Eliashberg theory of superconductivity

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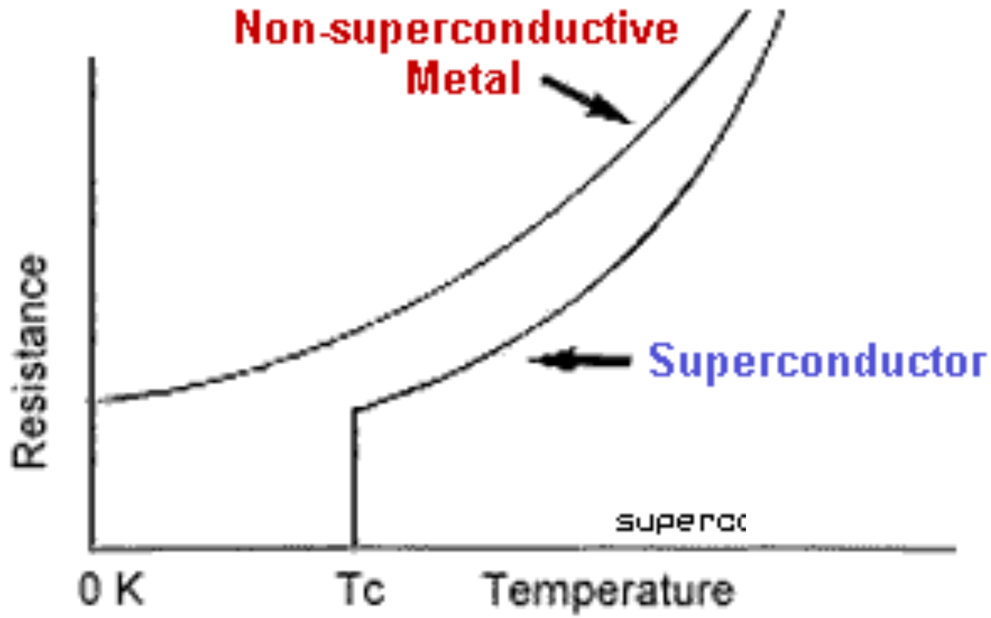
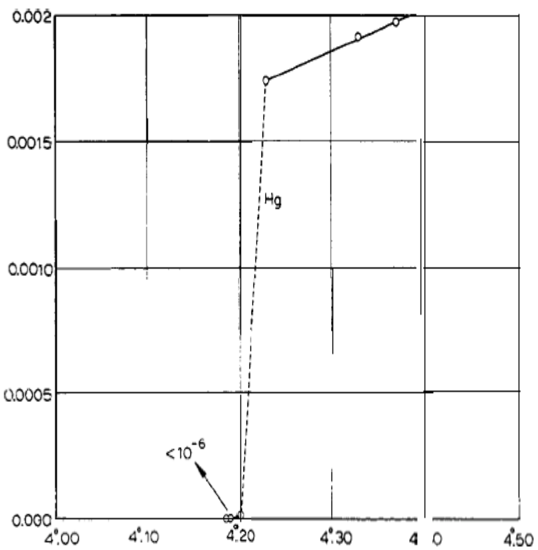
Superconductivity:

Zero-resistance state of interacting electrons



It started in 1911!



Nobel Prize 1913

What we need for superconductivity?

Drude theory for metals predicts that resistance should remain finite at T=0

Ohm's law

$$j = \frac{ne^2\tau}{m_e}E$$

$$j = \sigma E = \frac{E}{\rho}$$

$$\rho = \frac{m_e}{ne^2\tau}$$

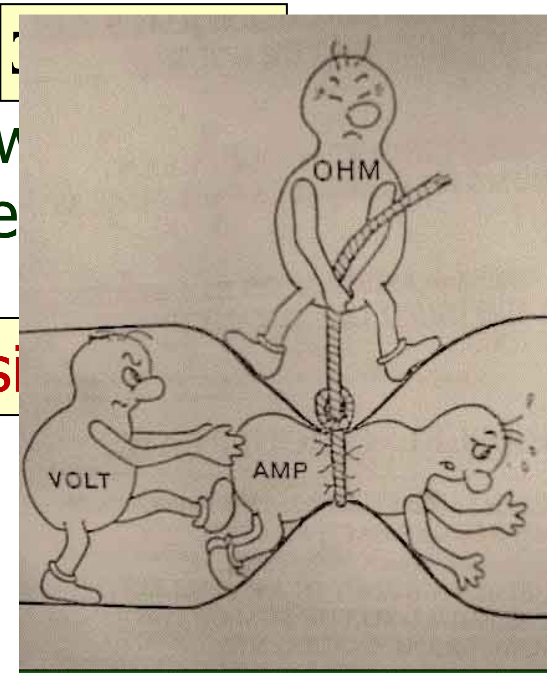
This is a dissipative current: to sustain j we need to borrow energy $(\sim\sigma E^2)$ from the source of an electric field

If the system had a macroscopic condensate

there would be an additional current $j \propto \nabla\varphi$, w

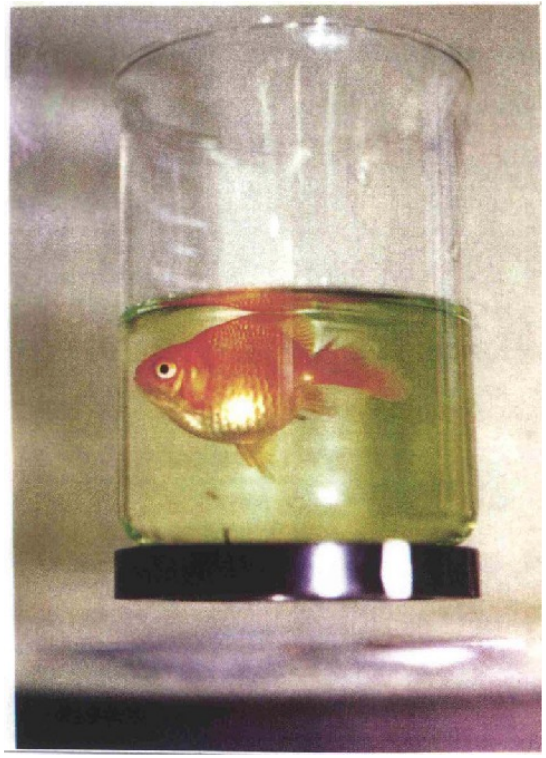
accompanied by energy dissipation and would e

A nonzero current at E =0 means that resi



The condensate $\boxed{\Xi = |\Xi| e^{i\varphi}}$ breaks U(1). By Anderson- Higgs mechanism, a vector potential field becomes massive, and this leads to an expulsion of a magnetic field from a superconductor

Meissner effect



Once we have a macroscopic condensate,
we have superconductivity

For bosons, the appearance of a condensate is natural,
because bosons tend to cluster at zero momentum
(Bose-Einstein condensation)

But electrons are fermions, and two fermions simply
cannot exist in the same quantum state.

However, if two fermions form a bound state,
a bound pair becomes a boson, and bosons do condense.

We need to pair fermions into a bound state.

For pair formation, there must be an attraction between fermions!

Two issues

How to get an attraction?

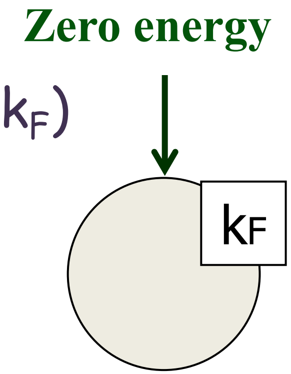
How strong attraction should be for SC?

How strong attraction should be for SC?

Leon Cooper

An arbitrary small attraction between fermions is already capable to produce bound pairs with zero total momentum in any spatial dimension, because the pairing susceptibility is logarithmically singular at vanishing temperature (Cooper logarithm)

Reason: low-energy fermions live not near $k=0$, but near a Fermi surface at a finite $k=k_F$, $d^3k = 4\pi(k_F)^2\, d(k-k_F)$



Let's look into this argument from Fermi liquid perspective

Fermi liquid analysis

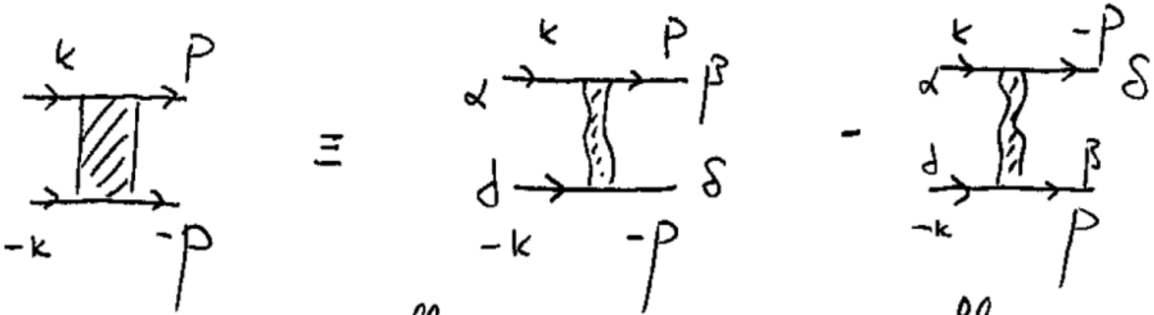
*) poles of the vertex function
(an antisymmetrized interaction)
determine bosonic excitations.

**) for stability, poles must be in the lower ½ plane

An example: zero-sound pole at $\underline{\underline{\omega = \pm v_0 q - i\delta}}$

Construct

$\Gamma =$



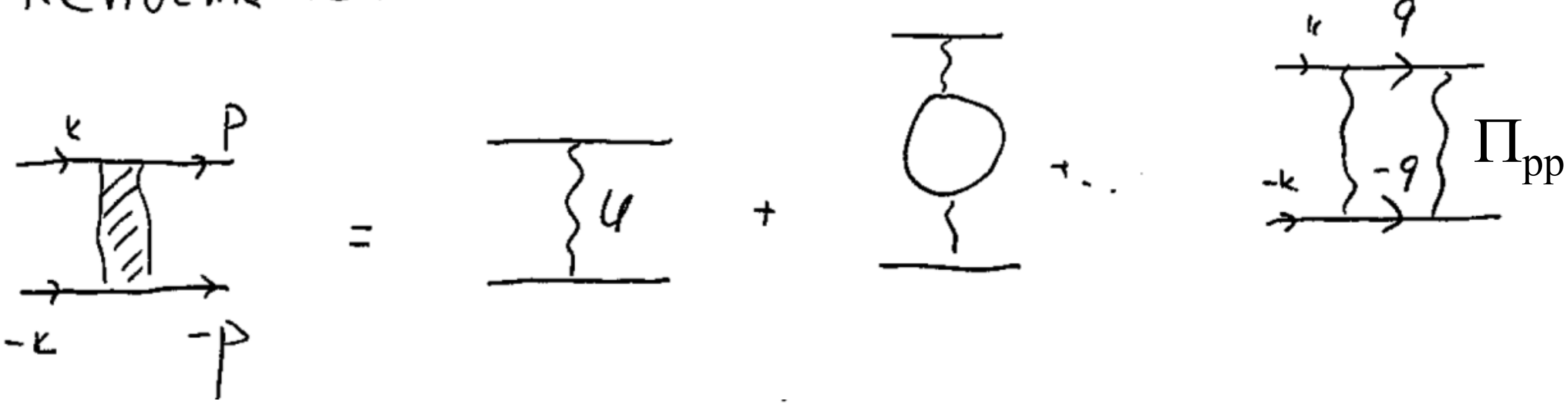
$U^{eff}(k-p)\,\delta_{\alpha\beta}\delta_{\gamma\delta} \qquad U^{eff}(k+p)\,\delta_{\alpha\delta}\delta_{\beta\gamma}$

Let's assume for simplicity that the interaction is momentum-independent U (Hubbard)

To first order in U

$$\Gamma = U\left[\underbrace{\delta_{\alpha\beta}\delta_{\gamma\delta} - \delta_{\alpha\delta}\delta_{\beta\gamma}}_{\text{singlet}}\right]$$

Renormalization



*) At zero incoming total frequency and zero incoming total momentum, Π_{pp} logarithmically diverges at $T=0$.

Cooper logarithm

**) At zero total momentum and finite total frequency Ω

$$\Pi_{pp}(q=0,\Omega) = \frac{i m p_F}{4\pi^2}\left[\log\frac{\omega_0}{\Omega+i\delta} + \log\frac{\omega_0}{-(\Omega+i\delta)}\right]$$

ω_0 is put by hand, I just assumed

$$U = \begin{cases} U, & \text{at energies below } \omega_0 \\ 0, & \text{at energies above } \omega_0 \end{cases}$$

Π_{pp} is a complex function – don't expect a pole infinitesimally close to real frequency

Once we sum up ladder diagrams keeping only Π_{pp} at each order, we get

$$\Gamma(\Omega) = \frac{U}{1 - \frac{\lambda}{2}\left[\log\frac{\omega_0}{\Omega + i\delta} + \log\frac{\omega_0}{-\Omega - i\delta}\right]}$$

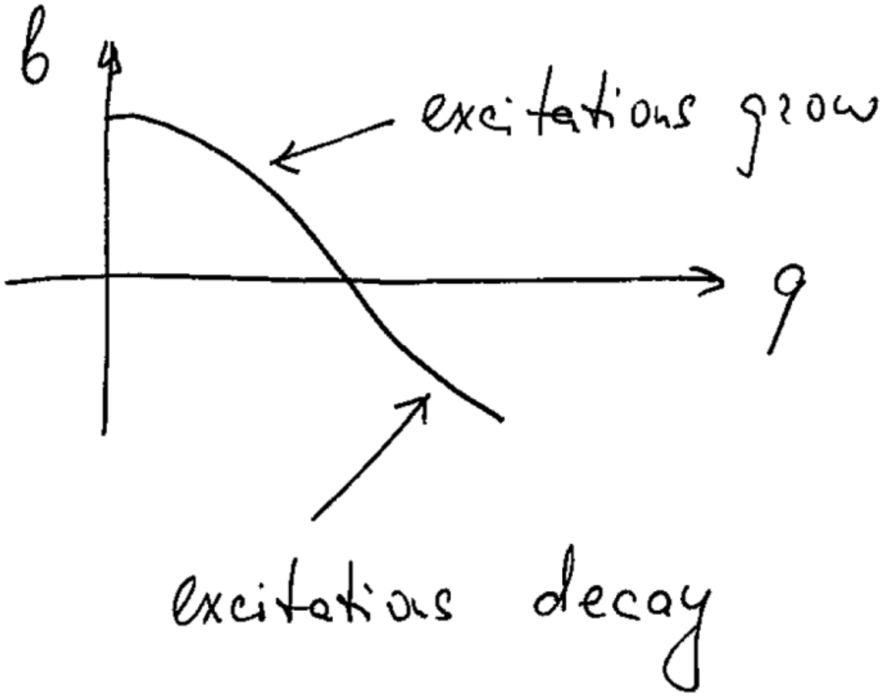
$$\boxed{\lambda = -\frac{m U p_F}{2\pi^2}}$$

To check where the poles of $\Gamma(\Omega)$ is in the complex plane of frequency, replace $\Omega + i\delta$ by $z = a + ib$

Near the pole,

$$\Gamma(\Omega) \sim \frac{1}{\Omega - i b}$$

If momentum q is finite



Instability only for excitations with near-zero total momentum of a pair.

BCS theory of superconductivity

J. Bardeen, L. Cooper, R. Schrieffer

Nobel Prize 1972

BSC-I:
what the instability of the normal state means

Suppose $U<0$.

which state is stable?

$$\mathcal{H} = \sum_{k,\alpha} \varepsilon_k \, b^+_{k\alpha} b_{k\alpha} + \frac{U}{N} \sum_{1234} b^+_{k_1\uparrow} b^+_{k_2\downarrow} b_{k_3\downarrow} b_{k_4\uparrow}$$

$(1+2=3+4)$

momentum spin

Idea of BCS: assume and verify that

$$\left.\begin{aligned} -\frac{U}{N}\sum_k \langle b^+_{k\uparrow}\, b^+_{-k\downarrow}\rangle &= \Delta \neq 0 \\ \text{Simultaneously,}\quad -\frac{U}{N}\sum_k \langle b^+_{k\downarrow}\, b^+_{-k\uparrow}\rangle &= -\Delta \end{aligned}\right\}\ \text{spin singlet}$$

This physically implies that fermions form bound pairs

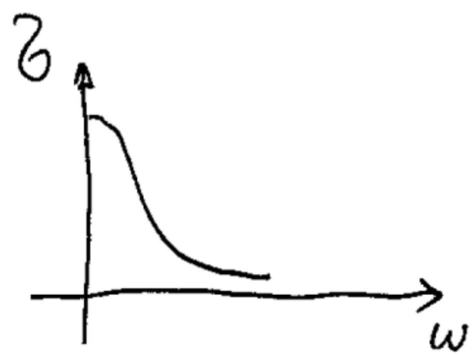
$$\Delta = |\Delta|\, e^{i\varphi}$$

assume $\varphi = 0$ for simplicity

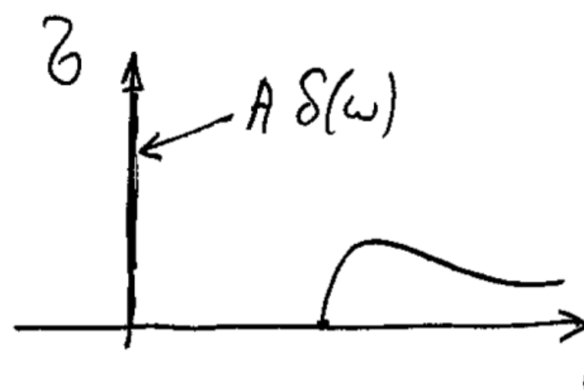
$$\frac{U}{N}\sum b^+ b^+ b b \rightarrow -\frac{\Delta^2 N}{U} - \Delta \sum_k \left(b^+_{k\uparrow} b^+_{-k\downarrow} + b_{k\downarrow} b_{-k\uparrow} \right) + \dots$$

Re-diagonalize the quadratic form and find Δ from self-consistency condition

Most relevant: conductivity



Drude conductivity in the normal state



The presence of a δ-function implies that resistivity $= 0$.

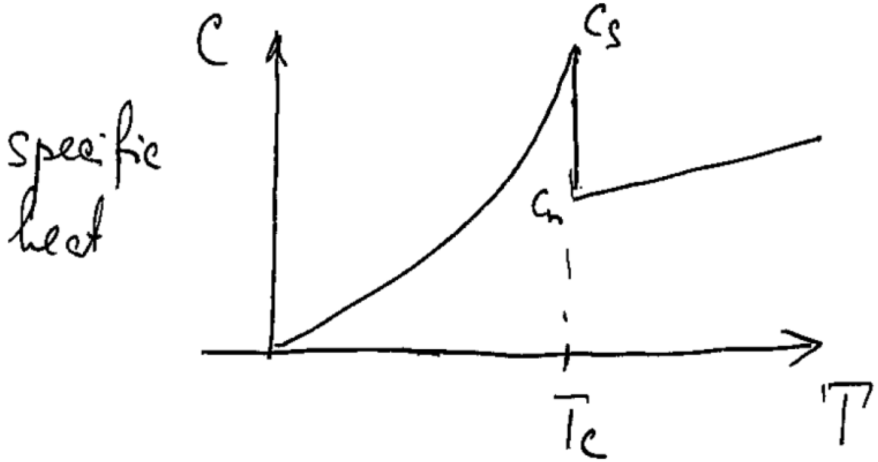
BCS: actual computations

$$\Delta \sim \omega_0\, e^{-1/\lambda}$$

$$T_c \sim \omega_0\, e^{-1/\lambda}$$

both depend on the upper cutoff ω_0

However, $\dfrac{2\Delta}{T_c} = 3.53 \quad \left(\dfrac{\Delta}{T_c} = 1.76 \right)$



$$\frac{c_s - c_n}{c_n} = 1.43 \quad \left(= \frac{12}{7\zeta(3)} \right)$$

One can improve the analysis already within BCS-I

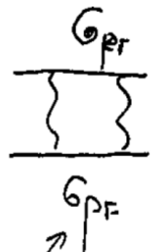
Q: can one obtain T_c & Δ without information about the upper cutoff?

A: yes, if the upper cutoff is larger than the Fermi energy

Gorkov
Melik-Barkhudarov

Point: the measured quantity is not U but rather a scattering length a

$$a = \frac{mU}{4\pi}\left(1 + \text{all corrections at } p_F = 0\right)$$



Green's function at $p_F \neq 0$

Write $$G_{p_F} G_{p_F} = \underbrace{\left(G_{p_F} G_{p_F} - G_0 G_0\right)}_{\text{momentum/frequency integral converges and doesn't require a cutoff}} + \underbrace{G_0 G_0}_{\text{replaces } U \text{ by } a}$$

interaction

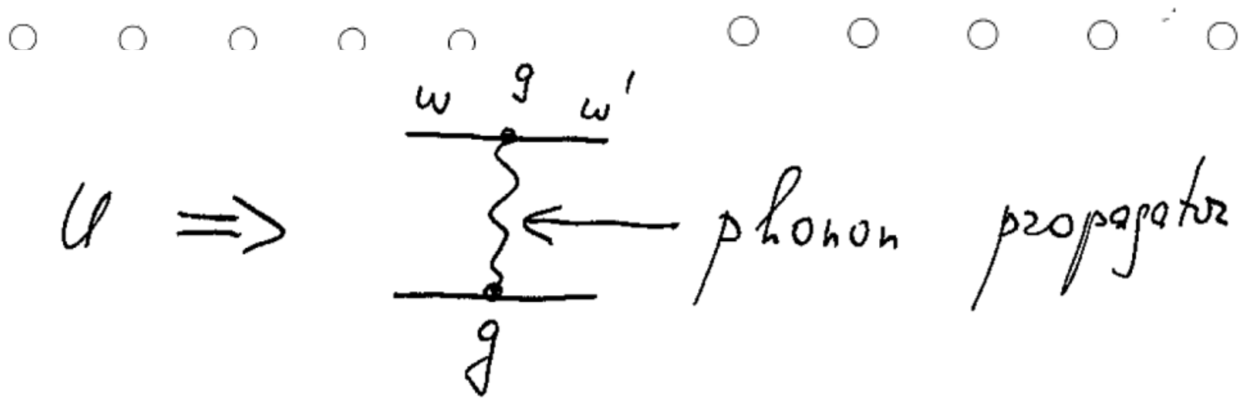
scattering length

BCS theory of superconductivity.

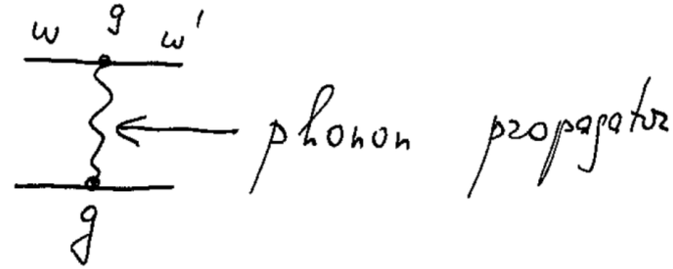
$\boxed{\text{BCS II}}$

What causes an attraction?

BCS: it is electron-phonon interaction



$U \Longrightarrow$



Consider a simple Einstein phonon

$$U_{eff} = g^2 D = \frac{g^2}{(\omega-\omega')^2 - \omega_D^2}$$

BCS for electron - phonon interaction:

Keep $U_{eff} = -g^2/\omega_D^2$ at $\omega, \omega' < \omega_D$

Set $U_{eff} = 0$ at $\omega, \omega' > \omega_D$

(a hard cutoff).

All previous BCS results hold, as long as

$\omega_D \ll E_F$

We still don't know T_c and Δ

separately. Only their ratio is known

Eliashberg theory of SC

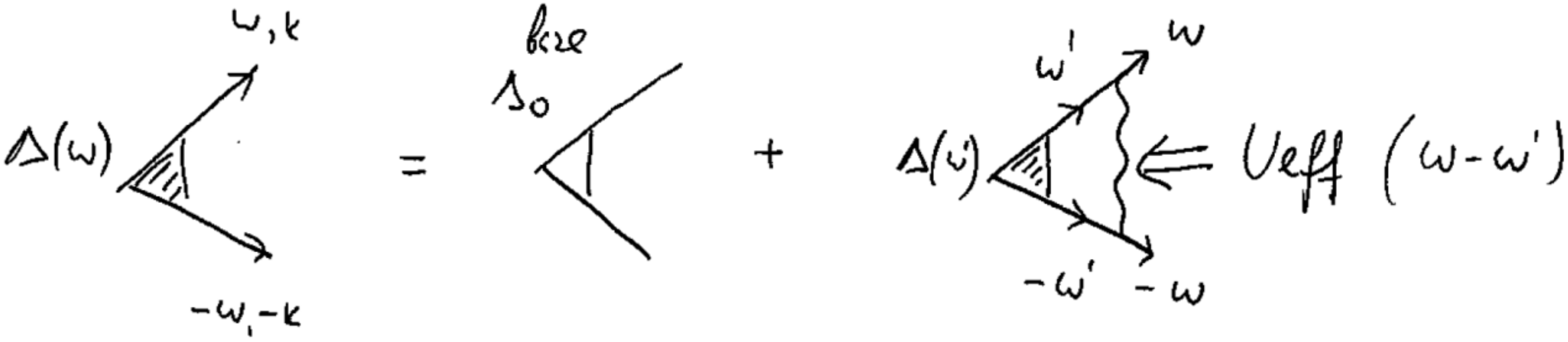
*) week coupling limit

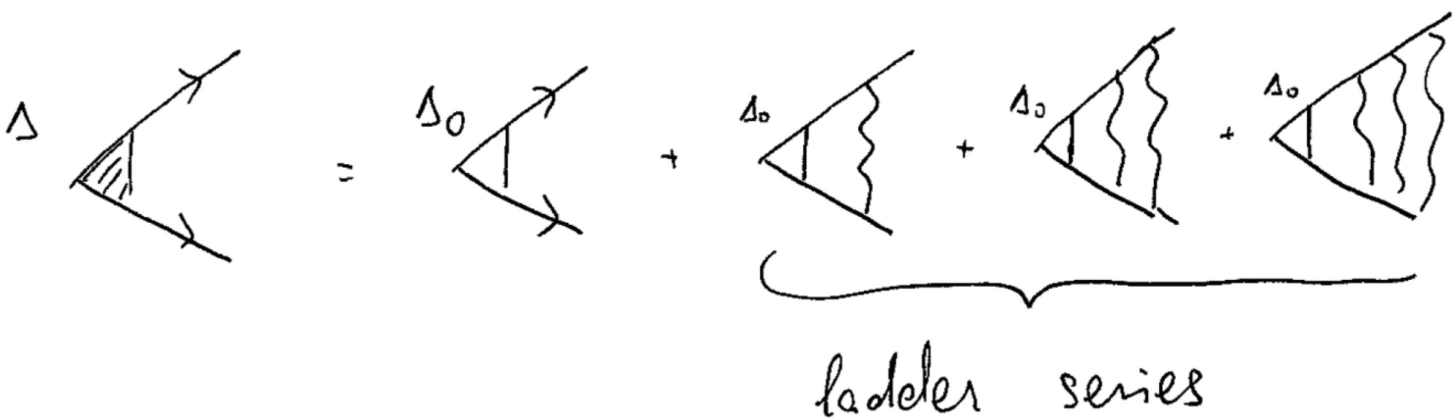
Don't approximate U_{eff} by a step function, keep its frequency dependence

Once $U_{eff} = U_{eff}(\omega - \omega')$, the gap

Δ becomes $\Delta(\omega)$

Graphically, self-consistent equation on $\Delta(\omega)$ is





For T_c and $\Delta(T<T_c)$, we can set $\Delta_0 = 0$

Superconductivity is easier to analyze on the Matsubara axis

$$(\omega \to i\omega_m)$$

$$U_{eff} = - \frac{g^2}{(\omega_m - \omega_m')^2 + \omega_D^2}$$

Eliashberg gap equation on Matsubara axis

$$\Delta(\omega) = \pi T \sum_{\omega'} \frac{\Delta(\omega')}{\sqrt{(\omega')^2 + \Delta^2(\omega')}} \frac{g^2}{(\omega-\omega')^2 + \omega_D^2}$$

Solution:

$$T_c = 0.69\, \omega_D\, e^{-1/\lambda}$$

$$\left(T_c^{BCS} = 1.13\, \omega_D\, e^{-1/\lambda} \right)$$

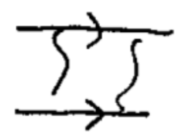
if cutoff is right at ω_D

Q: Is this the correct result at week coupling ?

It turns out, we have to include fermionic self-energy

$$G^{-1} = i(\omega+\Sigma) - \varepsilon_p$$

$$\Sigma = \Sigma(\omega) = \lambda\omega \quad \text{at} \quad \omega < \omega_D \qquad G^{-1} = i\omega(1+\lambda) - \varepsilon_p$$

A simple calculation of  $\Pi_{pp} \propto \frac{1}{1+\lambda}$

$$\lambda \to \lambda/1+\lambda$$

$$e^{-\frac{1}{\lambda}} \Rightarrow e^{-\frac{1+\lambda}{\lambda}} = e^{-1/\lambda}\ \underset{\approx}{1/e}$$

$$\boxed{T_c^{\text{correct}} = 0.25\, \omega_D\, e^{-1/\lambda}}$$

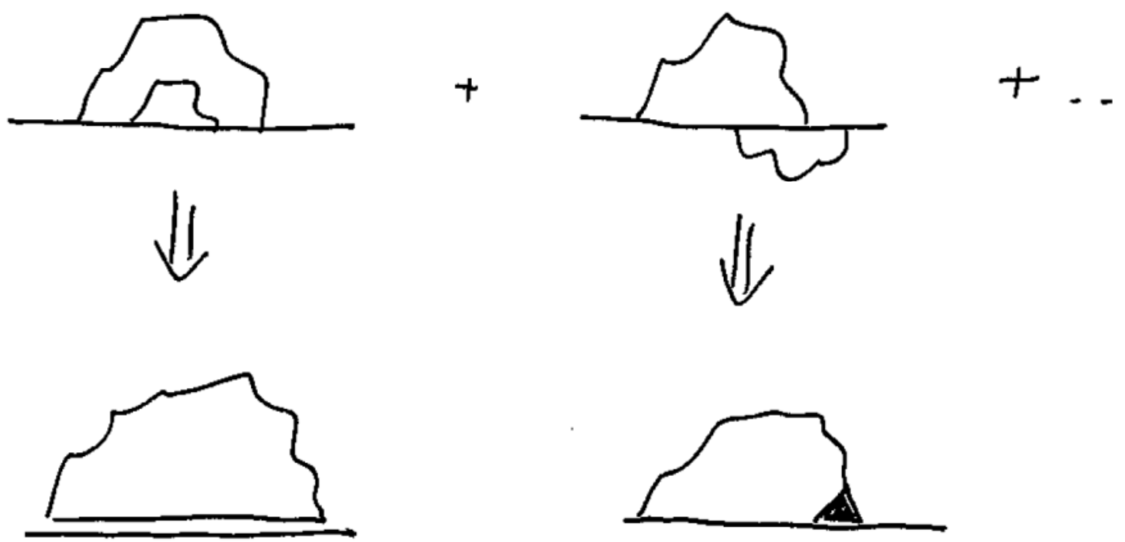
For an Einstein phonon

This calculation has been extended to other forms of a phonon propagator, and became the "standard" computational procedure for superconducting Tc and the gap function

Eliashberg theory of SC

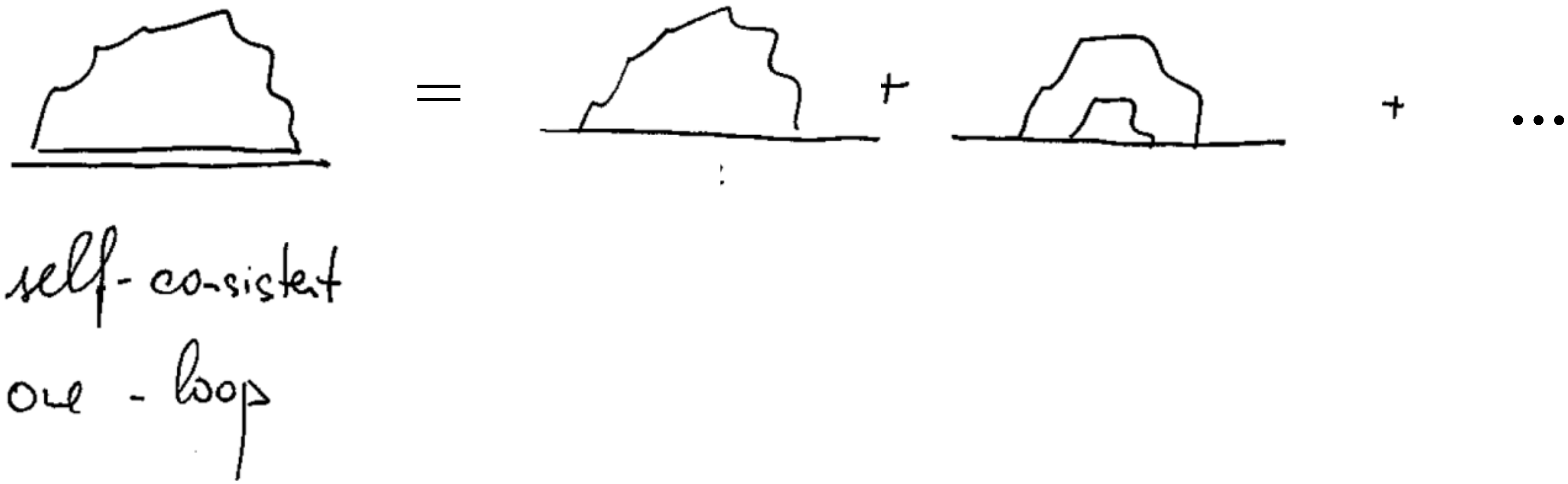
**) extension to stronger coupling

Add higher-order self-energy diagrams



self-consistent one-loop

vertex corrections



These diagrams form series
in the dimensionless coupling λ.
When the coupling is large, they
are all relevant

Let's look at vertex correction

*) At $q = 0$, Ω finite,

vertex correction is of the same order as $\frac{\partial \Sigma}{\partial \omega} = \lambda$ by Ward id.

$$\Gamma = 1 + \frac{\partial \Sigma}{\partial \omega} = 1 + \lambda$$

**) At $q \sim p_F$ $(v_F q \sim E_F)$ and $\Omega \sim \omega_D \ll E_F$,

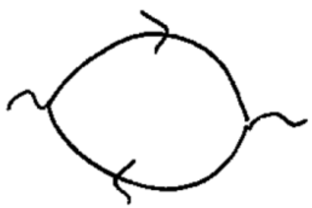
$$\Gamma = 1 + \lambda_E$$

$$\lambda_E = \lambda \times \frac{\omega_D}{E_F} \ll \lambda$$

Concept: fast electrons & slow bosons

In the process leading to a vertex correction, fast electrons vibrate at slow phononic frequencies, far away from their own resonance

There is another effect of electron-phonon interaction: Landau damping of a phonon

Ω, q  $\sim \dfrac{|\Omega|}{v_F q} \qquad (q_{typ} \sim p_F)$

"Strong coupling"
Eliashberg theory:

$\lambda > 1$

$\lambda_E < 1$

$\omega_D^2 \ll g^2 \ll \omega_D E_F$

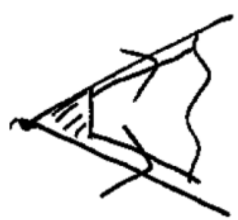
$\to$ a set of coupled integral eqs for normal state properties and superconductivity

The gap equation involves fully renormalized Green's function but no vertex corrections to ladder series

Fermionic self-energy is obtained in self-consistent one-loop approximation, without vertex corrections

Diagrammatically

$\Sigma =$  $\Rightarrow$ full Green's function (matrix in a SC state)

Φ  $=$ 

The actual Eliashberg equations (written by him)

$$\Sigma(\omega) = \pi T \sum_{\omega'} \frac{\omega' + \Sigma(\omega')}{\sqrt{(\omega' + \Sigma(\omega'))^2 + |\Phi(\omega')|^2}} \frac{g^2}{|\omega - \omega'|^2 + \omega_D^2}$$

$$\Phi(\omega) = \pi T \sum_{\omega'} \frac{\Phi(\omega')}{\sqrt{(\omega' + \Sigma(\omega'))^2 + |\Phi(\omega')|^2}} \frac{g^2}{|\omega - \omega'|^2 + \omega_D^2}$$

$$\boxed{T_c = \omega_D * f(\lambda)}$$

$$\lambda = g^2 / \omega_D^2$$

A trick:

$$\Delta(\omega) = \frac{\varphi(\omega)\cdot\omega}{\omega + \Sigma(\omega)} = \frac{\varphi(\omega)}{Z(\omega)}$$ gap function

$$Z(\omega) = 1 + \frac{\Sigma(\omega)}{\omega}$$ quasiparticle renormalization factor

$$\frac{\varphi(\omega)}{\sqrt{|\varphi(\omega)|^2 + (\omega+\Sigma(\omega))^2}} = \frac{\Delta(\omega)}{\sqrt{\omega^2 + \Delta^2(\omega)}} \qquad \frac{\omega + \Sigma(\omega)}{\sqrt{(\omega+\Sigma(\omega))^2 + |\varphi(\omega)|^2}} = \frac{\omega}{\sqrt{\omega^2 + \Delta^2(\omega)}}$$

$$\Delta(\omega) = \pi T \sum_{\omega'} \frac{\Delta(\omega') - \frac{\Delta(\omega)}{\omega}\omega'}{\sqrt{\Delta^2(\omega') + (\omega')^2}} \frac{g^2}{|\omega-\omega'|^2 + \omega_D^2}$$

$$Z(\omega) = 1 + \frac{\pi T}{\omega} \sum_{\omega'} \frac{\omega'}{\sqrt{(\omega')^2 + \Delta^2(\omega')}} \frac{g^2}{|\omega-\omega'|^2 + \omega_D^2}$$

The same equations can be obtained from Luttinger-Ward-Eliashberg free energy

$$F = F_{el} + F_{int}$$

$$F_{el} = -2T \sum_p \log\left(-\det \hat{G}_p\right) - i\,\mathrm{Tr}\left(\hat{\Sigma}_p \hat{G}_p\right)$$

$$F_{int} = g^2 T^2 \sum_{pp'} G_p\, D_{ph}(p-p')\, G_{p'} - F_p^*\, D_{ph}(p-p')\, F_{p'}$$

$$\hat{G} = \begin{pmatrix} G & F \\ F^* & -G \end{pmatrix} \qquad \hat{\Sigma} = \begin{pmatrix} \Sigma & \varphi \\ -\varphi^* & \Sigma \end{pmatrix}$$

and

$$G_p = (-)\,\frac{\varepsilon_{-p} - i\tilde{\Sigma}_{-p}}{(\varepsilon_p - i\tilde{\Sigma}_p)(\varepsilon_{-p} - i\tilde{\Sigma}_{-p}) + |\varphi_p|^2}$$

$$\boxed{\tilde{\Sigma} = \omega + \Sigma}$$

$$F_p = i\,\frac{\varphi_p}{(\varepsilon_p - i\tilde{\Sigma}_p)(\varepsilon_{-p} - i\tilde{\Sigma}_{-p}) + |\varphi_p|^2}$$

Stationary solutions:

$$\frac{\delta F}{\delta \Sigma_p} = \frac{\delta F}{\delta \varphi_p} = 0$$

$$\varphi^*_p = i T \sum_{p'} g^2 D(p-p') F^*_{p'}$$

$$\Sigma_p = i T \sum_{p'} g^2 D(p-p') G_{p'}$$

another form of Eliashberg eqs.

Integrate over momenta

$\Downarrow$

"Conventional" Eliashberg eqs: only in frequency domain.

Eliashberg computational procedure

*) Add eqn. for bosonic polarization

$$\Pi(q,\Omega) \qquad \left[\, D^{-1} = D_0^{-1} - \Pi \,\right]$$

$$F = F_{el} + F_{bos} + F_{int}$$

A set of 3 coupled eqs for

$$\left.\begin{aligned} \Sigma &= \Sigma(p,\Omega) \\ \Phi &= \Phi(p,\Omega) \\ \Pi &= \Pi(q,\Omega) \end{aligned}\right]$$

keep Landau damping and momentum dependence of the self-energy, neglect vertex corrections

Superconductivity at strong coupling $\left(\begin{array}{l}\lambda \gg 1 \\ \lambda_E \ll 1\end{array}\right)$

$$\boxed{T_c = \omega_D * f(\lambda)}$$

* at $\lambda \ll 1$, $\quad T_c \sim 0.25\, \omega_D\, e^{-1/\lambda}$

* at $\lambda \gg 1$?

Allen-Dynes 1975 : $\quad T_c \propto \omega_D \sqrt{\lambda}$

Recall : $\quad \lambda = g^2/\omega_D^2 \implies T_c \sim g$

It looks, I can set $\omega_D = 0$ and still get a finite T_c $\left(\text{say, } \omega_D \to 0,\ E_F \to \infty,\ \lambda_E = \frac{g^2}{\omega_D E_F} \right.$ remains small $\left. \right)$

$\omega_D=0$

Eliashberg eq for T_c

$$\Delta(\omega) = \pi T \sum_{\omega'} \frac{\left(\Delta(\omega') - \Delta(\omega)\frac{\omega'}{\omega}\right)}{|\omega'|} \frac{g^2}{|\omega-\omega'|^2}$$

*) divergent term at $\omega = \omega'$ cancels out by 0 in the numerator

[analog of the Anderson theorem]

**) after this, the eqn. has no singularities, and g is the only parameter

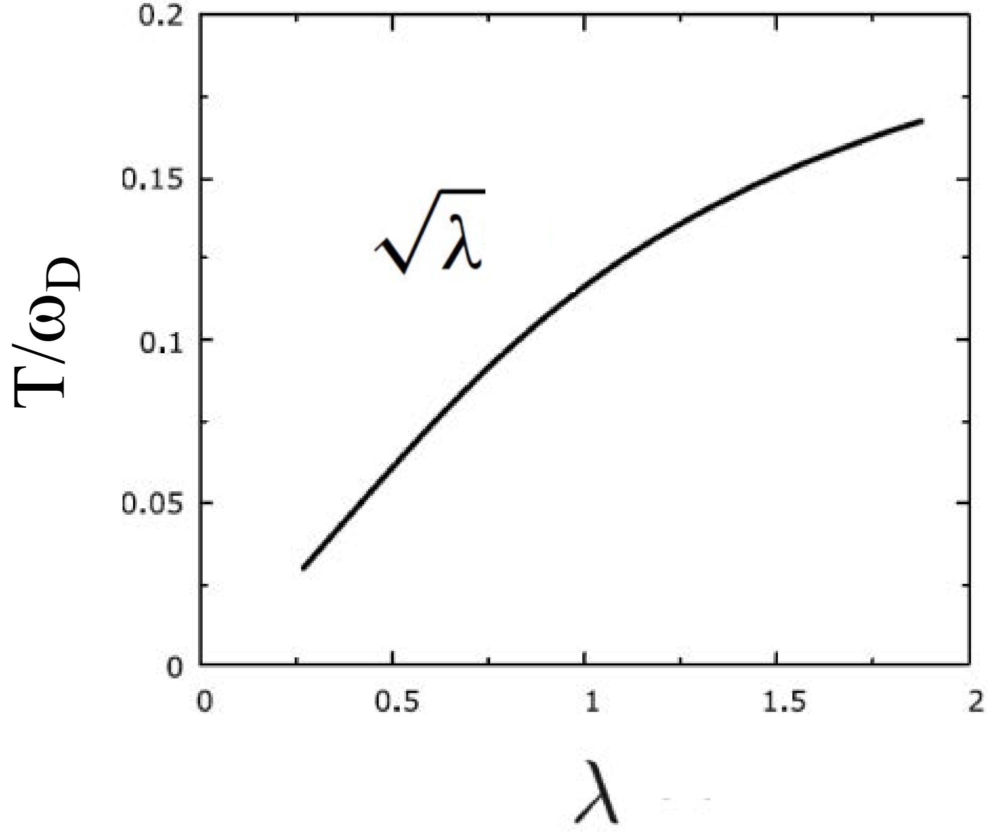
$$T_c = 0.18\, g \quad \left(= 0.18\, \omega_D \sqrt{\lambda} \right)$$

And what about $Z(\omega)$:

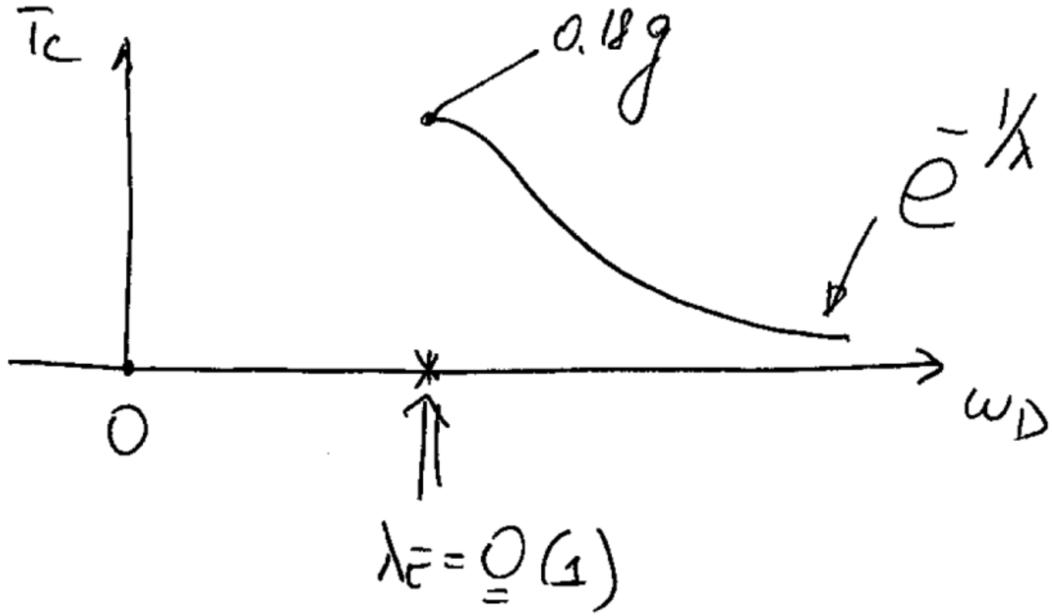
$$Z(\omega) = 1 + \frac{\Sigma(\omega)}{\omega} = 1 + \frac{\pi T}{\omega} \sum_{\omega'} \frac{\mathrm{sgn}(\omega')}{|\omega-\omega'|^2} \quad \text{at } \omega_D = 0$$

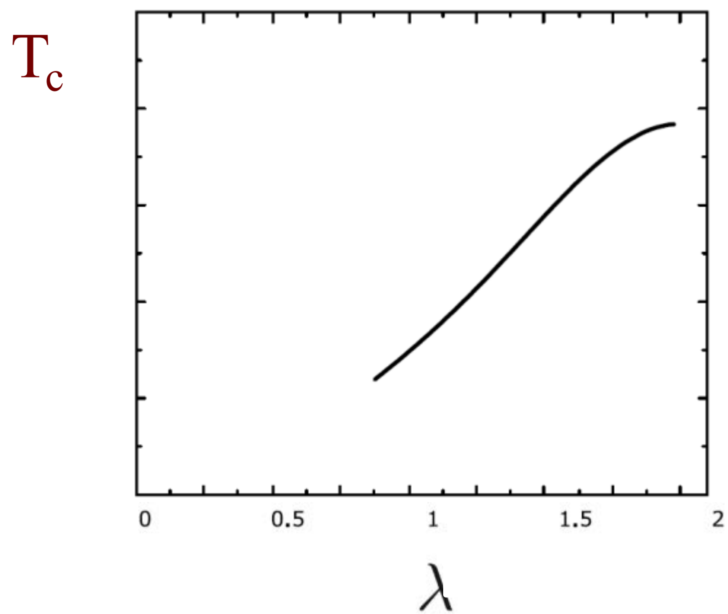
Self-energy diverges.

Numerics:



I. Esterlis et al





Numerics: saturation of T_c at strong coupling

Conclusions:

1. Superconductivity develops at strong coupling, even when fermionic self-energy diverges

2. Superconducting Tc saturates at a finite value when ω_D vanishes

THANK YOU

The last point for today:

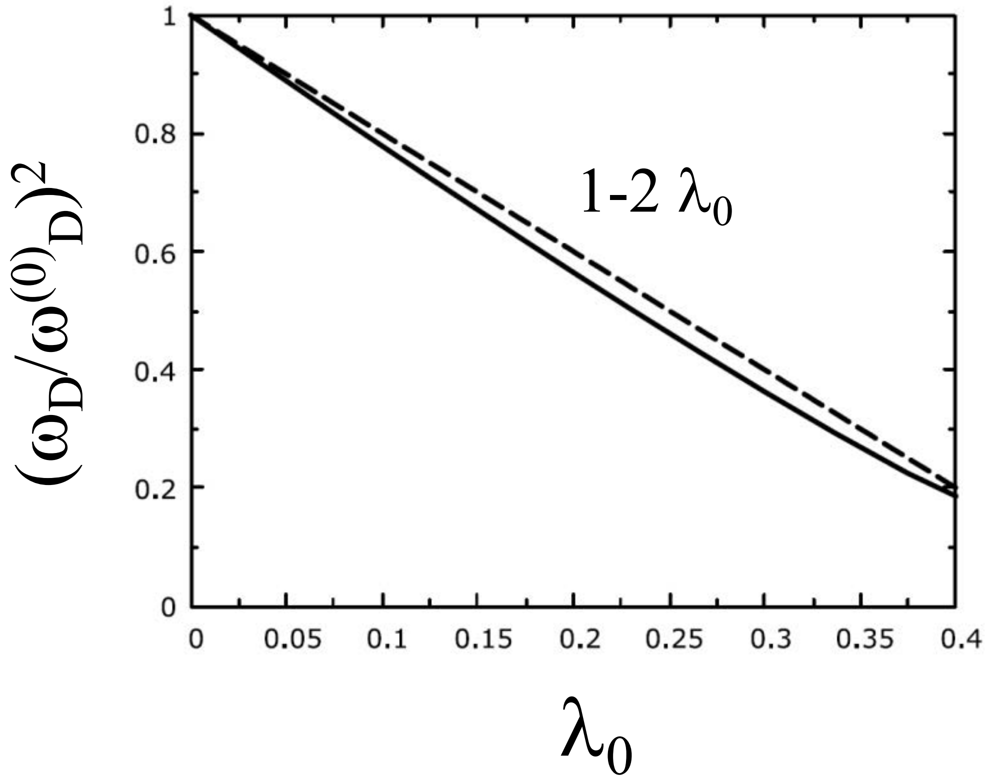
*) I previously said that bosonic polarization can be neglected.

**) This is true for Landau damping, however there is a frequency-independent shift from bare Debye frequency $\omega_D^{(0)}$ to the actual ω_D

***) For a circular Fermi surface in 2D,

$$\omega_D = \omega_D^{(0)} \sqrt{1-2\lambda_0}\ , \quad \lambda_0 = g^2 / \left(\omega_D^{(0)}\right)^2$$

Numerics for lattice dispersion (2D t-t' model)



Superconducting $T_c^{\text{Eliashberg}}$

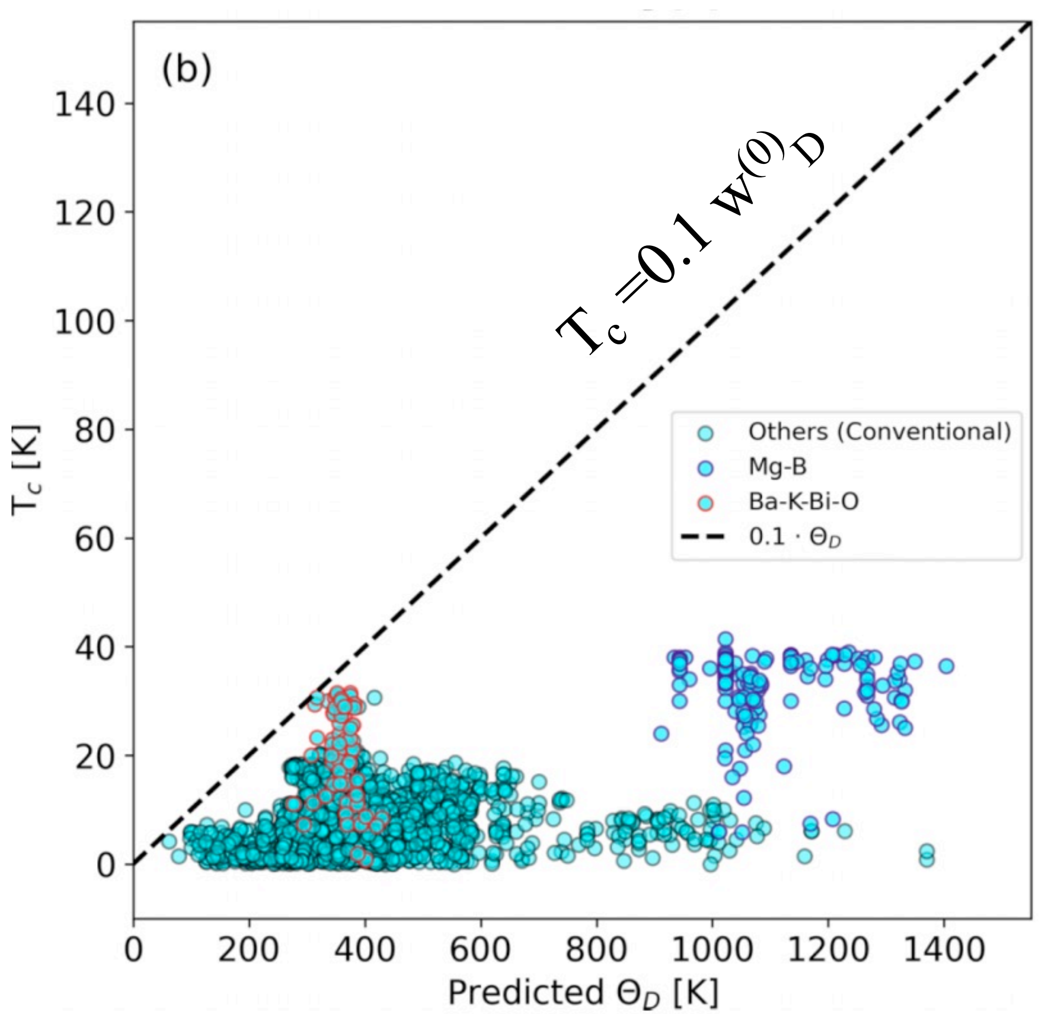
$$T_c^{max} = 0.18\, g \sim 0.13\, \omega_0$$

$T_c^{\text{Eliashberg}}$ cannot be larger than $0.13\,\omega_0$

Machine learning the relationship between Debye temperature and superconducting transition temperature

2023

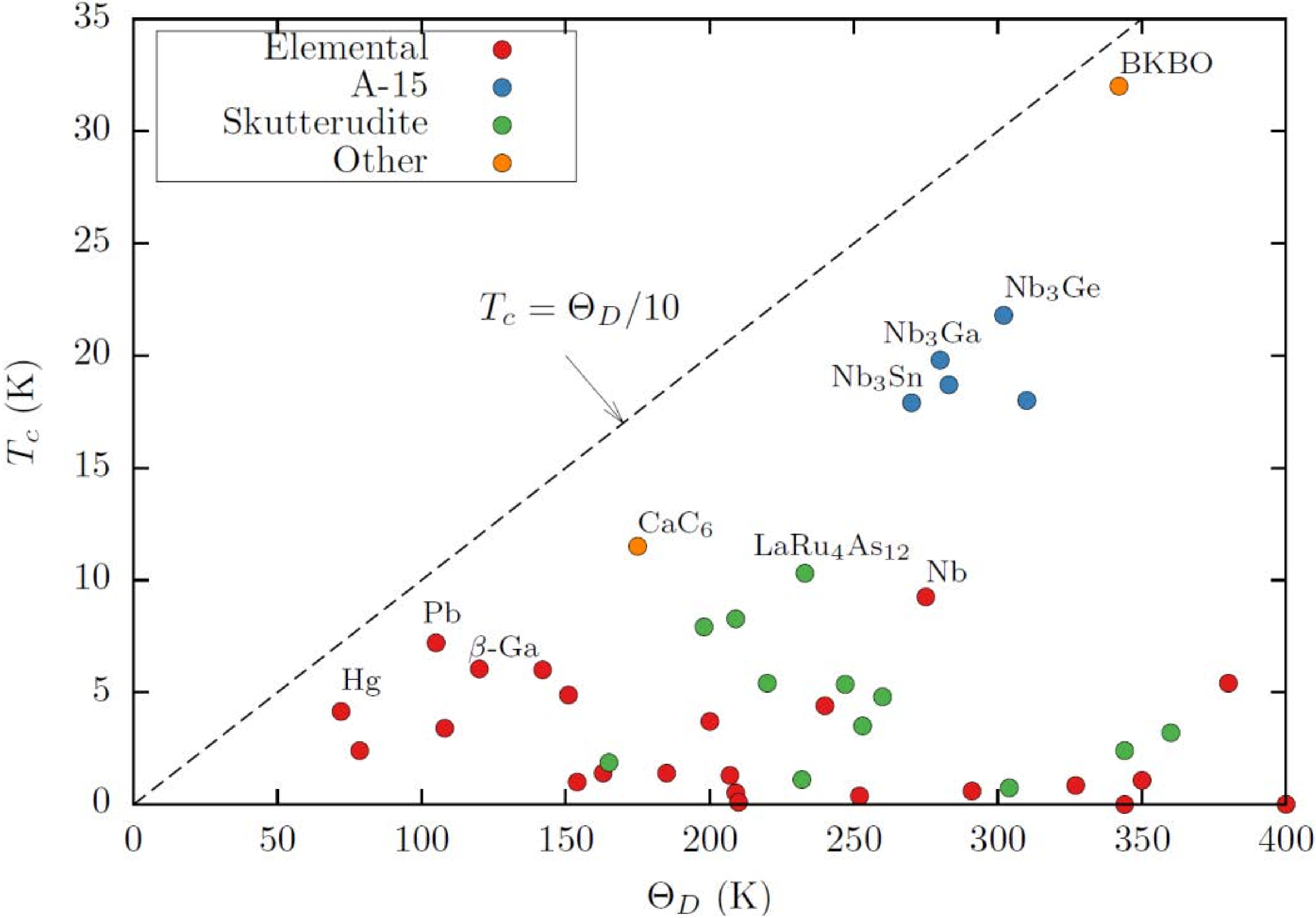
Adam D. Smith,[1,*] Sumner B. Harris,[2] Renato P. Camata,[1] Da Yan,[3] and Cheng-Chien Chen[1,†]



A bound on the superconducting transition temperature

2018

I. Esterlis[1], S. A. Kivelson[1] and D. J. Scalapino[2]



Superconducting stiffness

$$\Delta = |\Delta| e^{i\varphi}, \qquad \text{SC when } \langle \Delta \rangle \neq 0$$

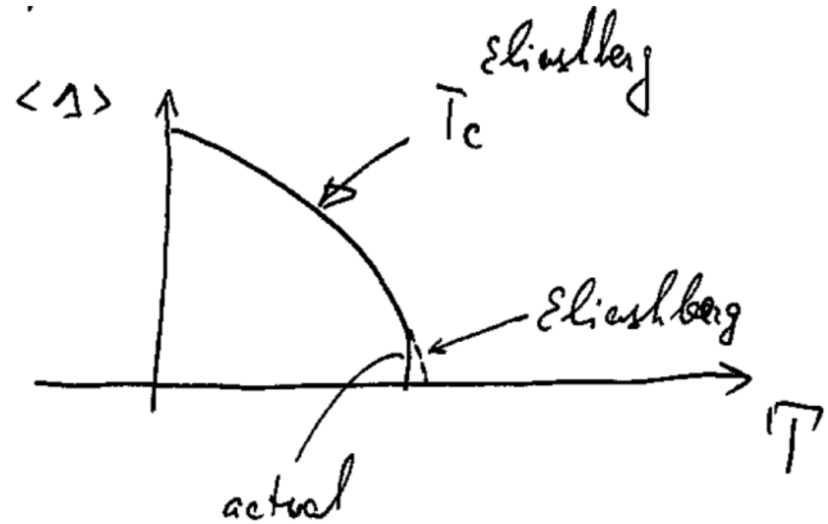
Previously, our discussion was about $|\Delta|$.

Energy cost to change phase

$$\delta F = \rho_s \int (\nabla \varphi)^2 d^2 r \qquad \} \text{ in 2D}$$

ρ_s has dimension of energy

If $\rho_s^{T=0} \gg T_c^{\text{Eliashberg}}, \Delta$



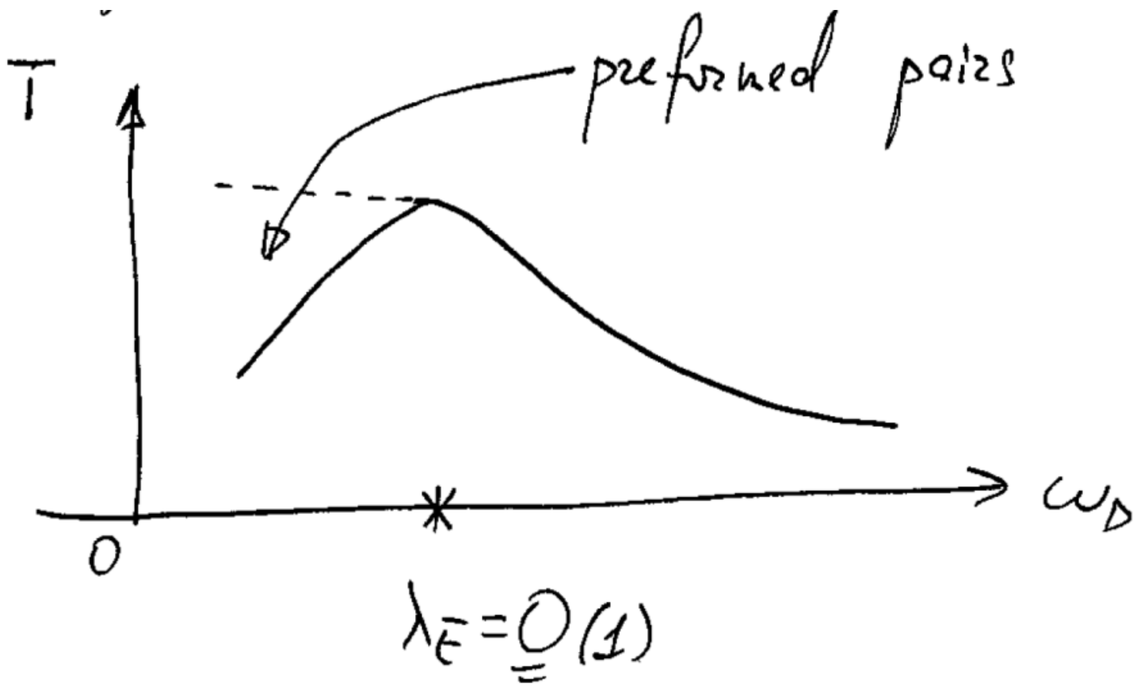
In Eliashberg theory:

$$\frac{\rho_s^{T=0}}{T_c^{\text{Eliashberg}}} \sim 1/\lambda_E \gg 1$$

As long as Eliashberg theory is justified, phase fluctuations are weak.

However, at the boundary of the applicability of Eliashberg theory,

$$\rho_s \sim T_c^{\text{Eliashberg}}$$

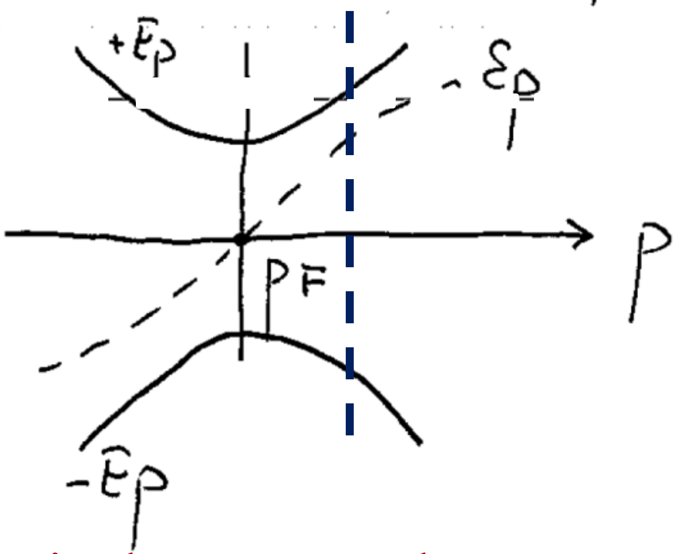


$$\mathcal{H} \Rightarrow \mathcal{H}_0 + \sum_k E_k \left(c^+_{k\alpha} c_{k\alpha} \right)$$

$$E_k = \sqrt{\varepsilon_k^2 + \Delta^2}$$

In terms of original fermions ($b_{k\alpha}$)

there are two poles at $+E_k$ and $-E_k$



p is the same as k

The system lowers its ground state energy when $\Delta \neq 0$

$$E_{ground} = E_{ground}^{normal} + E_{cond}$$

$$E_{cond} = -\frac{N\Delta^2}{2} Q$$

$$Q = \frac{1}{N} \sum_{p} \frac{E_p - |\varepsilon_p|}{E_p \left(E_p + |\varepsilon_p| \right)} > 0$$