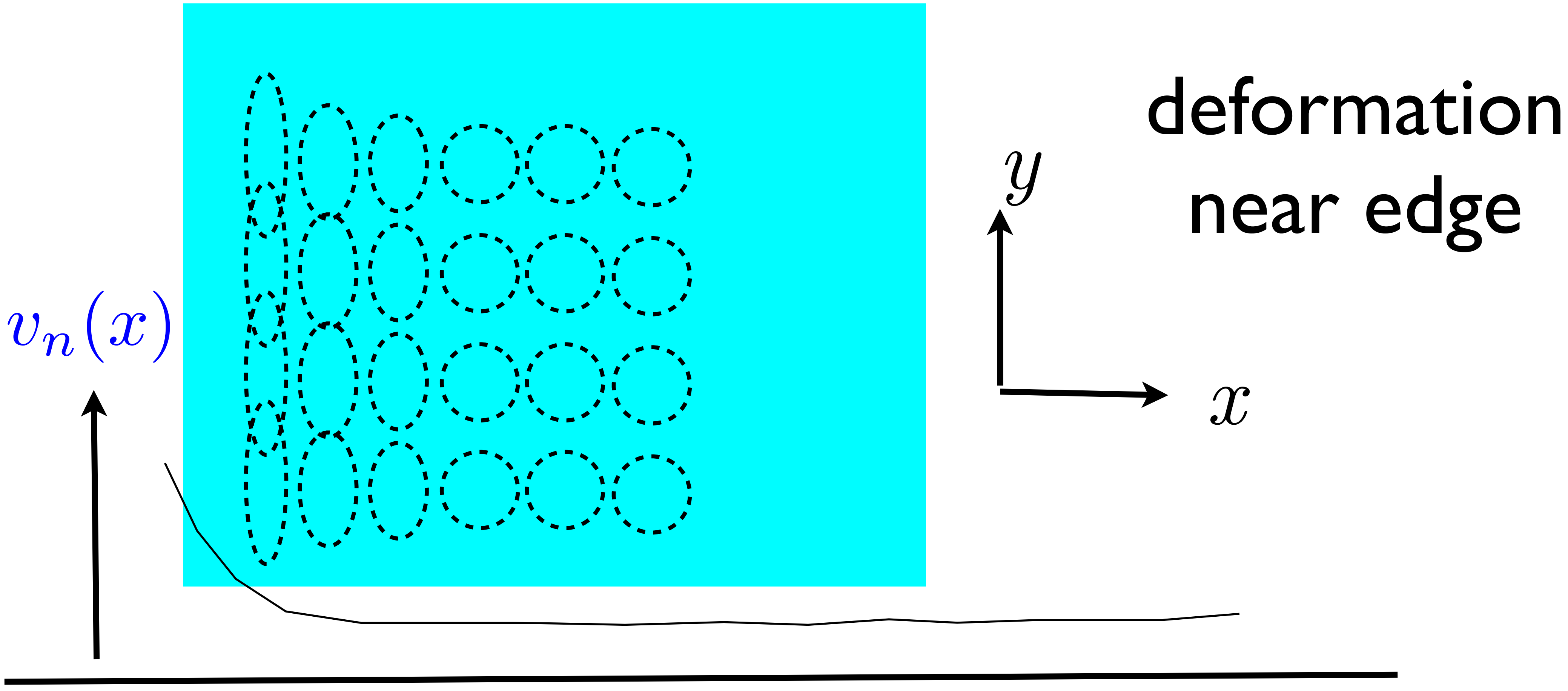
$$J_e^0(\boldsymbol{x}) = \frac{e}{2\pi q}\left(\frac{peB}{\hbar} - sK_g(\boldsymbol{x})\right)$$

Topologically quantized "guiding center spin" → s

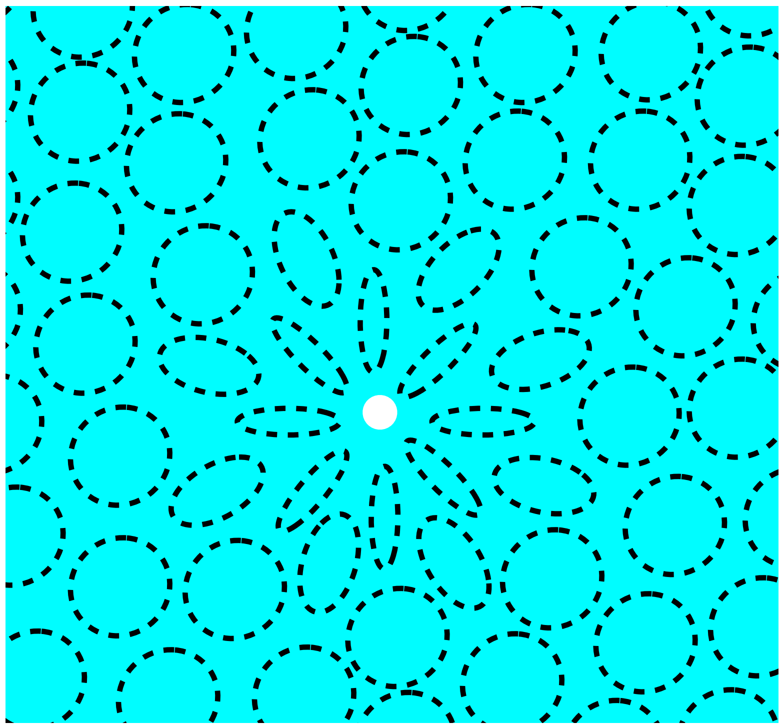
Gaussian curvature of the metric → $K_g(\boldsymbol{x})$

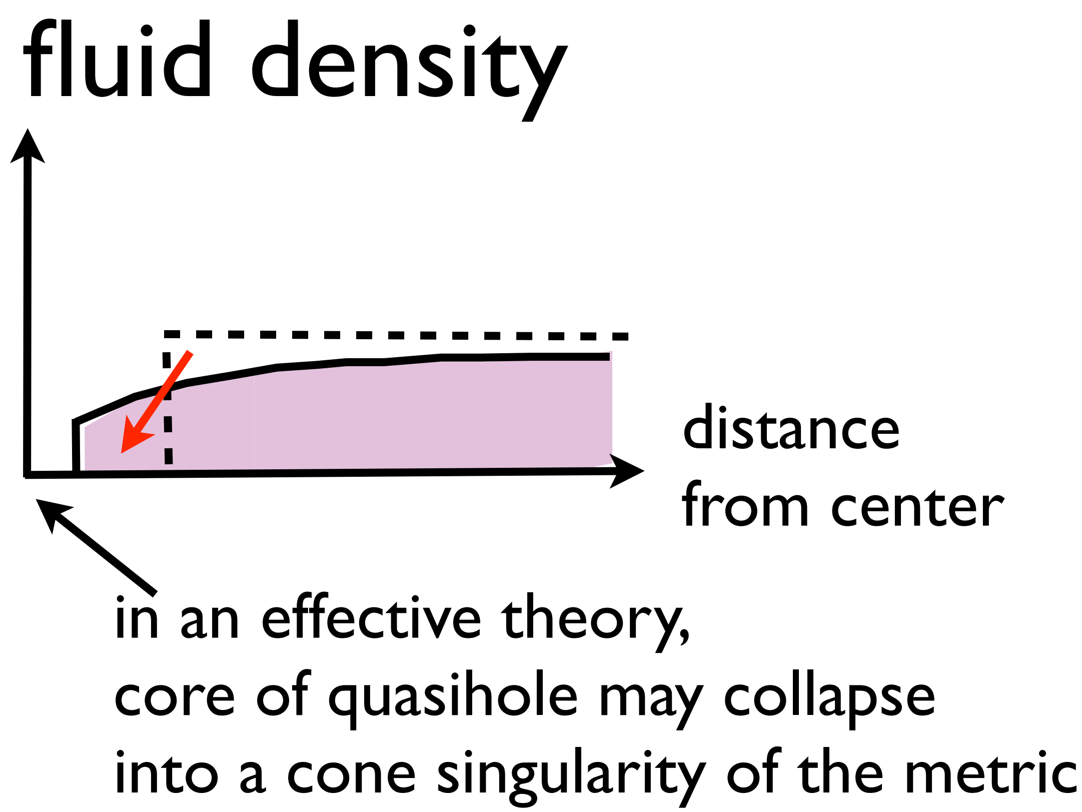
- In fact, it is locally determined, if there is an inhomogeneous slowly-varying substrate potential

$$H = \sum_i v_n(\boldsymbol{R}_i) + \sum_{i<j} V_n(\boldsymbol{R}_i - \boldsymbol{R}_j)$$



- “skyrmion”-like “cone”-like structure moves charge away from quasihole by introducing negative Gaussian curvature





- In the standard incompressible FQH states, the bulk interior of the fluid is described by a gapped topological field theory (TQFT).
- The gapless edge degrees of freedom are a direct sum of unitary representations of the Virasoro algebra.
- Can there be continuous second order transitions between FQH states at which the bulk gap collapses?

- The (fermion) “Gaffnian” model (Steve Simon et al)
- This is a model 2/5 state that (a) is an exact zero-energy state of a (three-body) interaction (b) has a non-unitary representation of the Virasoro algebra on its edge and (c) as a consequence is believed to have bulk gapless neutral excitations (Read).

- It is a Jack polynomial with a “root configuration exclusion statistics rule” of “not more than two particles in five consecutive orbitals”

- The “Gaffnian” interaction penalizes three-body states

$$(z_1 - z_2)(z_2 - z_3)(z_3 - z_1)$$ 11100

$$(z_1 - z_2)(z_2 - z_3)(z_3 - z_1) \times \left((z_1 - z_2)^2 + (z_2 - z_3)^2 + (z_3 - z_1)^2)\right)$$ 11001

$$H = V_0 P_{111} + V_2 P_{11001}$$

- On the torus, the 2/5 Gaffnian zero-energy states has a 10-fold degeneracy corresponding to the two sets of 5 "motifs"

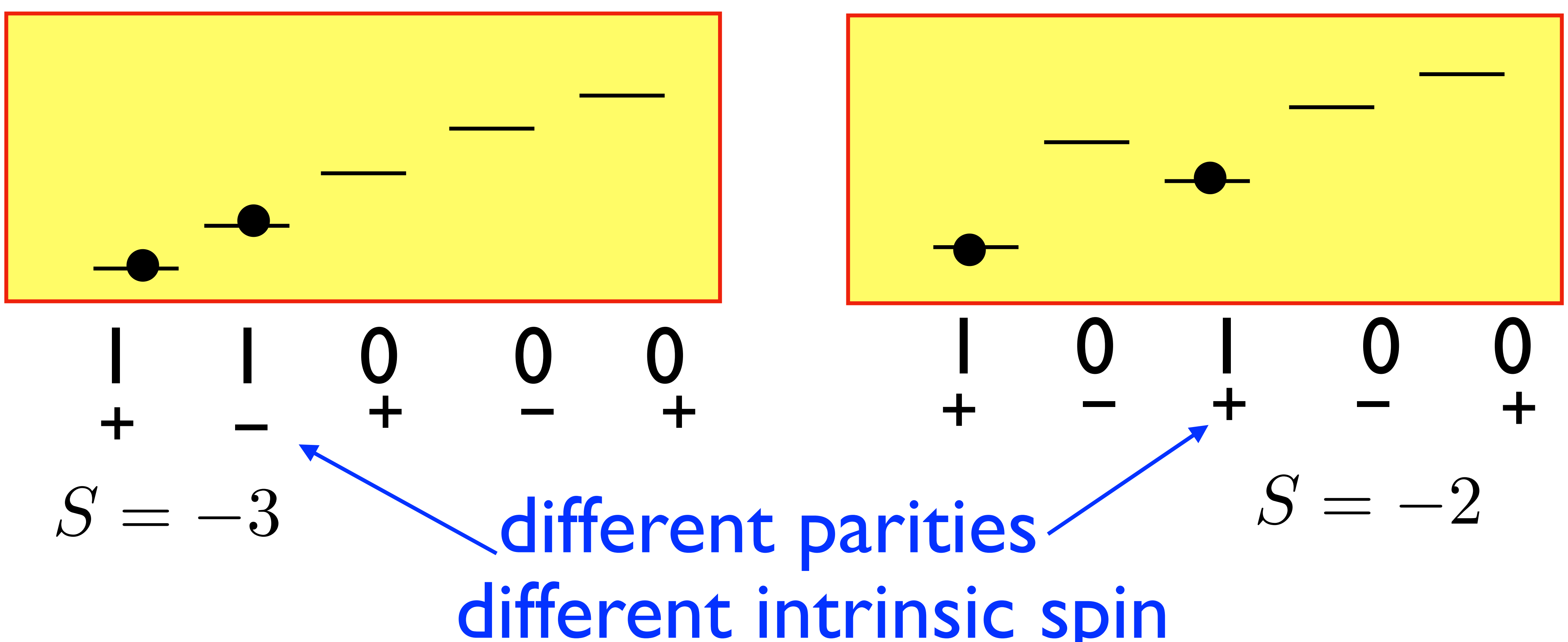
11000 01100 00110 00011 10001

10100 01010 00101 10010 01001

↑ lowest weight (most to left)

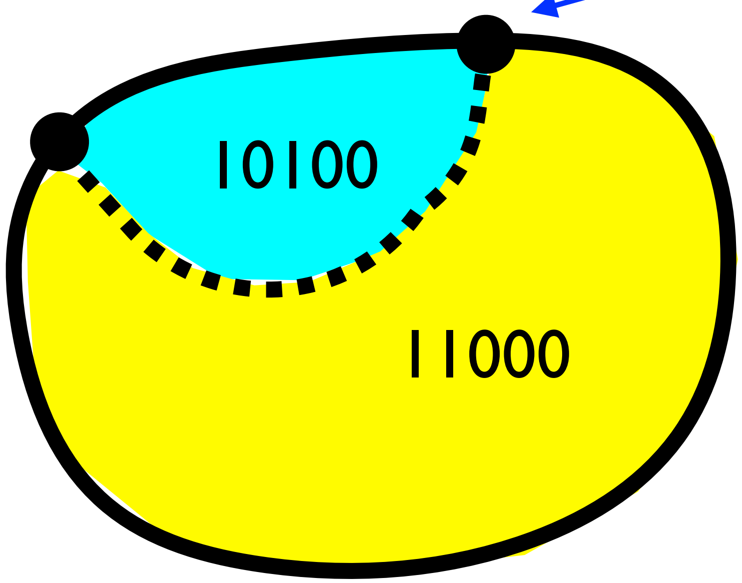
- A degeneracy beween two internal states of the 2/5 "composite boson" with different parity.

- In higher Landau levels the "10100" pattern may replace 11000 as the stable 2/5 pattern because of competition between the "vacancy potential" that favors putting the second particle in the second orbital, and repulsion from the first particle, which pushes it outwards



- Domain wall between states with different Wen-Zee term carries momentum density (electric dipole moment) but no chiral modes (no U(1) or Virasoro anomaly)

negative weight primary field of non-unitary CFT ???



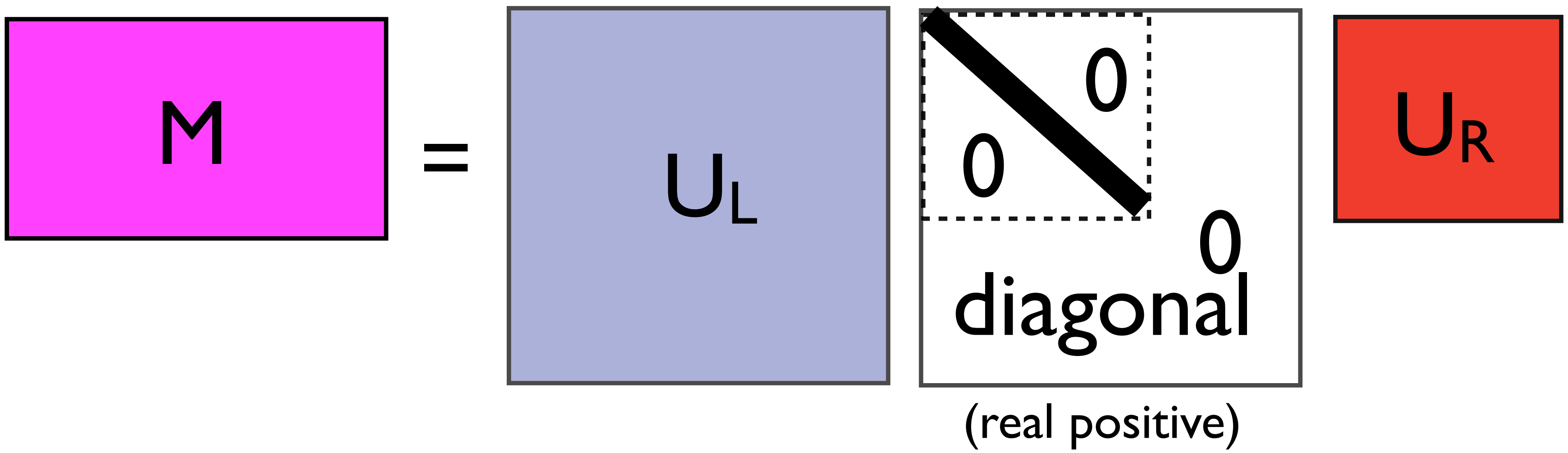
- gapless bulk if domain wall energy is zero or negative

transmits pressure!

- sliding of domain wall attachment point removes momentum from edge (non-unitary virasoro on edge)

- Many open questions about the gapless critical state (e.g. what is the dynamical critical exponent z (1or 2?))
- does charge gap exist for all ratios of the two parameters?
- develop a Full interpretation of the non-unitary Virasoro representation.

- Any matrix has a "singular value decomposition"



$$|\Psi\rangle = \sum_\nu e^{-\frac{1}{2}\xi_\nu} |\Psi^L_\nu\rangle \otimes |\Psi^R_\nu\rangle$$

"entanglement spectrum" eigenvalues

orthonormal basis of "Left" degrees of freedom

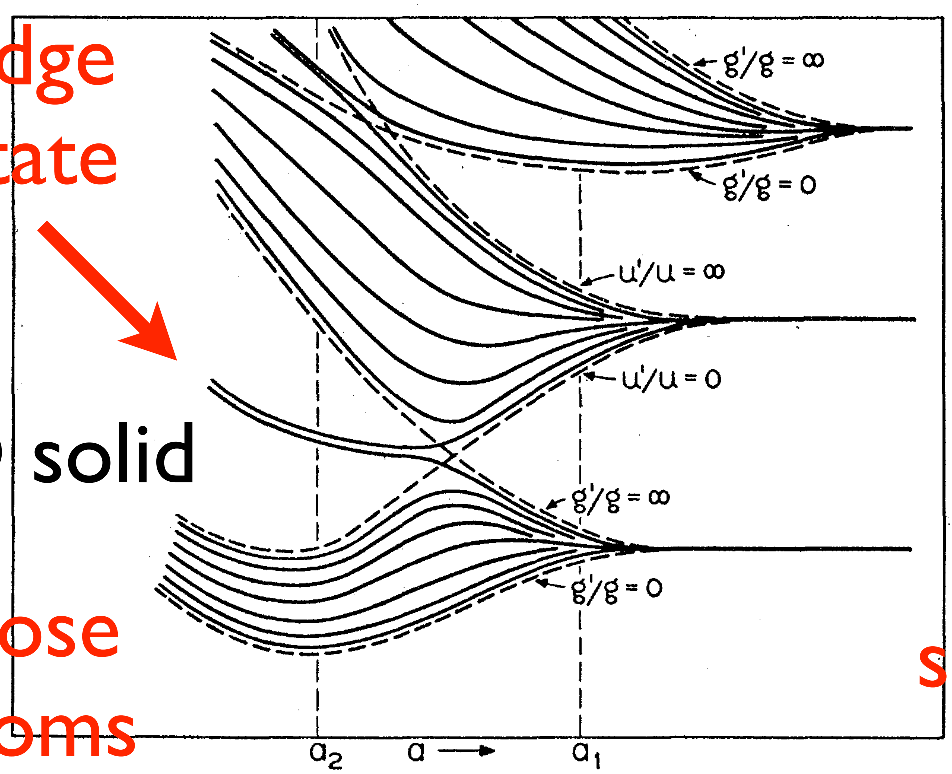
orthonormal basis of "Right" degrees of freedom

- One of the striking characteristic properties of band topological insulators (or “Symmetry-Protected Topological States”)is their edge states

Shockley 1939

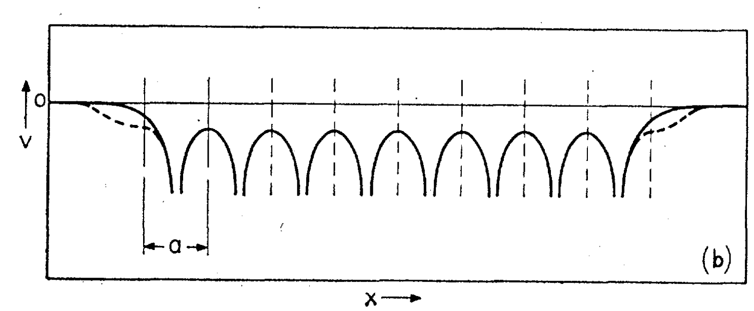
edge state

1D solid



close atoms

widely separated atoms



Fermi level pinned to edge state if neutral charge +1/2 electron if full , -1/2 if empty, per edge

protective symmetry: spatial inversion

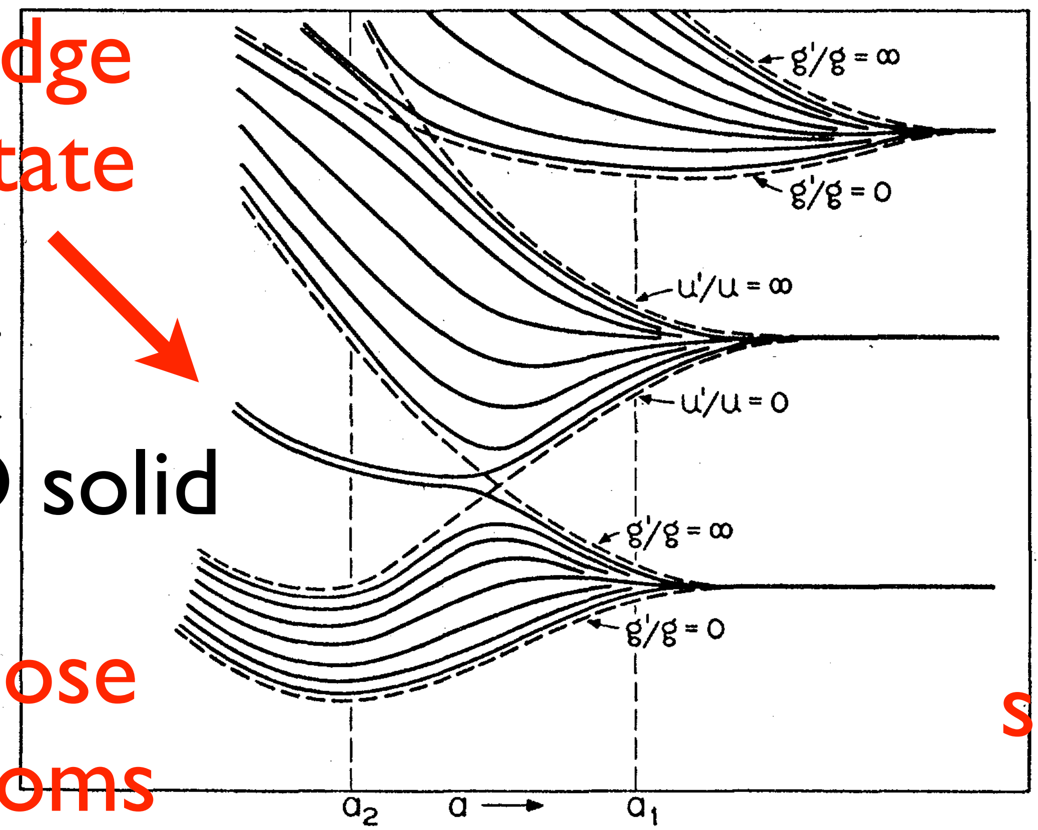
Z_2 invariant: $I_{k=0} \times I_{k=\pi} = \pm 1$

s-p band-inversion at k = π

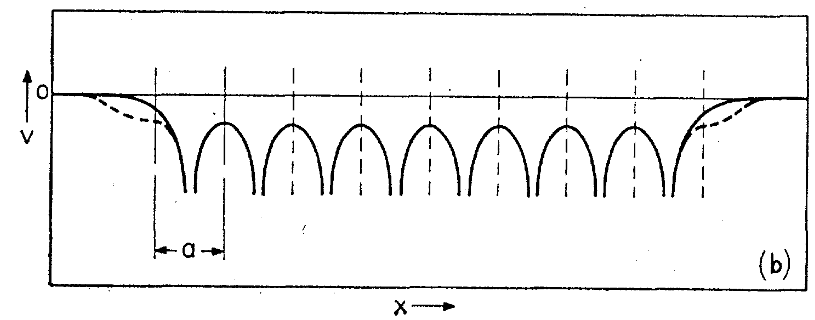
edge state

1D solid

close atoms

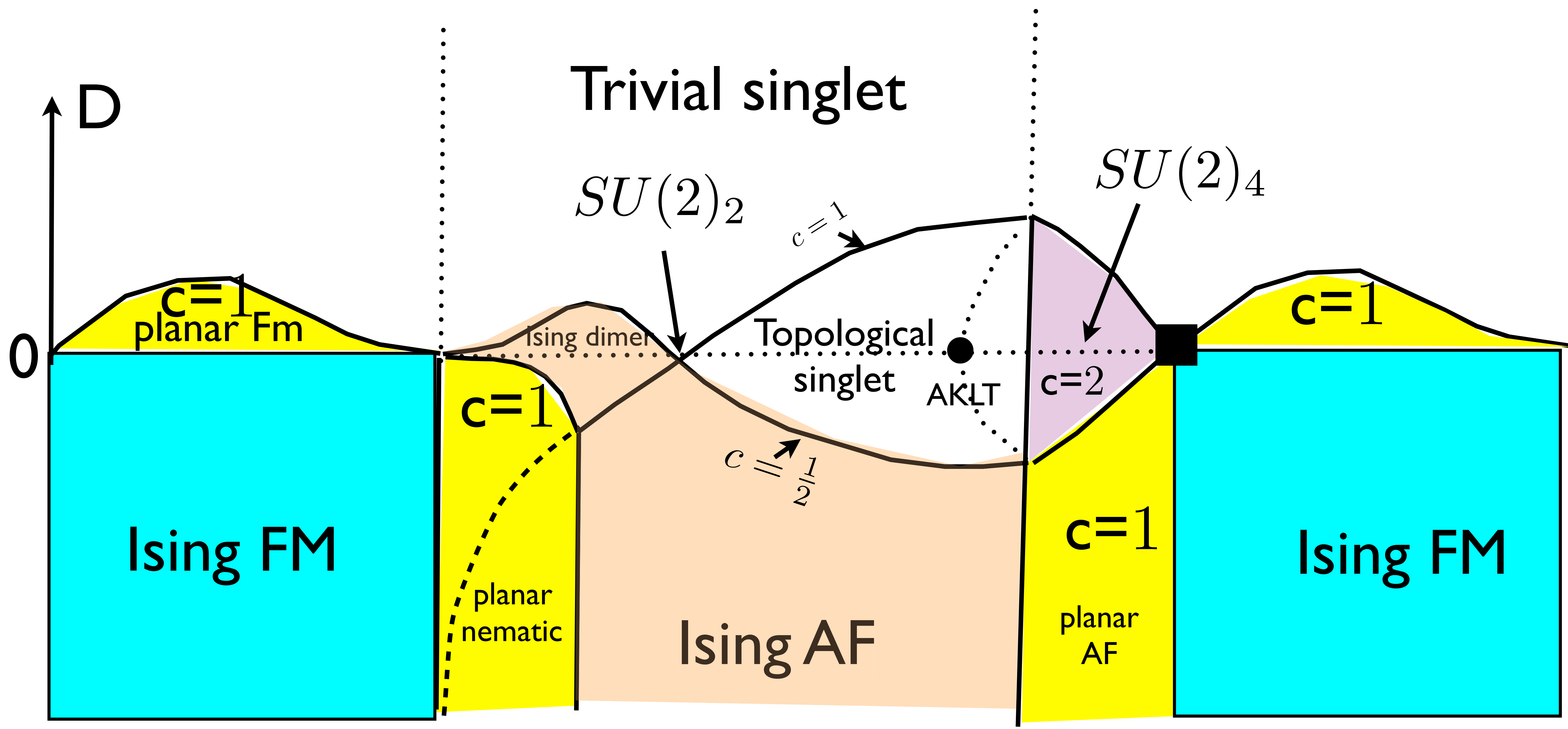


widely separated atoms



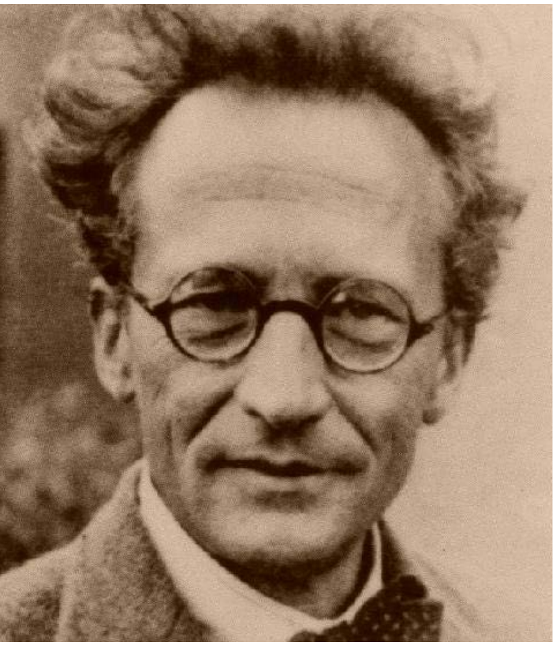
Fermi level pinned to edge state if neutral charge +1/2 electron if full , -1/2 if empty, per edge

- If both edge states are occupied, there is **<u>one</u>** extra electron, 50% at one edge, 50% at the other (half an electron at each edge)
- If both are empty there is half a hole at each edge



$$H = \sum_i J\vec{S}_i \cdot \vec{S}_{i+1} + K(\vec{S}_i \cdot \vec{S}_{i+1})^2 + D(S_i^z)^2$$

spin 1



Schrödinger vs Heisenberg

- Schrödinger's picture describes the system by a **wavefunction** $\psi(r)$ in real space
- Heisenberg's picture describes the system by a **state** $|\psi\rangle$ in Hilbert space
  - They are only equivalent if the basis $|\boldsymbol{r}\rangle$ of states in real-space are orthogonal:

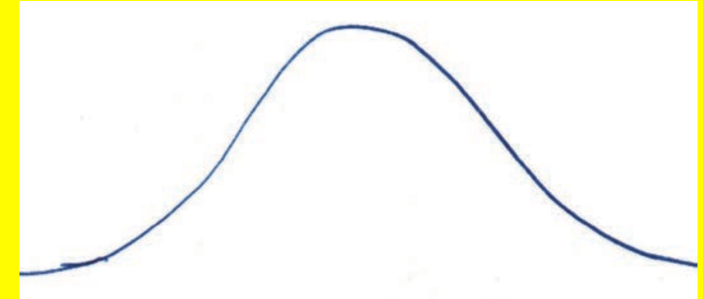
$$\psi(\boldsymbol{r}) = \langle \boldsymbol{r} | \psi \rangle$$

requires

$$\langle \boldsymbol{r} | \boldsymbol{r}' \rangle = 0 \quad (\boldsymbol{r} \neq \boldsymbol{r}')$$

← this fails in a quantum geometry

This is the **entire** problem: nothing other than this matters!



$$H = \sum_{i<j} U(\boldsymbol{R}_i - \boldsymbol{R}_j)$$

$$[R^x, R^y] = -i\ell_B^2$$

like phase-space, has Heisenberg uncertainty principle

- H has translation and inversion symmetry

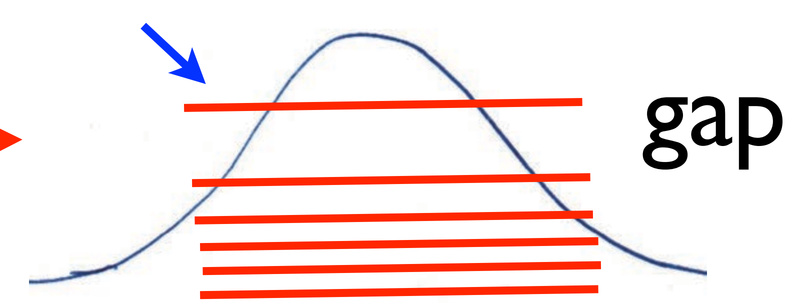
$$[(R_1^x + R_2^x), (R_1^y - R_2^y)] = 0$$

$$[H, \textstyle\sum_i \boldsymbol{R}_i] = 0$$

- generator of translations and electric dipole moment!

$$[(R_1^x - R_2^x), (R_1^y - R_2^y)] = -2i\ell_B^2$$

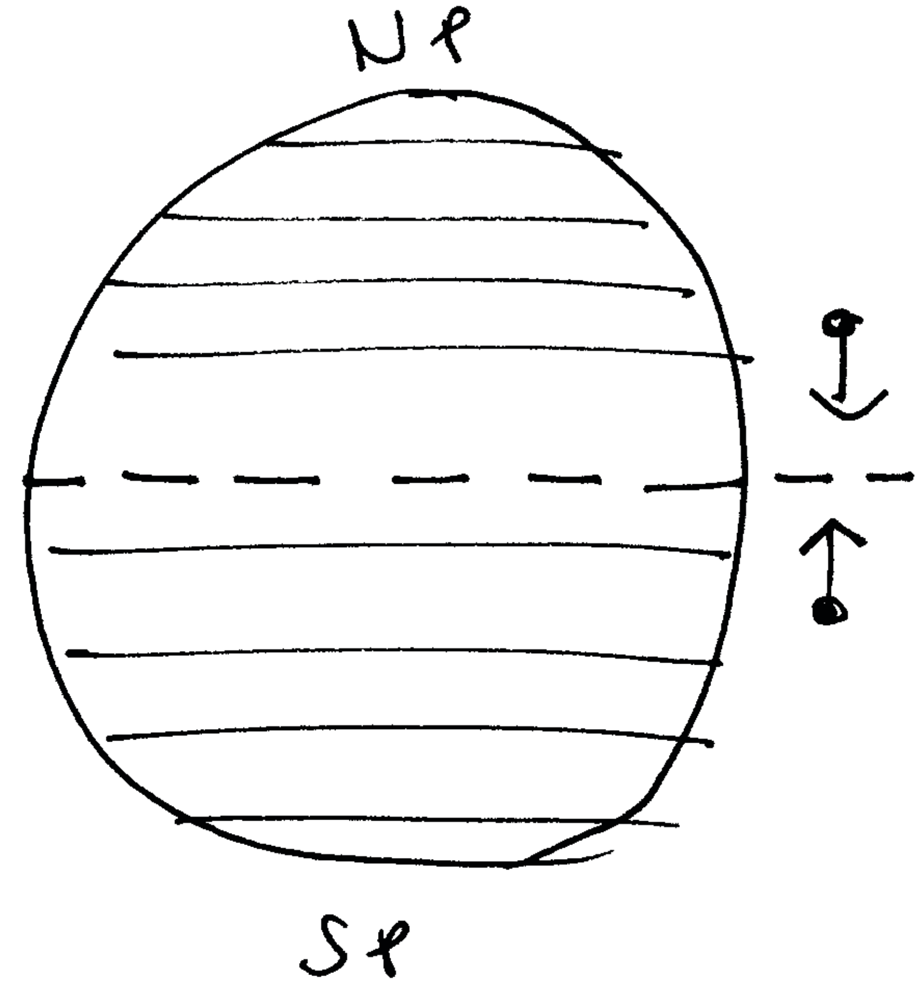
- relative coordinate of a pair of particles behaves like a single particle

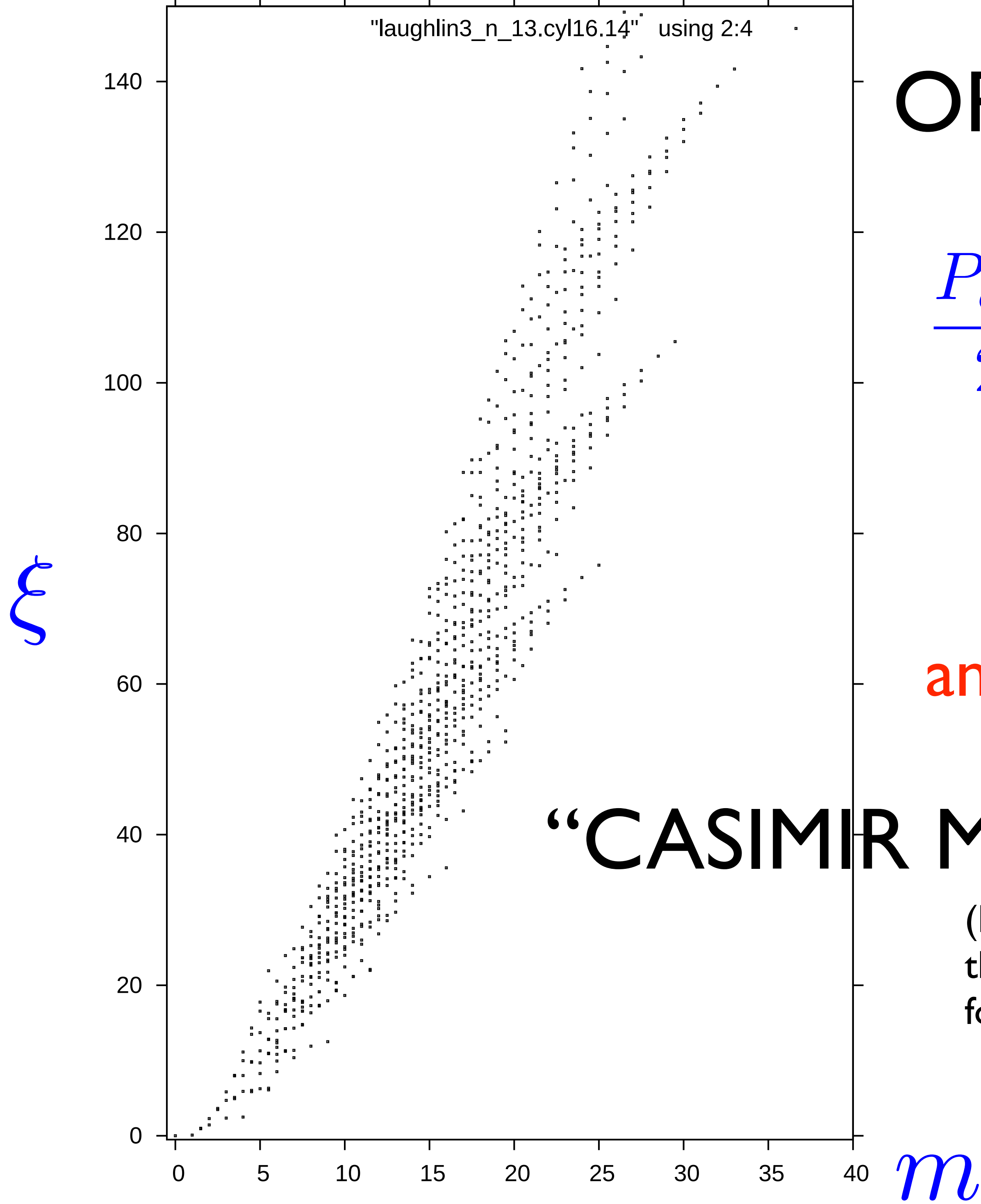


two-particle energy levels

Entanglement spectra and “dominance”

- Schmidt decomposition of Fock space into N and S hemispheres.
- Classify states by Lz and N in northern hemisphere, relative to dominant configuration. Lz always decreases relative to this (squeezing)





ORBITAL CUT

$$\frac{P_a L^a}{2\pi} = \frac{\sum_\alpha m_\alpha e^{-\xi_\alpha}}{\sum_\alpha e^{-\xi_\alpha}} = \eta_H^{cd}\epsilon_{ac}\epsilon_{bd}\frac{L^a L^b}{2\pi\ell_B^2}$$

$$+\tfrac{1}{24}(\tilde{c}-\nu) - h$$

signed conformal anomaly (chiral stress-energy anomaly)

chiral anomaly

virasoro level of sector

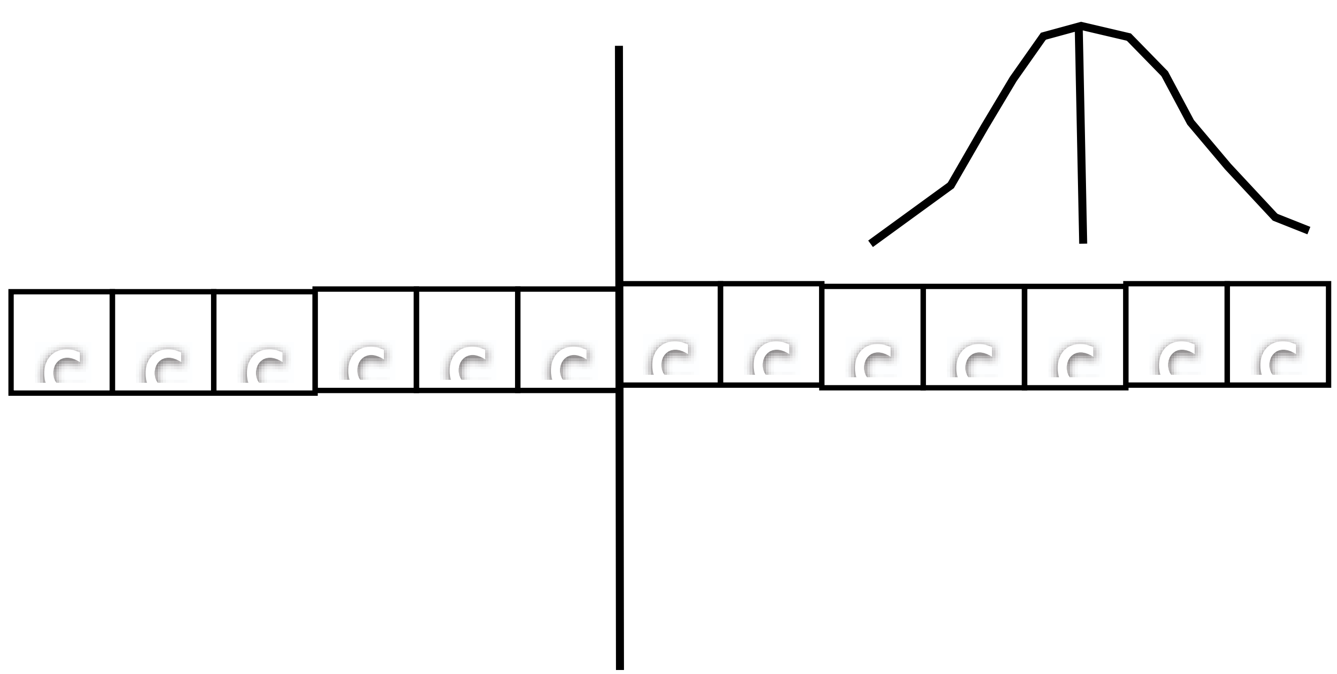
"CASIMIR MOMENTUM" term

(NOT "real-space cut" which requires the Landau orbit degrees of freedom and their form factor to be included

- Hall viscosity gives "thermally excited" momentum density on entanglement cut, relative to "vacuum", at von Neumann temperature T = 1

"fuzzy continuum" vs Lattice

- orbital vs "real space" cut



- The fundamental problem is in the projected space, not its extension to "real space"

- “skyrmion”-like “cone”-like structure moves charge away from quasihole by introducing negative Gaussian curvature