Tensor networks and entanglement spectroscopy

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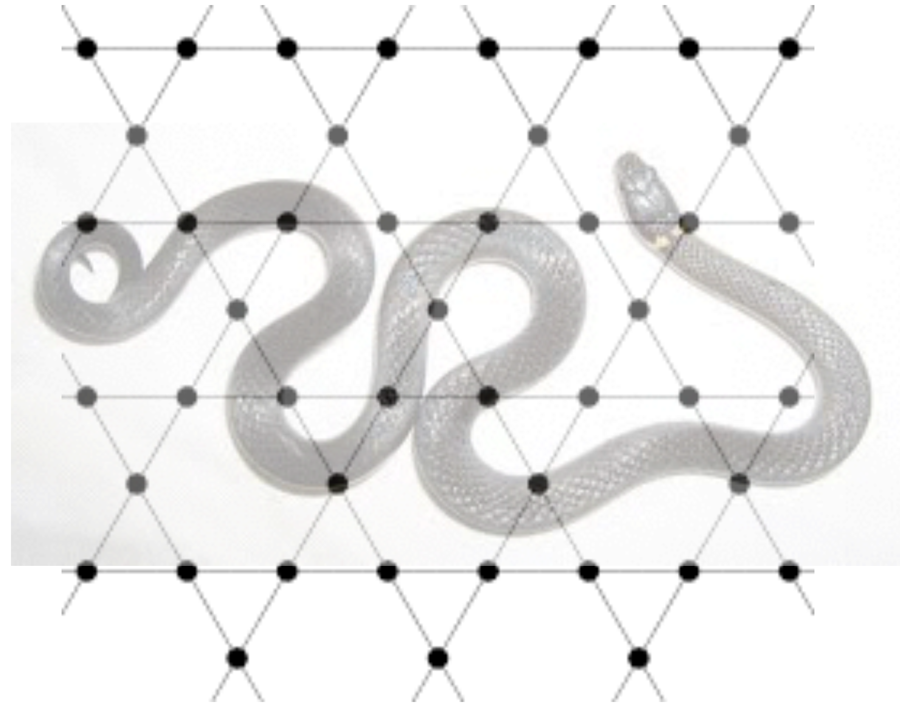
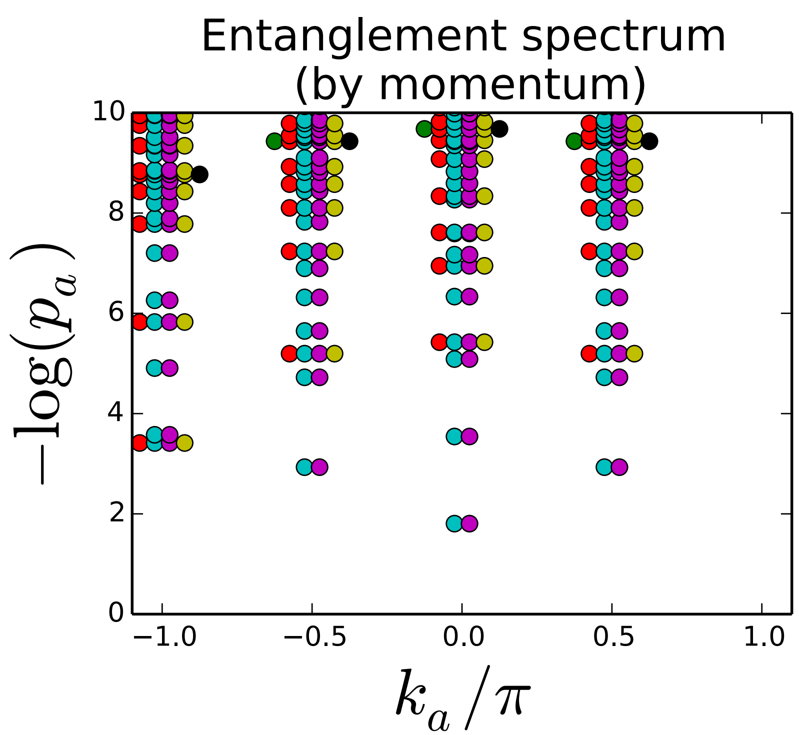
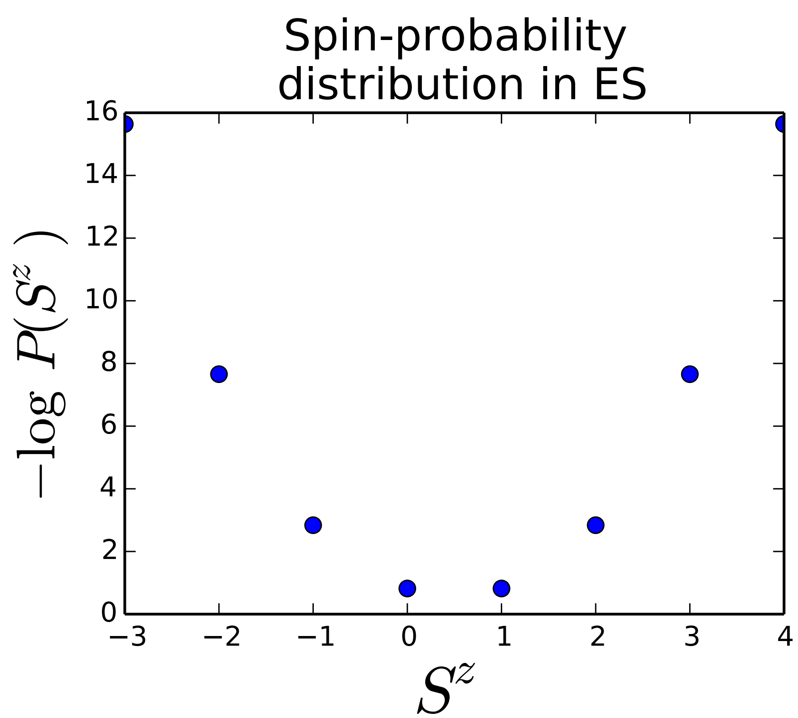
Station Q







MagLab Theory Winter School 2015

review of MPS and many of the figures in this talk: Kjäll, Zaletel, Mong, Bardarson & Pollmann 2012

Outline

- Day 1: Introduction to tensor network numerics
  - Entanglement and the Schmidt decomposition
  - 1D: the matrix product state ansatz
    - DMRG
  - 2D: the ‘tensor network’ ansatz
    - Dimer & RVB wavefunctions
    - Open problems
- Day 2: Entanglement spectroscopy: detecting emergent anyons in numerics

Goal:

Find an unbiased method for numerically calculating the low energy properties of any local (perhaps frustrated) quantum Hamiltonian in a time which is polynomial in the system size (or independent of system size with translation invariance).

Some amusing cold water first:

[David Pérez-Garcia, Toby Cubitt & Michael Wolf]

Our result (informal statement)

Problem (Spectral Gap):
Input: nearest-neighbor interaction h
Output: decide if H has a gap or not.

Theorem:
The Spectral Gap problem is undecidable.

⇕

There is no algorithm that on input h decides it

Corollary: There exist nearest neighbor interactions for which the existence or absence of gap cannot be proven within the axioms of mathematics.

[from David Pérez-Garcia]

The storage problem

Classical:

$$\overbrace{\uparrow\uparrow\downarrow\cdots\uparrow}^{L}$$
$$\Downarrow$$
$$110\cdots1 \quad\Rightarrow\quad S = \log_2(2^L) = L$$

Information **linear** in system size

Quantum:

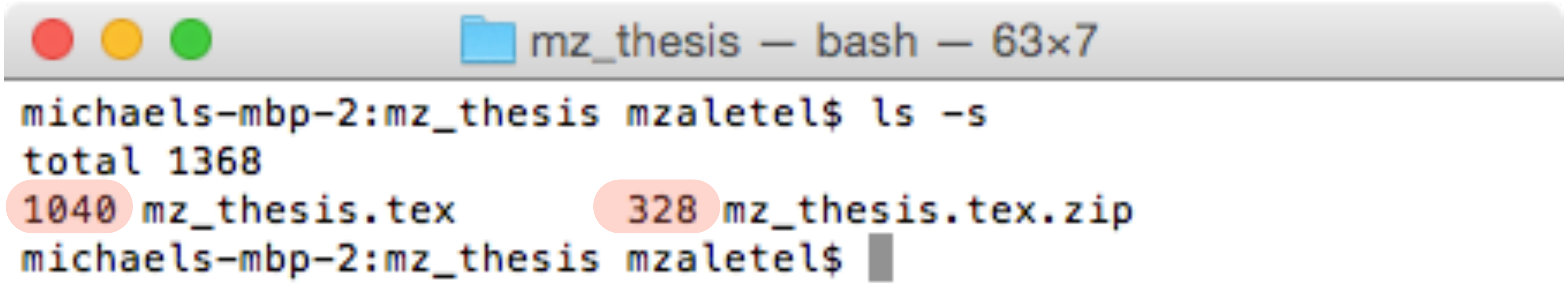
$$|\Psi\rangle = \Psi_0\,|\uparrow\uparrow\cdots\rangle + \Psi_1\,|\downarrow\uparrow\cdots\rangle + \Psi_2\,|\uparrow\downarrow\cdots\rangle + \cdots$$

(floating points)

$$\{\Psi_i\} \quad\Rightarrow\quad S \sim 4\cdot 8\cdot 2^L$$

Information **exponential** in system size (limits exact-diagonalization)

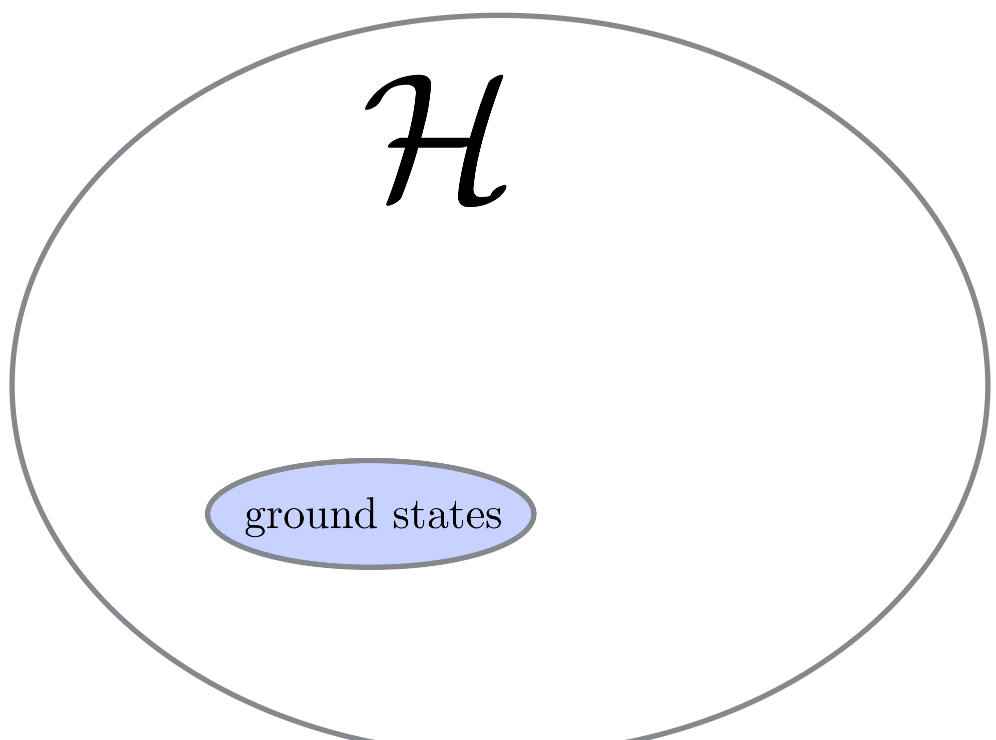
Quantum compression?



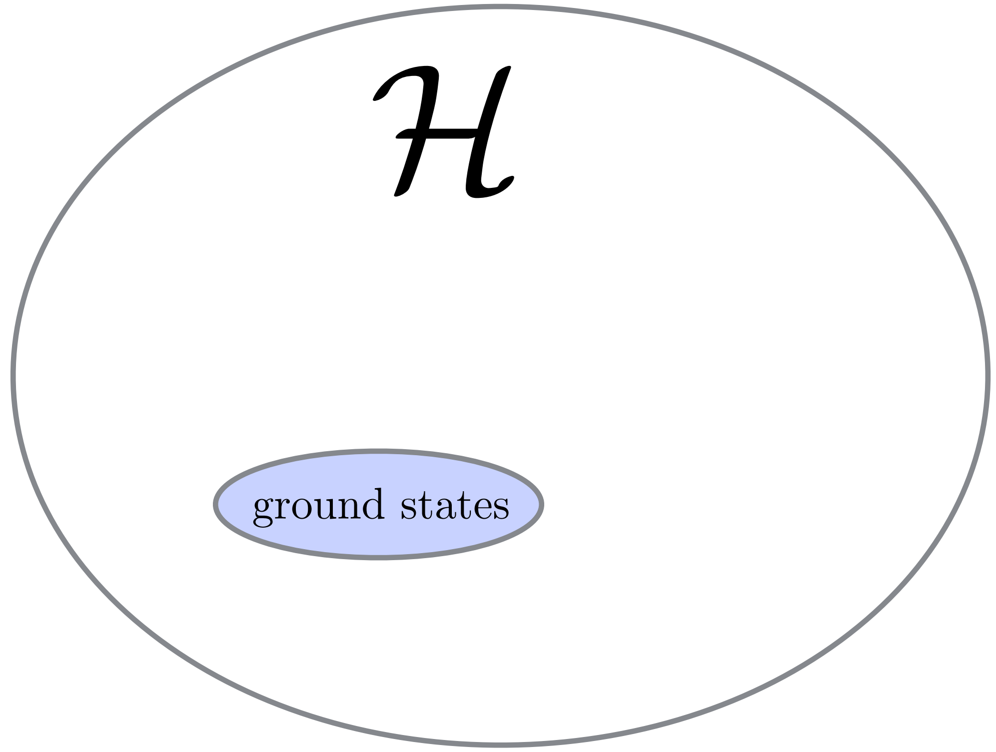
My thesis actually contains surprisingly little information…

We are interested in states which have **low energy** for **local** Hamiltonians

How big is the important space?



Hokey Estimate I



Parameterize space of ground states via space of local Hamiltonians:

$$\hat{H} = \sum_{i=1}^{L} \hat{H}_i$$

Finite info for each $\hat{H}_i$, so

$$S \propto L$$

Estimate II: the ‘convenient illusion of Hilbert space’

from Poulin, et al., 2011

The Setup:

Start in a product state: $|t = 0\rangle = \otimes_{n=1}^{L} |\uparrow\rangle$

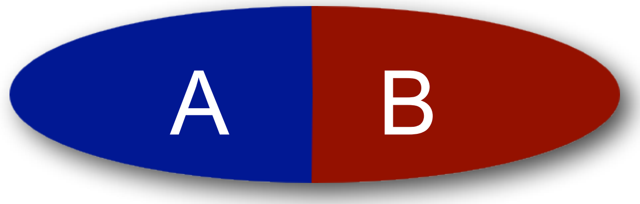
Time evolve under an *arbitrary* k-body Hamiltonian: $\hat{H}(t)$

After any time $t \sim \mathrm{poly}(L)$ we can only access a fraction

$$\frac{\mathrm{Vol}(\{|t\rangle\})}{\mathrm{Vol}(\mathcal{H})} < L^L \epsilon^{2^L}, \epsilon < 1$$

of the many-body Hilbert space

Schmidt decomposition



$$\mathcal{H} = \mathcal{H}_L \otimes \mathcal{H}_R$$

$$|\Psi\rangle = \sum_{i,j=1}^{D_L, D_R} \Psi_{ij} \, |i\rangle_L \, |j\rangle_R \qquad D_L \times D_R \text{ \#s}$$

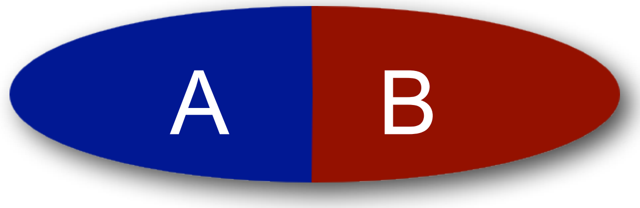
If A & B are uncorrelated (not-entangled), there is a special basis in which

$$|\Psi\rangle = |\alpha\rangle_L \, |\alpha\rangle_R \qquad D_L + D_R \text{ \#s}$$

More generally, there is a special basis - the **Schmidt basis** in which

$$|\Psi\rangle = \sum_{\alpha=1}^{\chi} \lambda_\alpha \, |\alpha\rangle_L \, |\alpha\rangle_R \qquad \chi \times (D_L + D_R) \text{ \#s}$$

Schmidt compression



$$\mathcal{H} = \mathcal{H}_L \otimes \mathcal{H}_R$$

$$|\Psi\rangle = \sum_{\alpha=1}^{\chi} \lambda_\alpha \, |\alpha\rangle_L \, |\alpha\rangle_R$$

$\chi \times (D_L + D_R)$ #s

The "entanglement entropy:"

$$\sum_\alpha \lambda_\alpha^2 = 1, \quad S_E = -\sum_\alpha \lambda_\alpha^2 \log(\lambda_\alpha^2)$$

When

$$e^{S_E} \ll D_L$$

we can keep only important contributions and compress the state!

A qubit of entanglement

$$|\psi\rangle = \frac{1}{2}\Big(|\uparrow\rangle_A + |\downarrow\rangle_A\Big)\Big(|\uparrow\rangle_B + |\downarrow\rangle_B\Big) \quad \Rightarrow S = 0$$

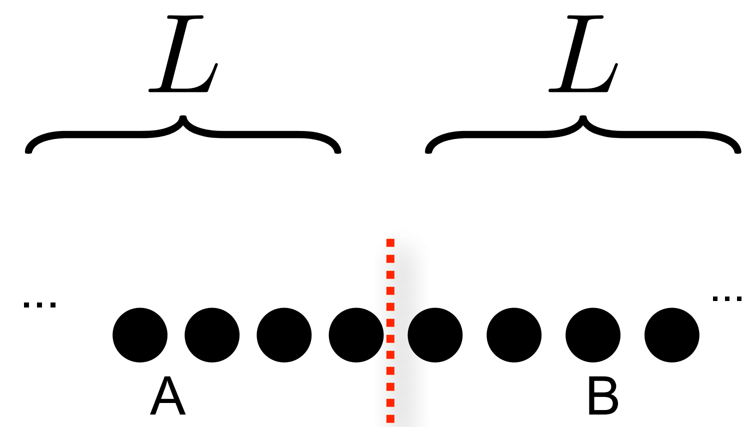
$$|\psi\rangle = \frac{1}{\sqrt{2}}\Big(|\uparrow\rangle_A|\downarrow\rangle_B + |\downarrow\rangle_A|\uparrow\rangle_B\Big) \quad \Rightarrow S = \log 2$$

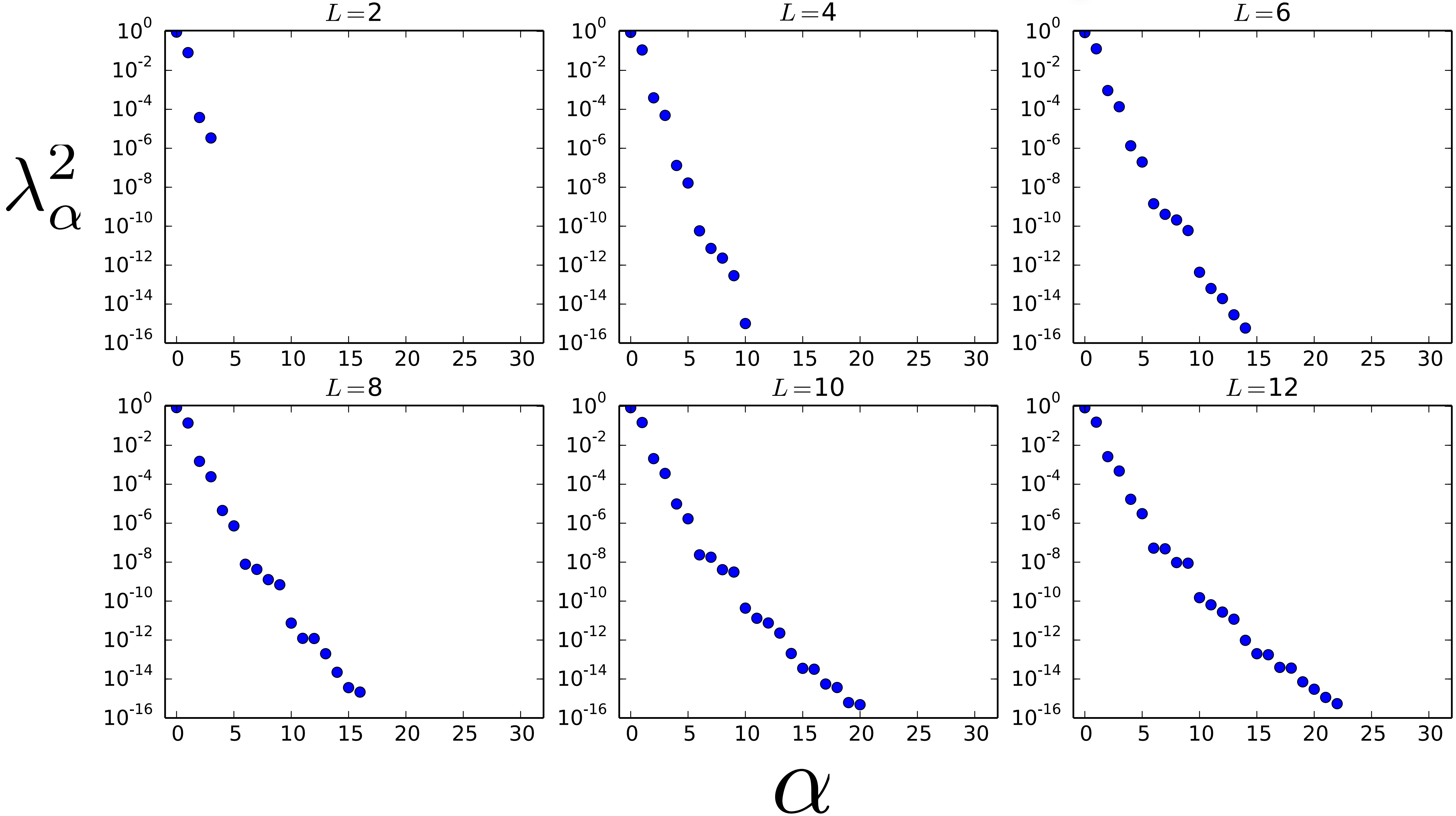
$$S = -\sum \lambda_\gamma^2 \log \lambda_\gamma^2$$

Example: 1D transverse field Ising model

Slightly perturbed from QCP

Cut length 2L chain in half:

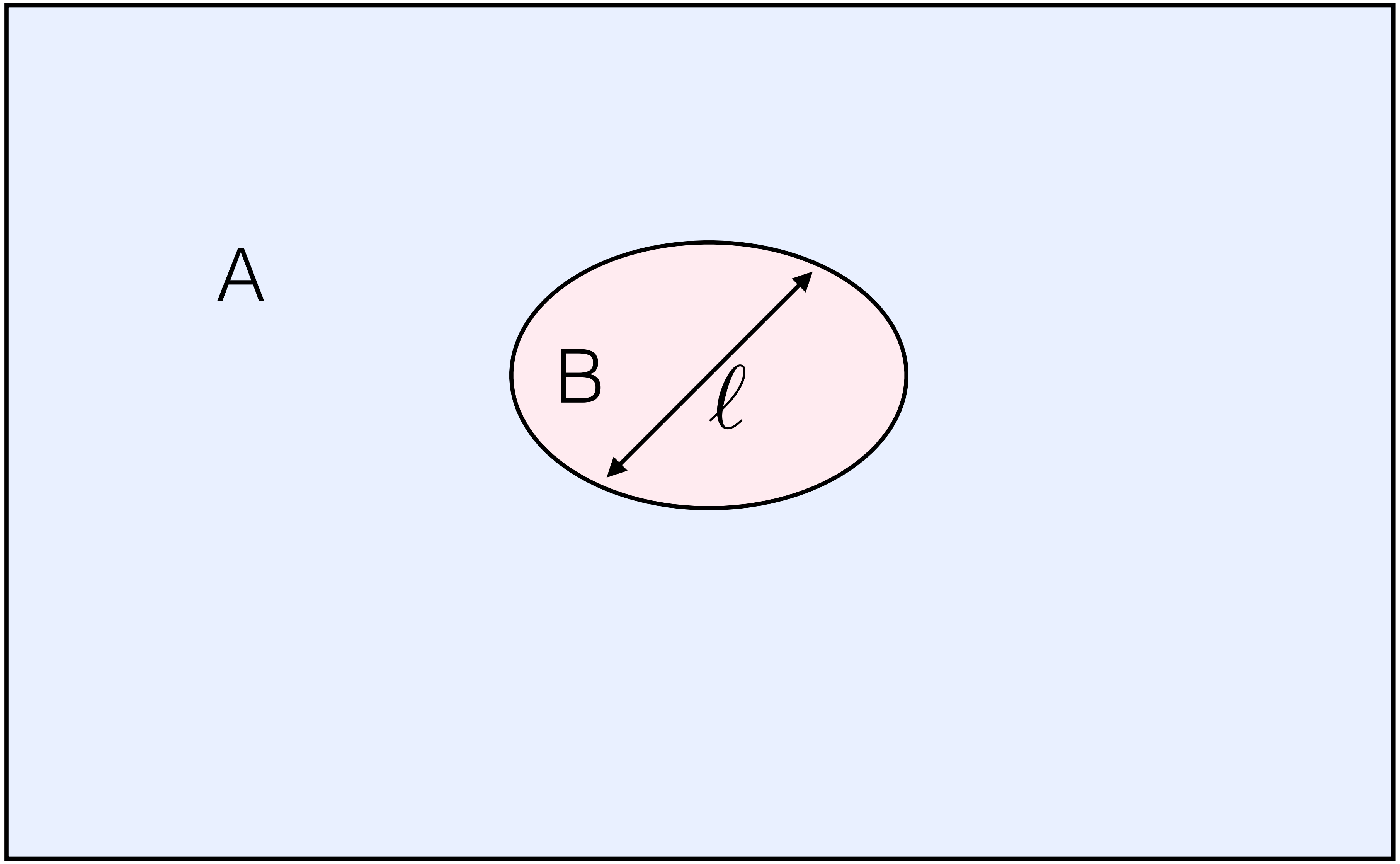




The Area Law

[Srednicki]

Ground states: $S_E(\ell) \sim \ell^{D-1}$

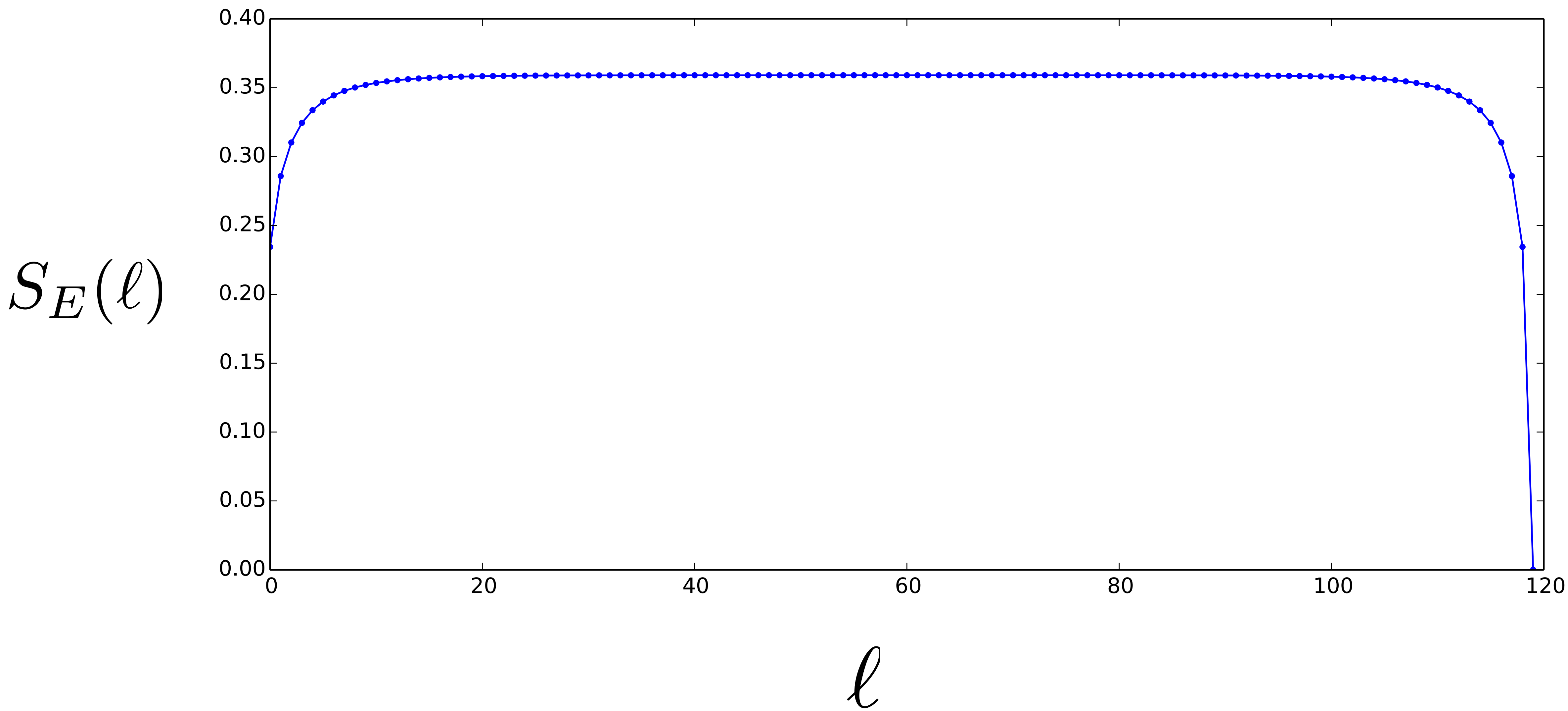


Proven in 1D for gapped states [Hastings 2005]

Mild violations for certain critical systems (1+1 CFT, Fermi surfaces…)

Volume law expected at finite energy density (eigenstate thermalization)

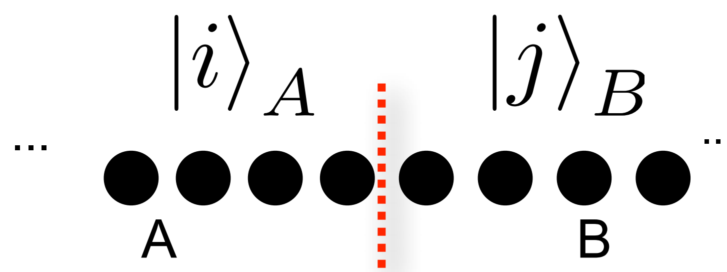
The Area Law: $S_E(\ell) \sim \ell^{D-1}$ for D = 1 spin chain



The MPS Ansatz

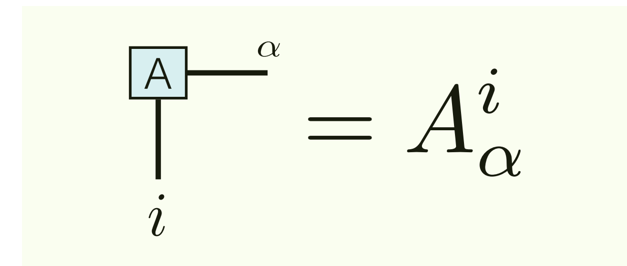
[Fannes et al. 1992; Östlund & Rommer 1995]

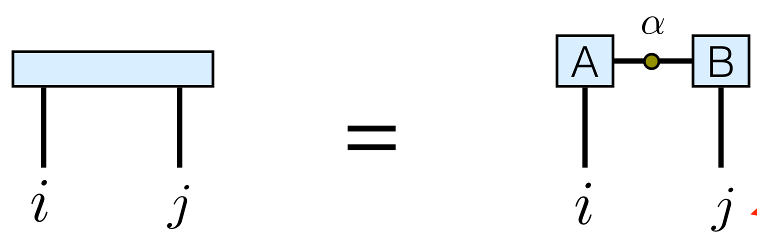
Step 1:
cut state in half



$$\Psi_{ij} = \sum_{\alpha}^{\chi} A^i_\alpha \lambda_\alpha B^j_\alpha$$

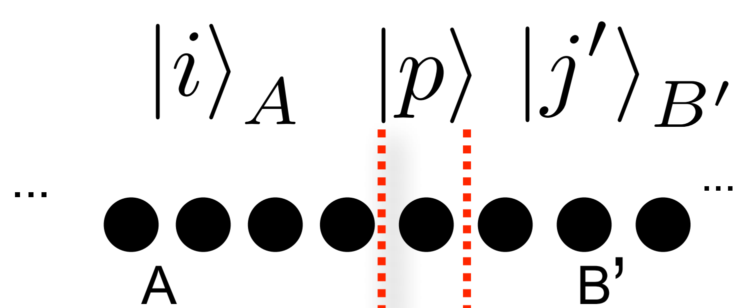
Penrose graphical notation:





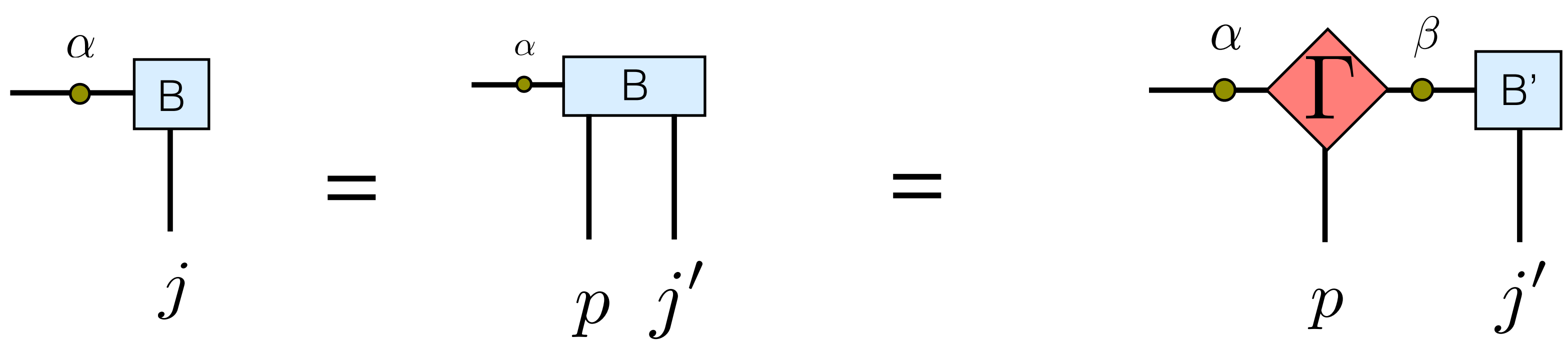
Still giant Hilbert space: half the chain

Step 2:
Split off 1 site from the right

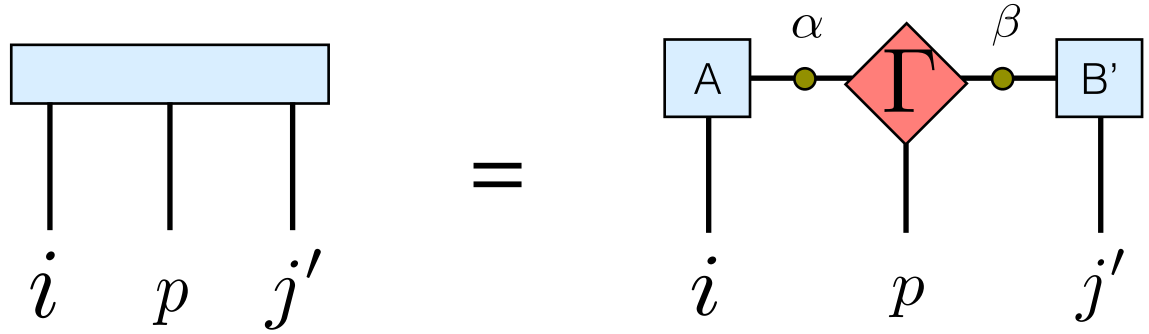


$$B^j_\alpha = \sum_{\beta=1}^{\chi} \Gamma^p_{\alpha\beta} \lambda'_\beta B'^{j'}_\beta$$

Schmidt coefficients 1 bond to right

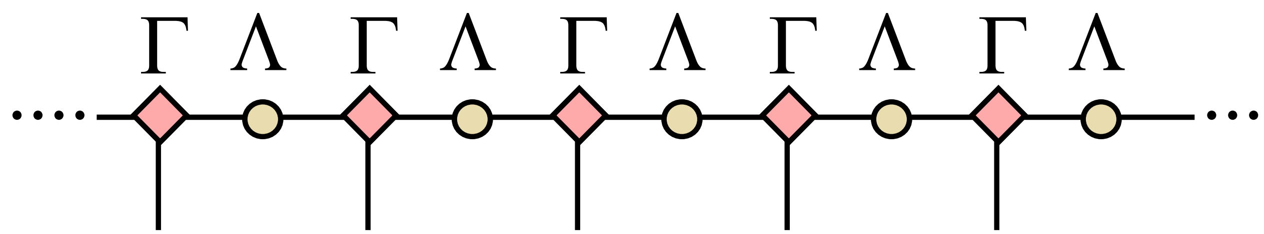


The MPS Ansatz



Step 3: $\Psi_{i_1 i_2 \cdots}$ =

repeat!



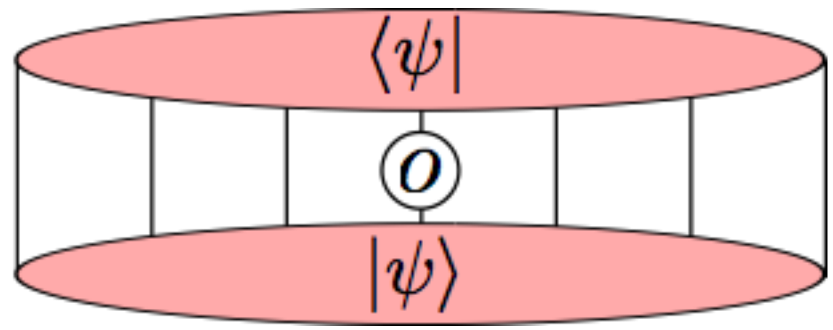
$$= \sum_{\alpha_1, \alpha_2, \ldots, \alpha_N}^{\chi} \Gamma^{[1] i_1}_{\alpha_1} \lambda^{[1]}_{\alpha_1} \Gamma^{[2] i_2}_{\alpha_1 \alpha_2} \lambda^{[2]}_{\alpha_2} \ldots \lambda^{[N-1]}_{\alpha_{N-1}} \Gamma^{[N] i_N}_{\alpha_N}$$

Compressed L-site wavefunction into L $d \cdot \chi \cdot \chi$ tensors

MPS: Computing observables

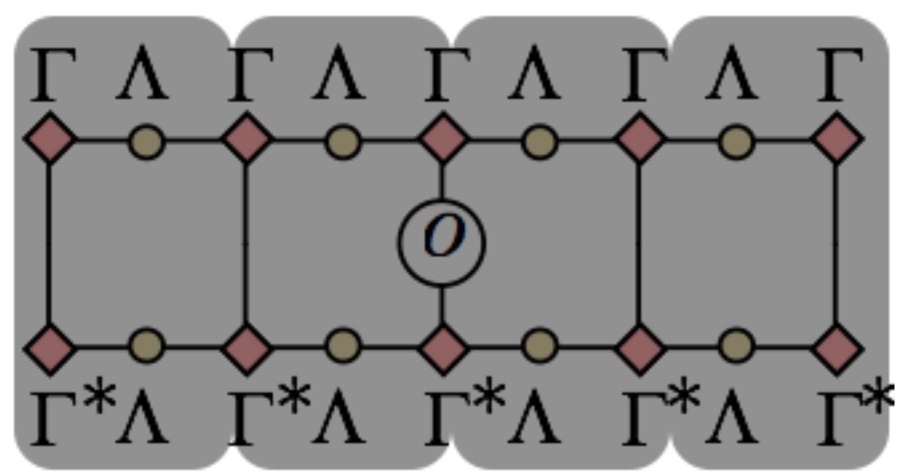
Exact diagonalization: $\sim \mathcal{O}(e^{\alpha L})$

$$\langle\psi|O|\psi\rangle =$$



MPS: $\sim \mathcal{O}(\chi^3)$

$$\langle\psi|O|\psi\rangle =$$

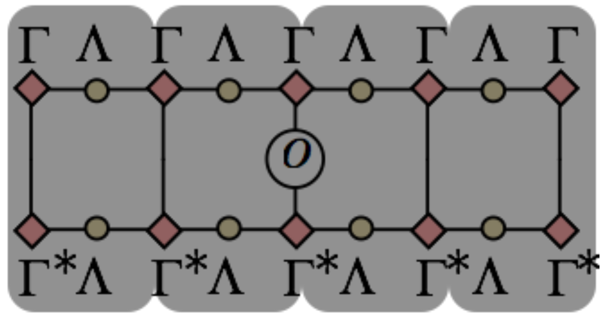


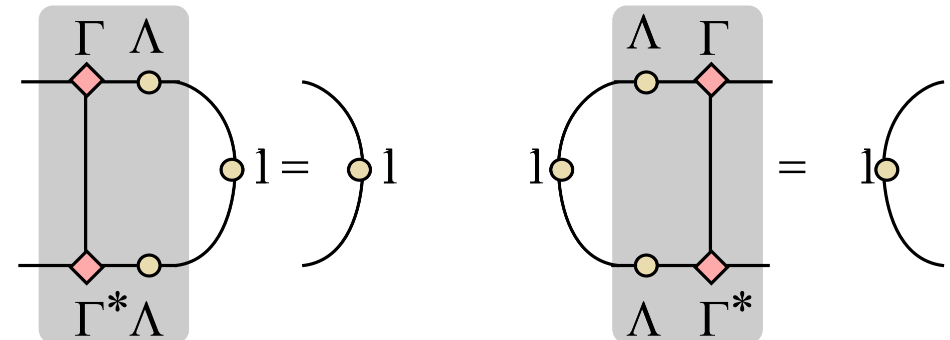
MPS: Computing observables

Simplification Rule:

[Vidal 2007]

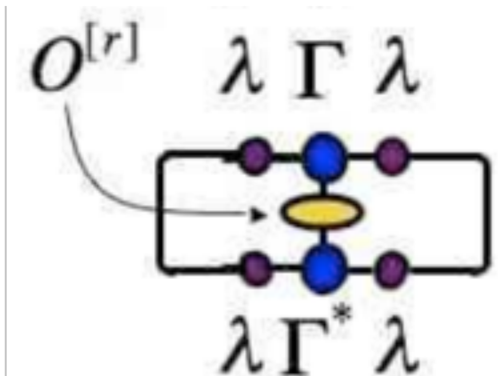
$\langle\psi|O|\psi\rangle =$





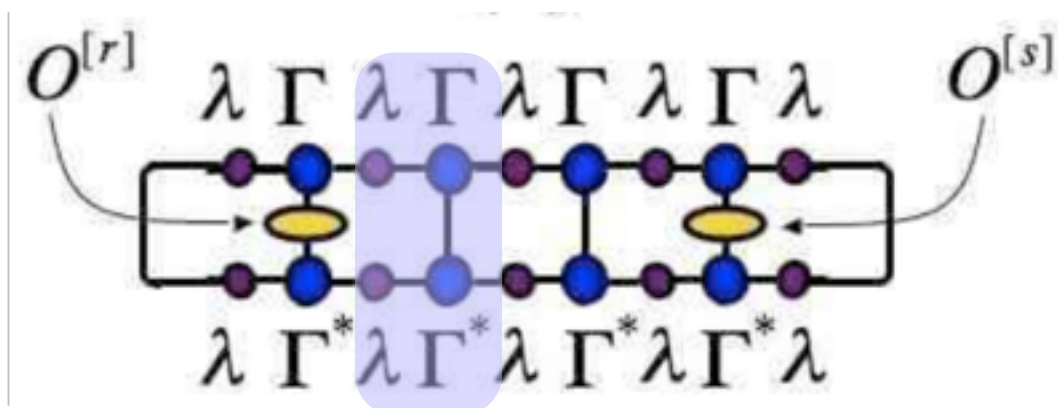
- Local expectation values

$$\langle\psi|O^{[r]}|\psi\rangle =$$



- Correlation functions

$$\langle\psi|O^{[r]}O^{[l]}|\psi\rangle =$$



- Correlation length: Second largest eigenvalue of the transfer matrix

DMRG : Density Matrix Renormalization Group

[White 1992; McCullough 2008]

Given $\hat{H}$, how do we find good a MPS approximations to the g.s.?

MPS: $\{\Gamma^{[j]}\lambda^{[j]}\} \to |\Psi\rangle$

Minimize $E(\Gamma^{[j]}\lambda^{[j]}) = \frac{\langle\Psi|\,\hat{H}\,|\Psi\rangle}{\langle\Psi|\Psi\rangle}$

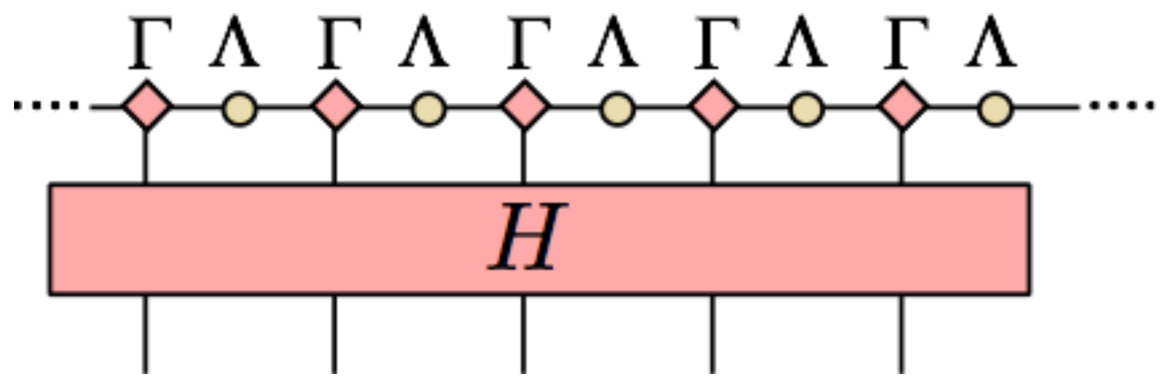
Non-linear minimization problem

Strategy:

1. Hold all tensors fixed but those at site j
2. Solve *quadratic* problem at site j
3. Move on to site j + 1; repeat

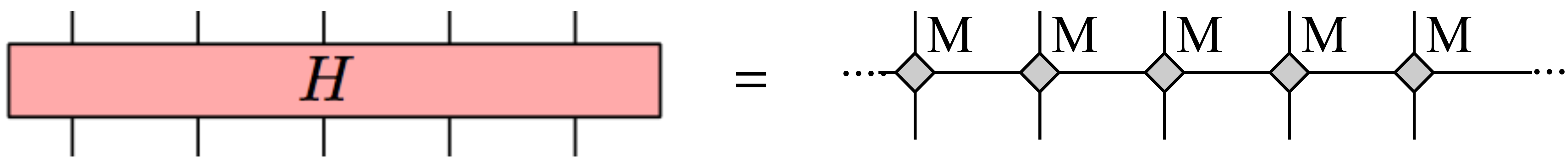
Review of algorithm: Kjäll, Zaletel, Bardarson, Mong & Pollmann 2012, 1212.6255

$\langle\Psi|\,\hat{H}\,|\Psi\rangle$



Screws up MPS structure!

For local-ish Hamiltonians, generalize MPS to Matrix Product Operator (MPO)



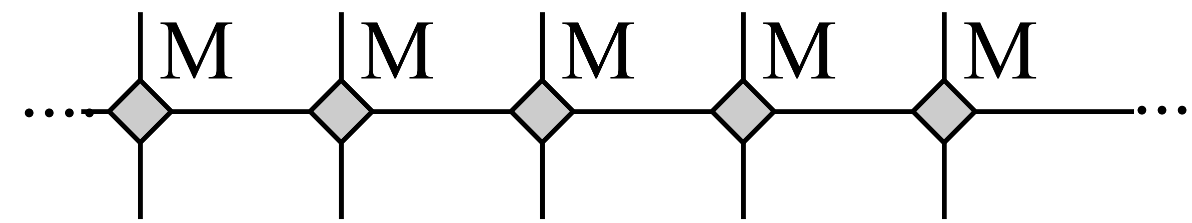
Transverse Field Ising: $M = \begin{pmatrix} 1 & 0 & 0 \\ \sigma_z & 0 & 0 \\ g\sigma_x & \sigma_z & 1 \end{pmatrix}$

Review of algorithm: Kjäll, Zaletel, Bardarson, Mong & Pollmann 2012, 1212.6255

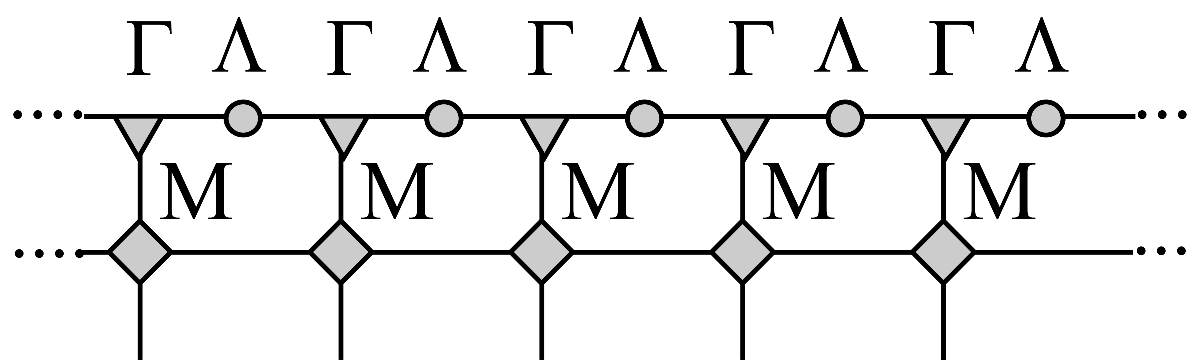
[Verstraete, Porras & Cirac 2004; Murg 2008]

$$\langle\Psi|\,\hat{H}\,|\Psi\rangle$$

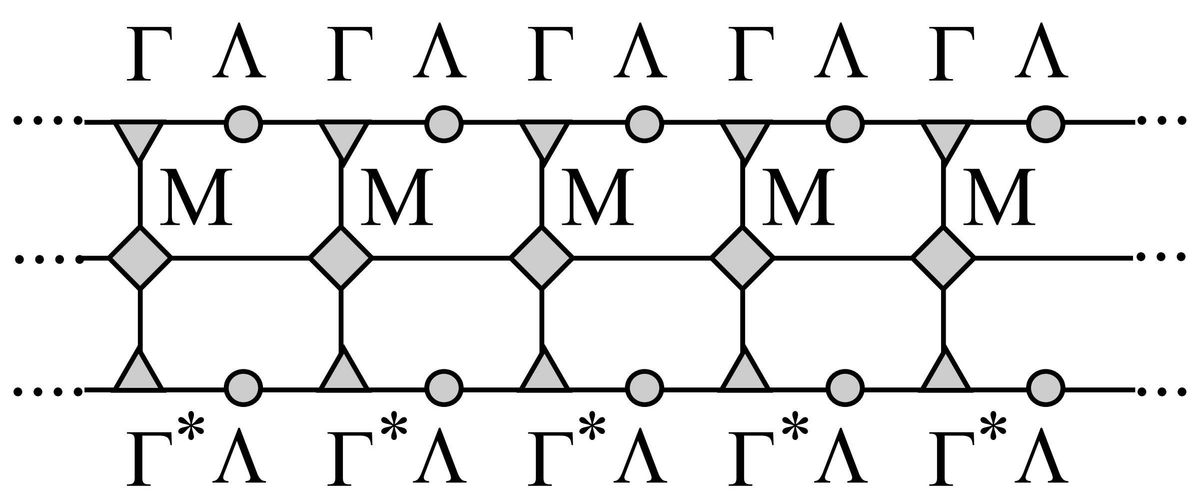
$$\hat{H}$$



$$\hat{H}\,|\Psi\rangle$$



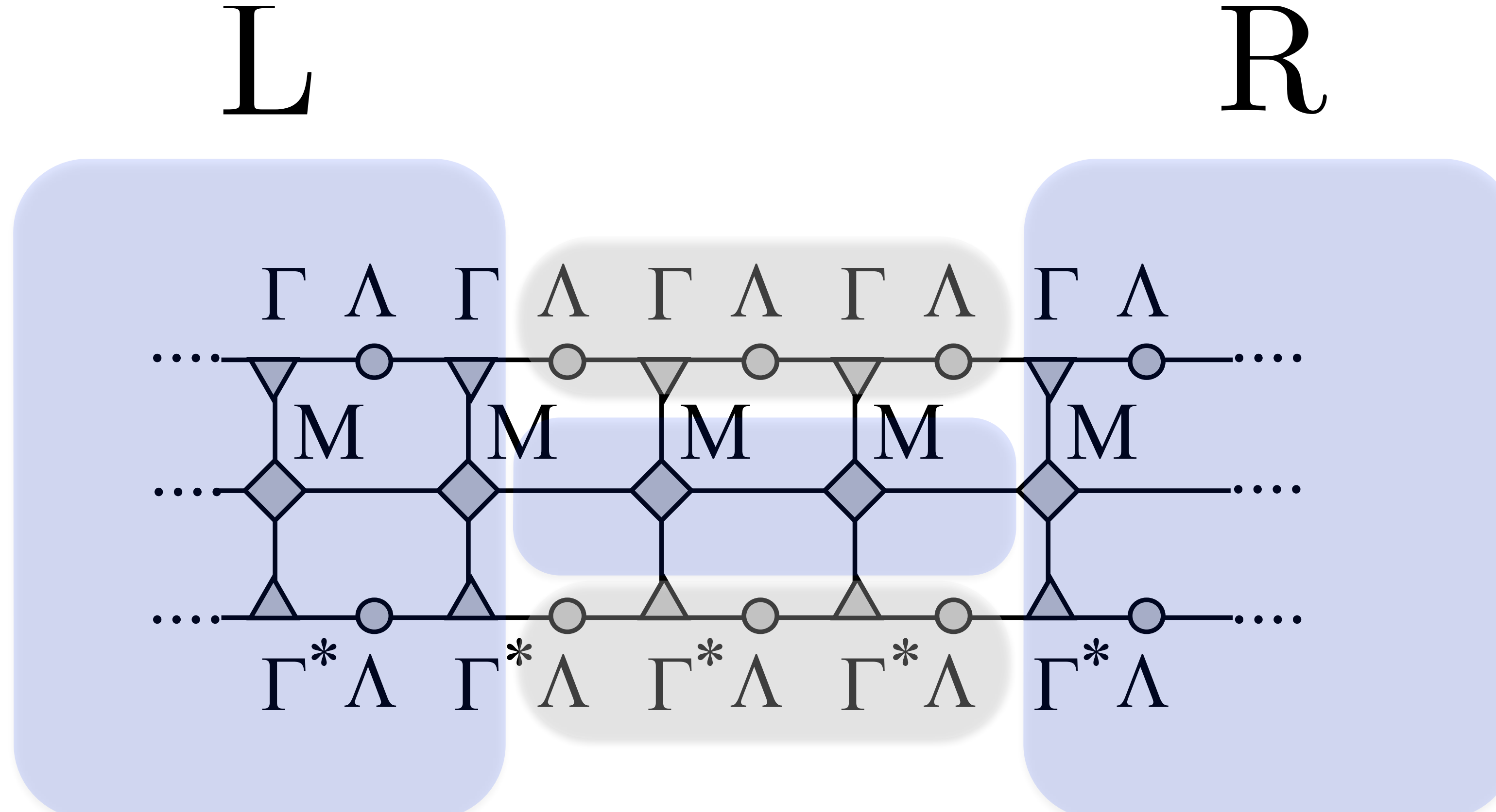
$$\langle\Psi|\,\hat{H}\,|\Psi\rangle$$

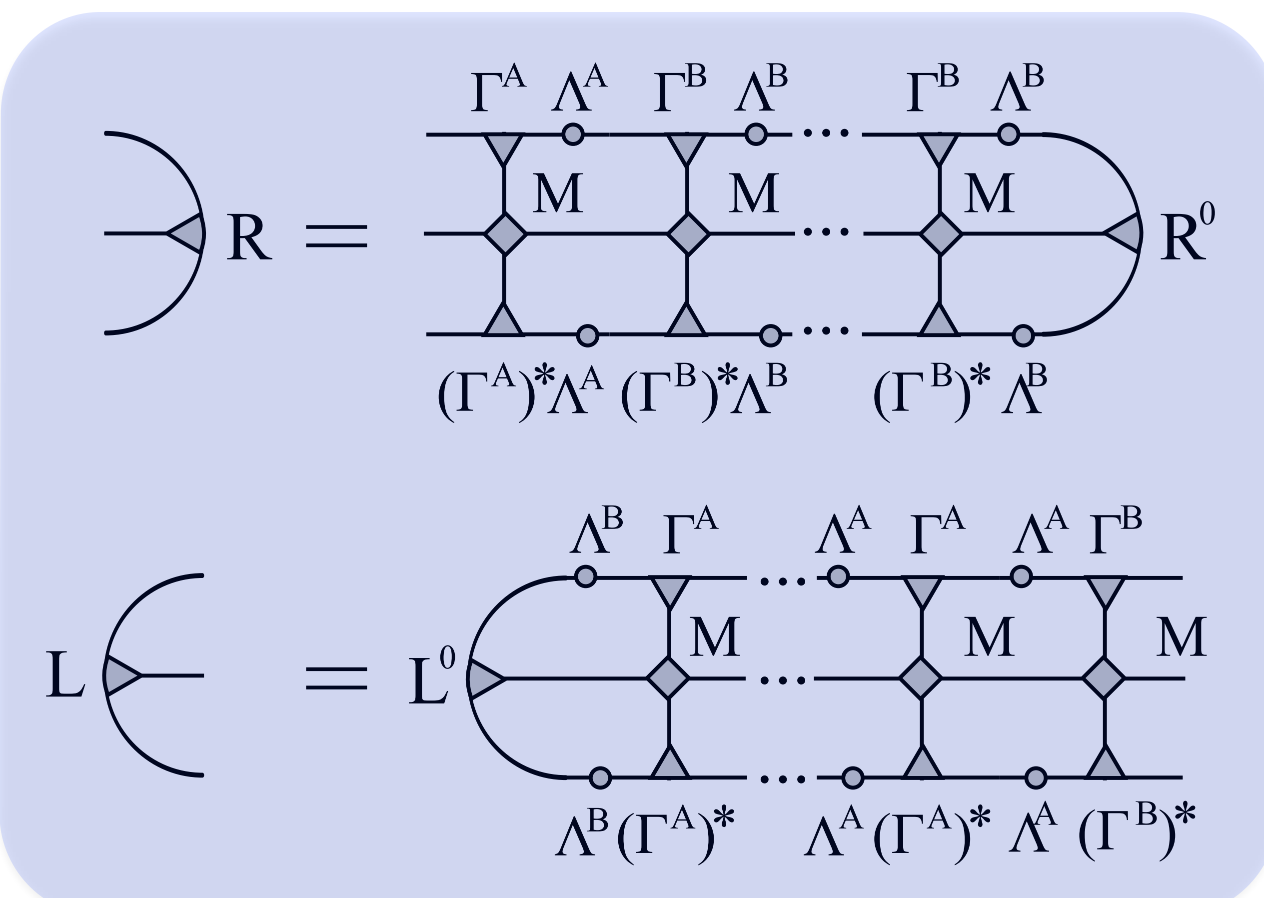


Review of algorithm: Kjäll, Zaletel, Bardarson, Mong & Pollmann 2012, 1212.6255

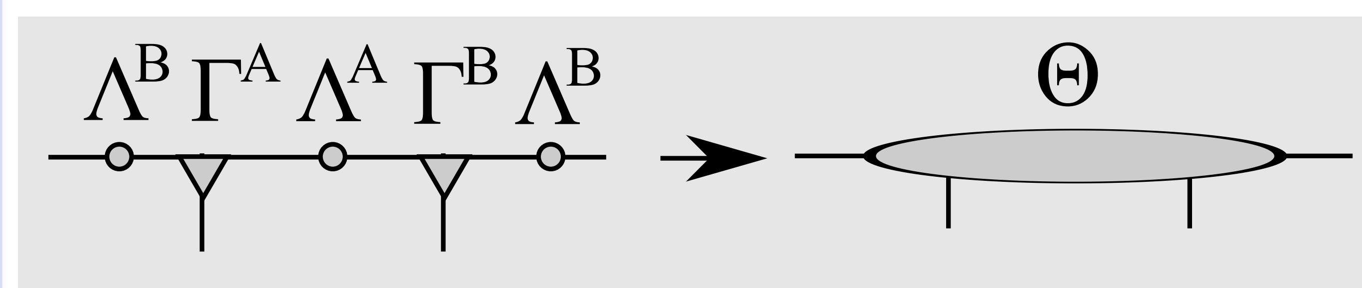
Focus on two sites

$$E = \langle \Theta | H_{eff} | \Theta \rangle =$$





Variational Wavefunction:

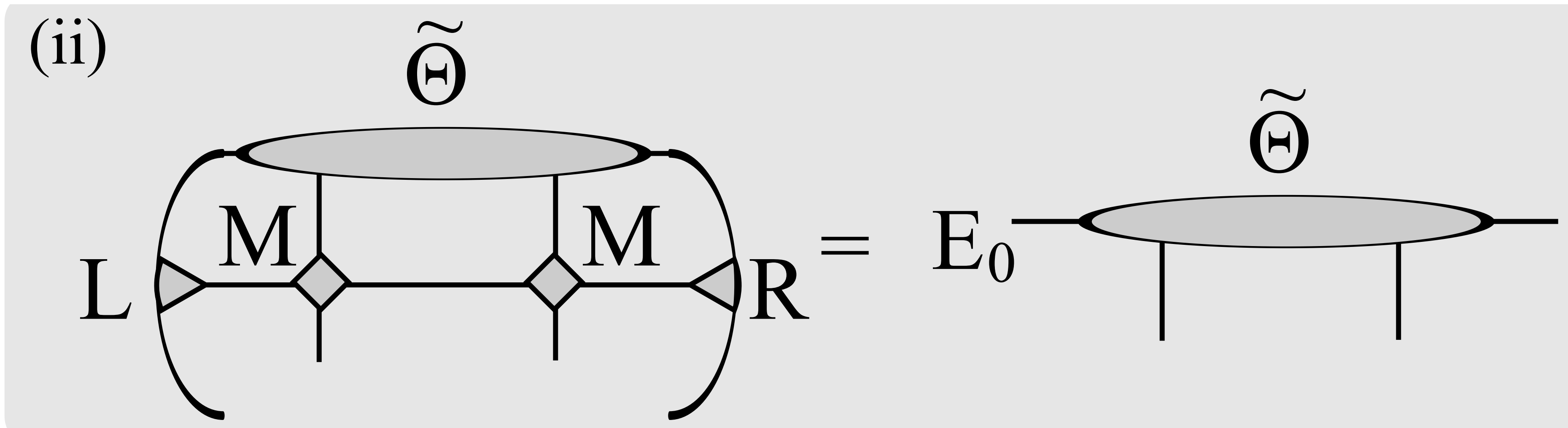


Orthonormal basis for

2 sites + L / R Schmidt states

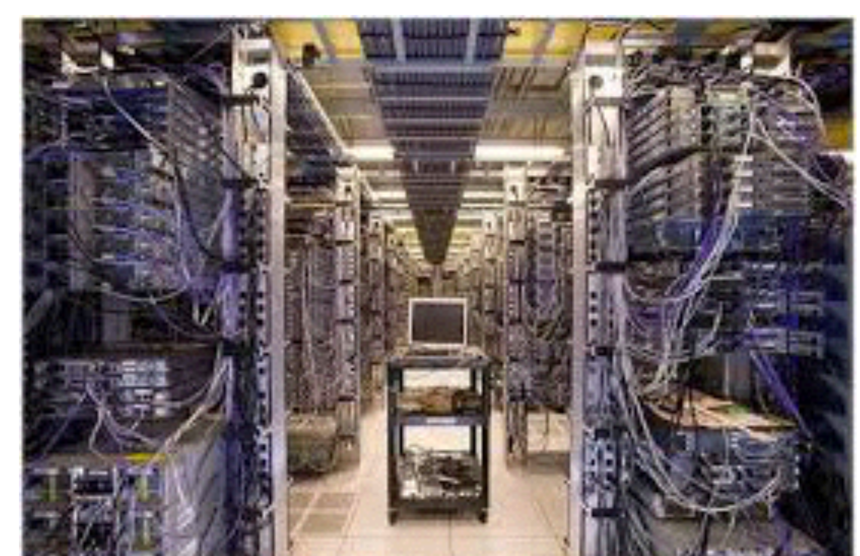
Review of algorithm: Kjäll, Zaletel, Bardarson, Mong & Pollmann 2012, 1212.6255

Lower the energy by finding the ground state of effective Hamiltonian (Lanczos, etc.):



$$H_{eff}\,|\tilde{\Theta}\rangle = E_0\,|\tilde{\Theta}\rangle$$

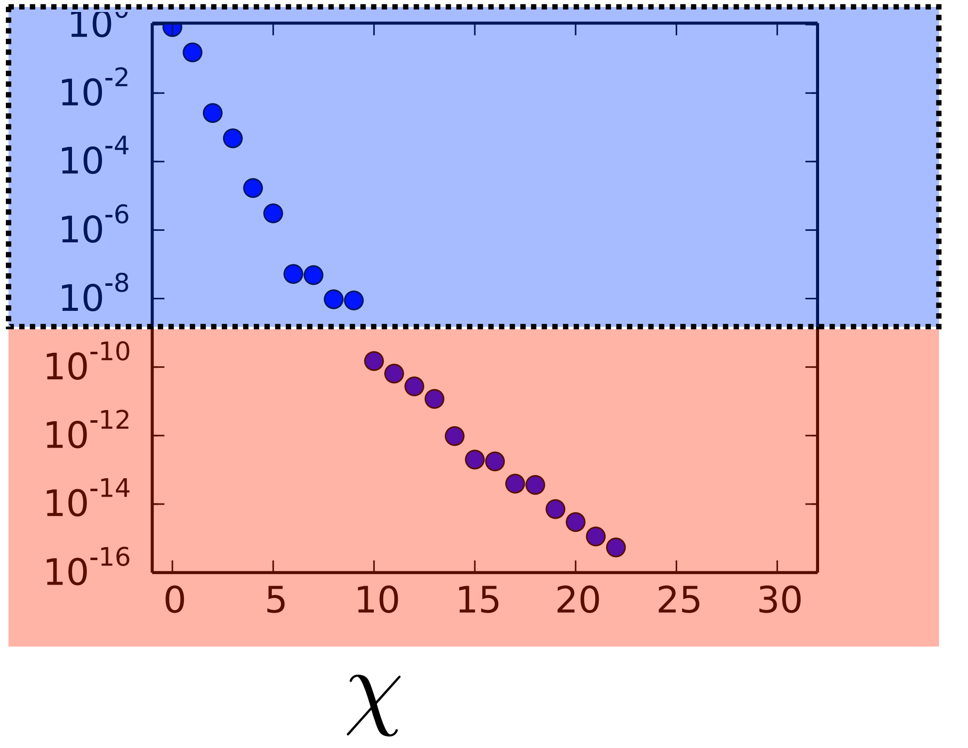
This is where you burn CPU hours:

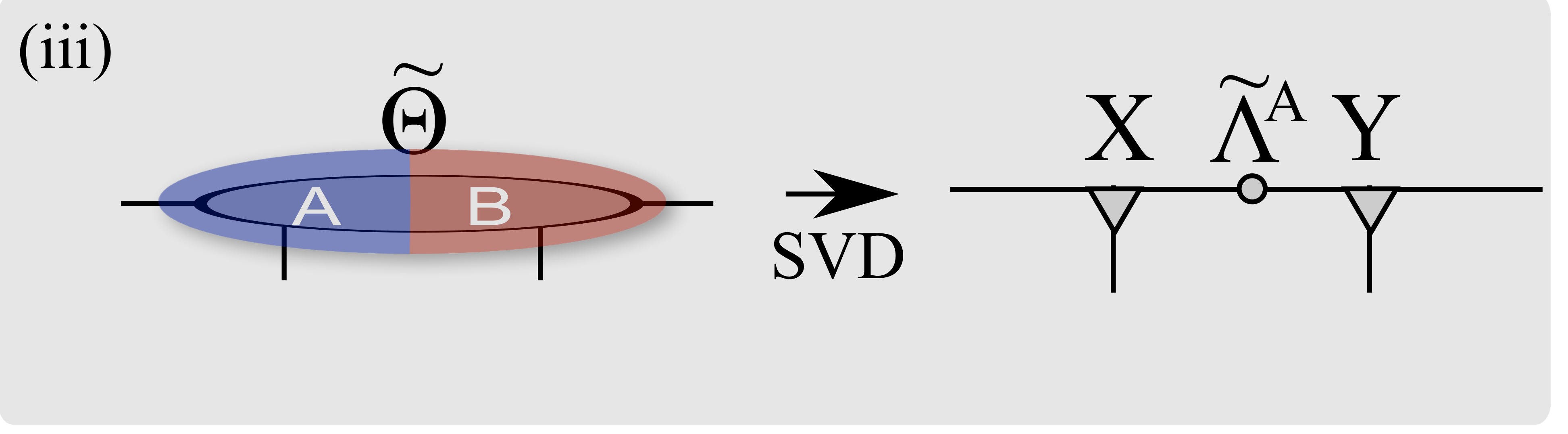


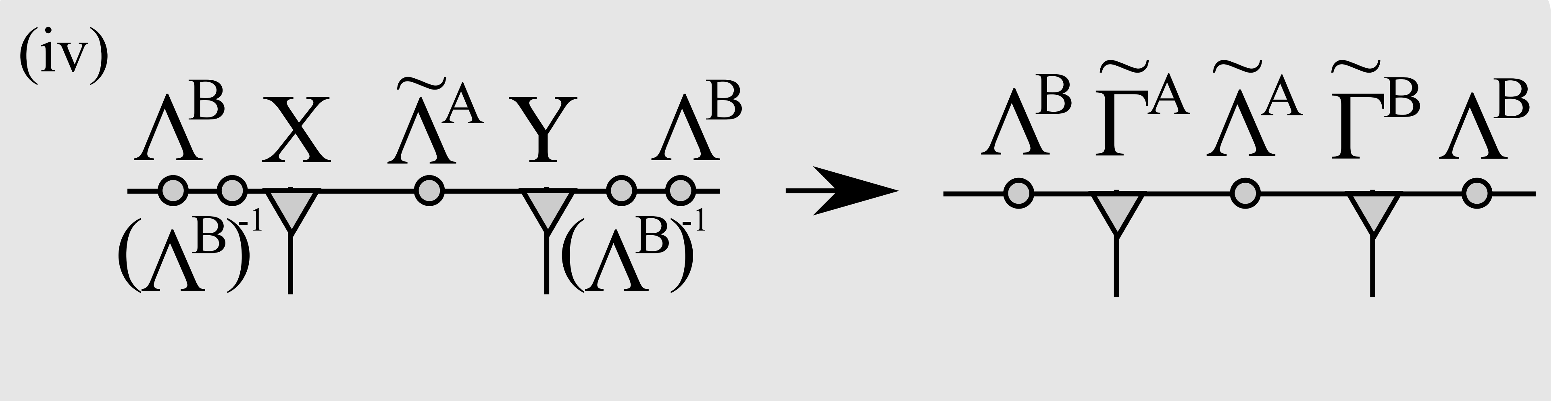
Review of algorithm: Kjäll, Zaletel, Bardarson, Mong & Pollmann 2012, 1212.6255

Bring ansatz back to MPS form

$$|\tilde{\Theta}\rangle = \sum_{\alpha=1}^{\chi} |\alpha\rangle_A \, \tilde{\Lambda}_\alpha \, |\alpha\rangle_B$$

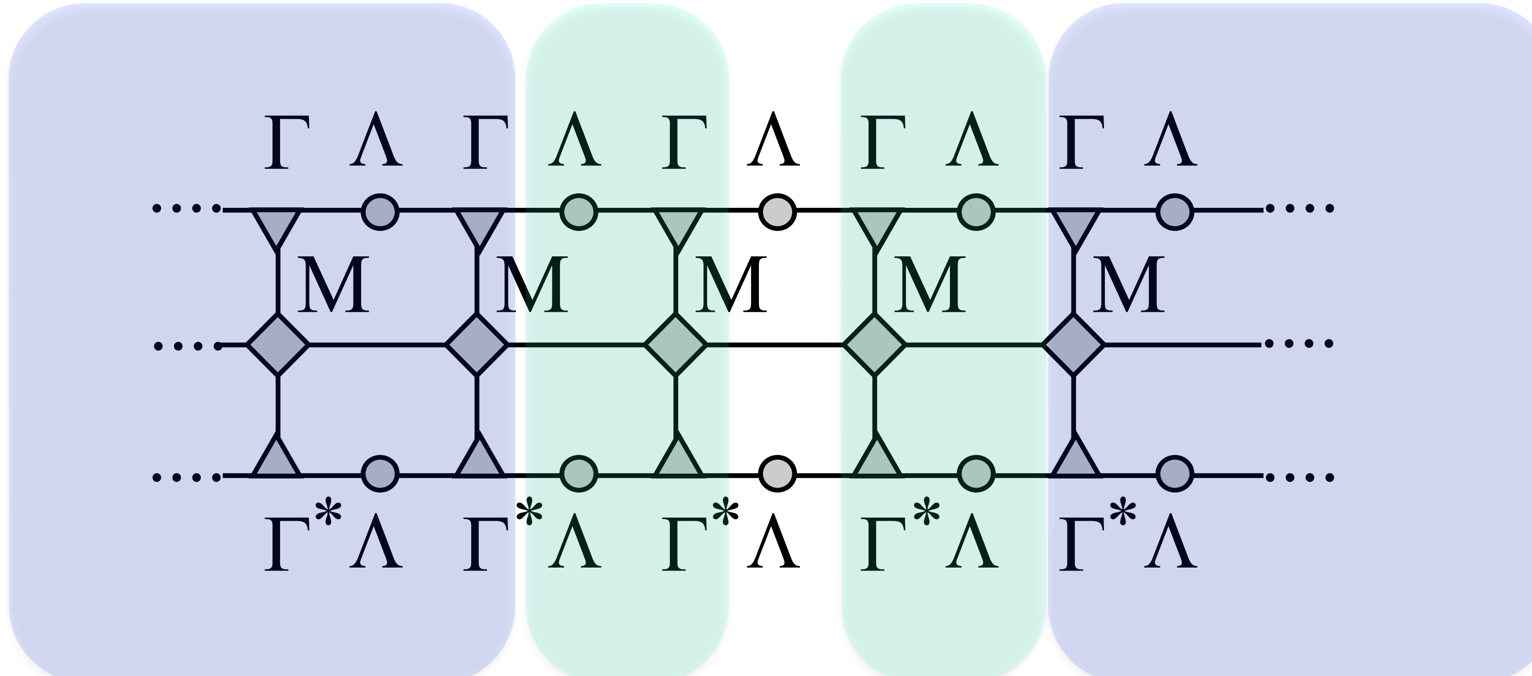


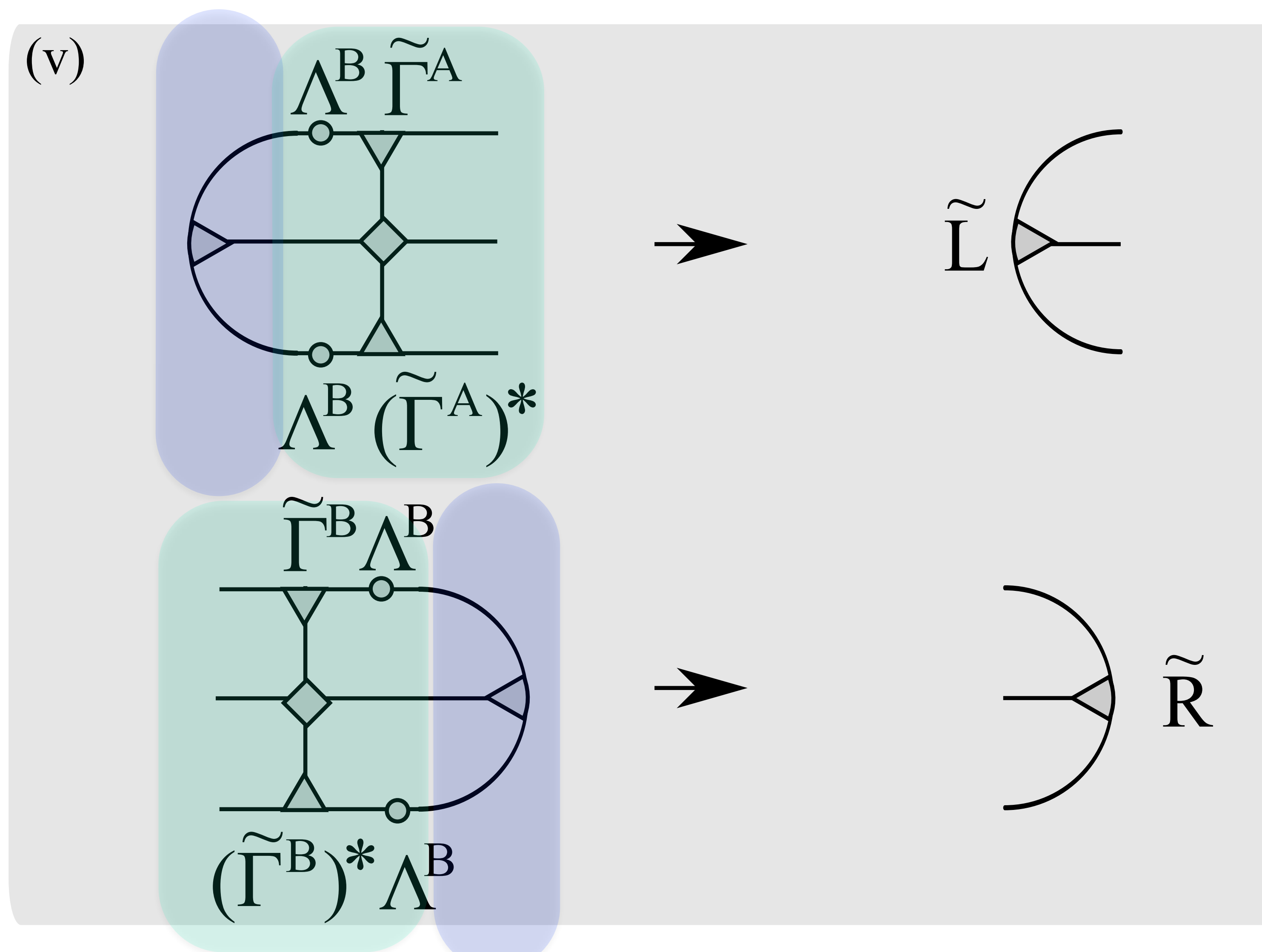




Review of algorithm: Kjäll, Zaletel, Bardarson, Mong & Pollmann 2012, 1212.6255

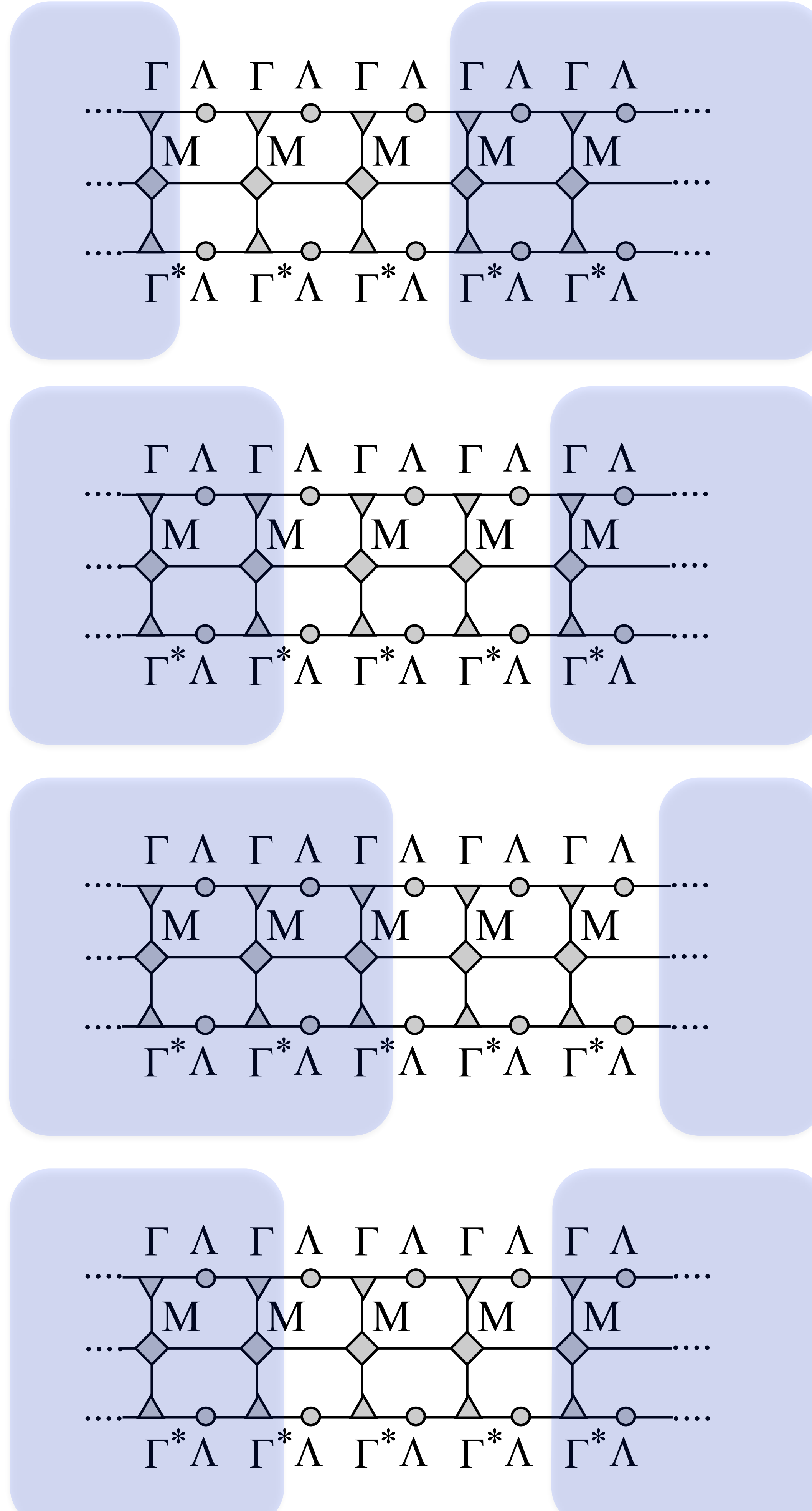
Update L / R environments





Review of algorithm: Kjäll, Zaletel, Bardarson, Mong & Pollmann 2012, 1212.6255

“Sweep” until convergence



Review of algorithm: Kjäll, Zaletel, Bardarson, Mong & Pollmann 2012, 1212.6255

Comments

Algorithm works *unchanged* on an infinitely long system with periodic unit cell: "iDMRG"

[McCullough 2008]

Complexity: length / unit cell = **L**

$$\chi \sim e^{S_E}$$

(holding Hamiltonian fixed)

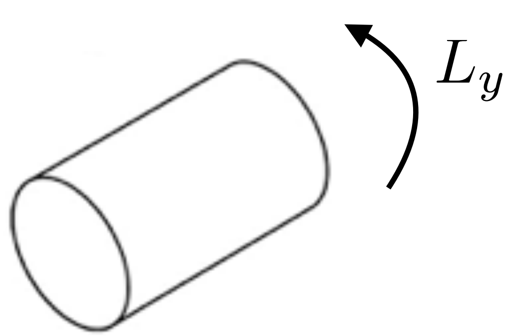
CPU: $L\chi^3$

RAM: $L\chi^2$

1D gapped: $S_E \sim \mathrm{const}$

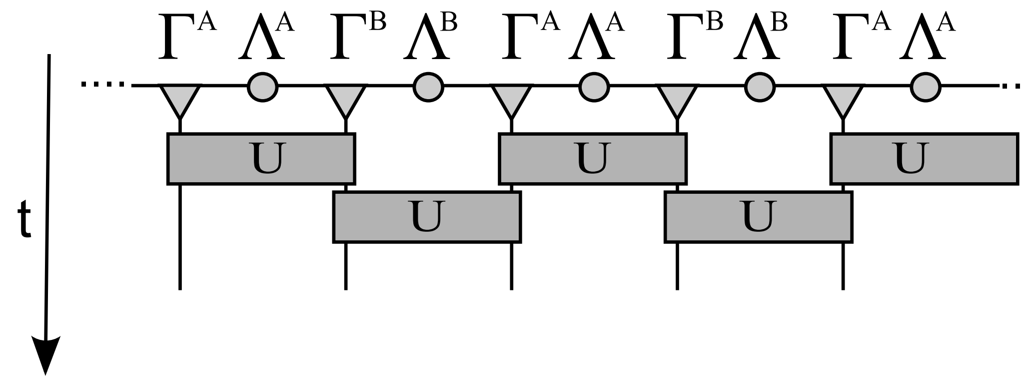
1D CFT: $S_E \sim \frac{c}{6}\log(\xi/a)$

2D: $S_E \sim L_y$

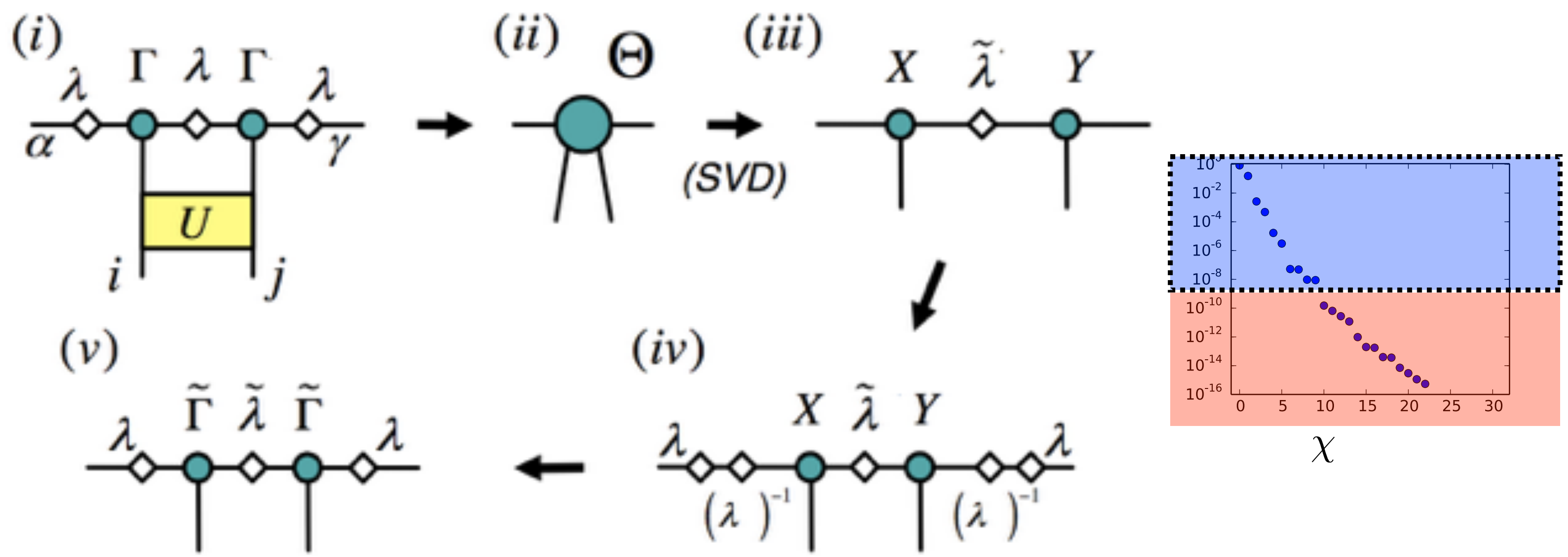


Time evolution

Trotter-decompose U(dt) into 2-site gates:



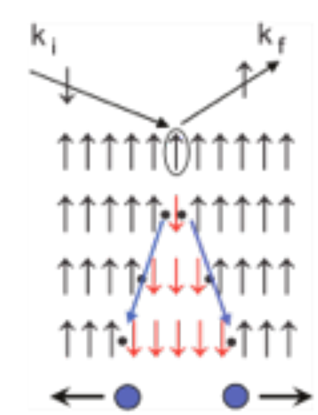
TEBD [Vidal 03]



Dynamical structure factor: $S(k,\omega)$

$$C(x,t) = \langle \psi_0 | S_x^-(t) S_0^+(0) | \psi_0 \rangle$$

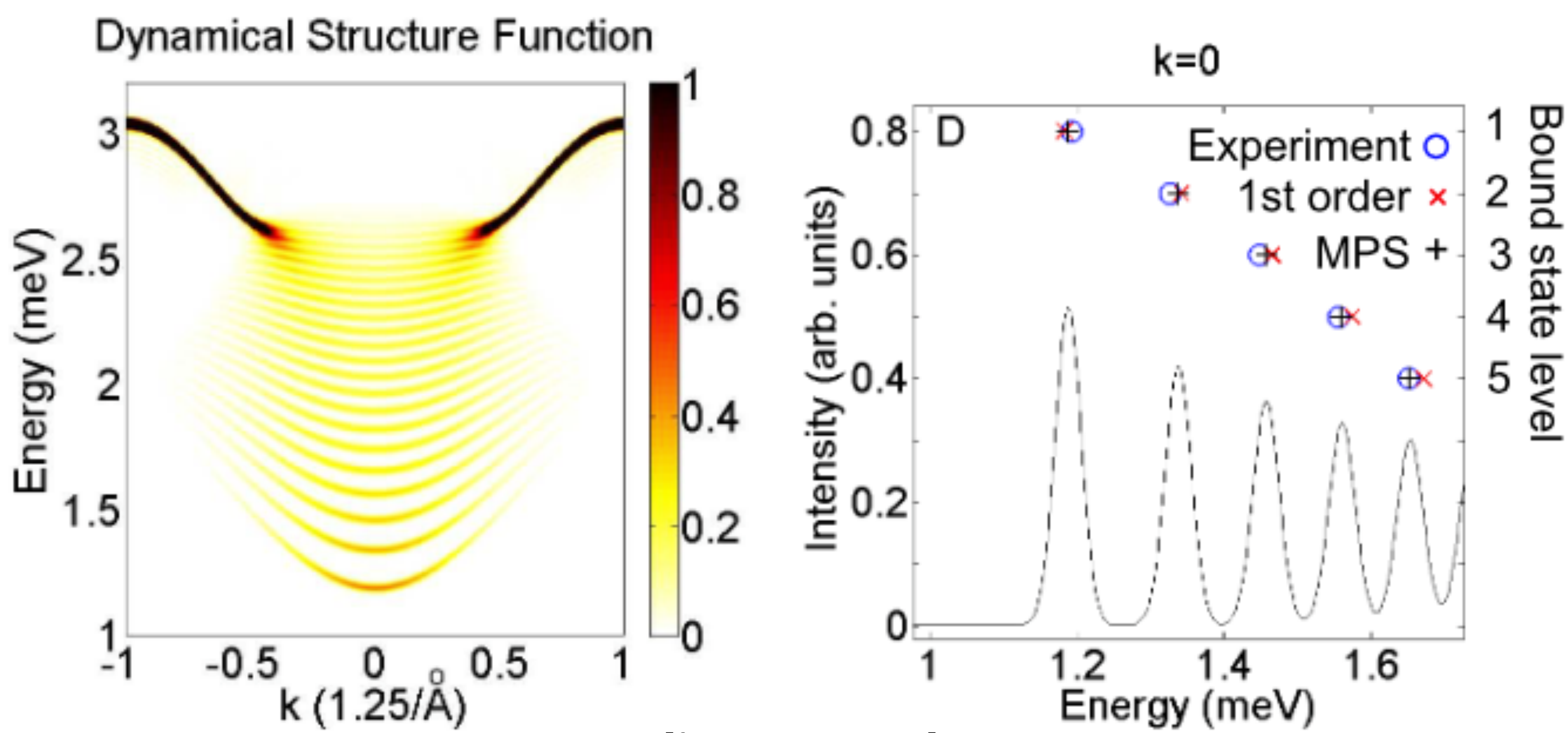
$$S(k,\omega) = \sum_x \int_{-\infty}^{\infty} dt e^{-i(kx+\omega t)} C(x,t)$$



Time evolution

Experiments of Coldea, et al.: 1D TFI perturbed by order parameter

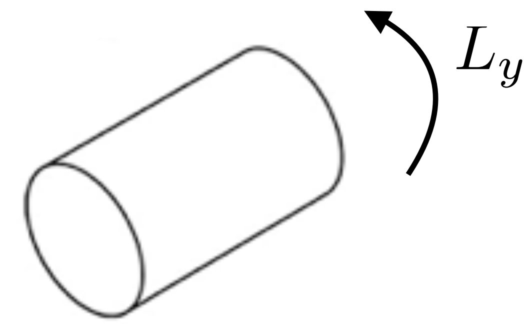
$$\begin{aligned} H &= -J' \sum_n S^z_n S^z_{n+1} - h^x \sum_n S^x_n - h^z \sum_n S^z_n \qquad (2) \\ &- J_p \sum_n \left(S^x_n S^x_{n+1} + S^y_n S^y_{n+1} \right) + J_B \sum_n S^z_n S^z_{n+2}. \end{aligned}$$



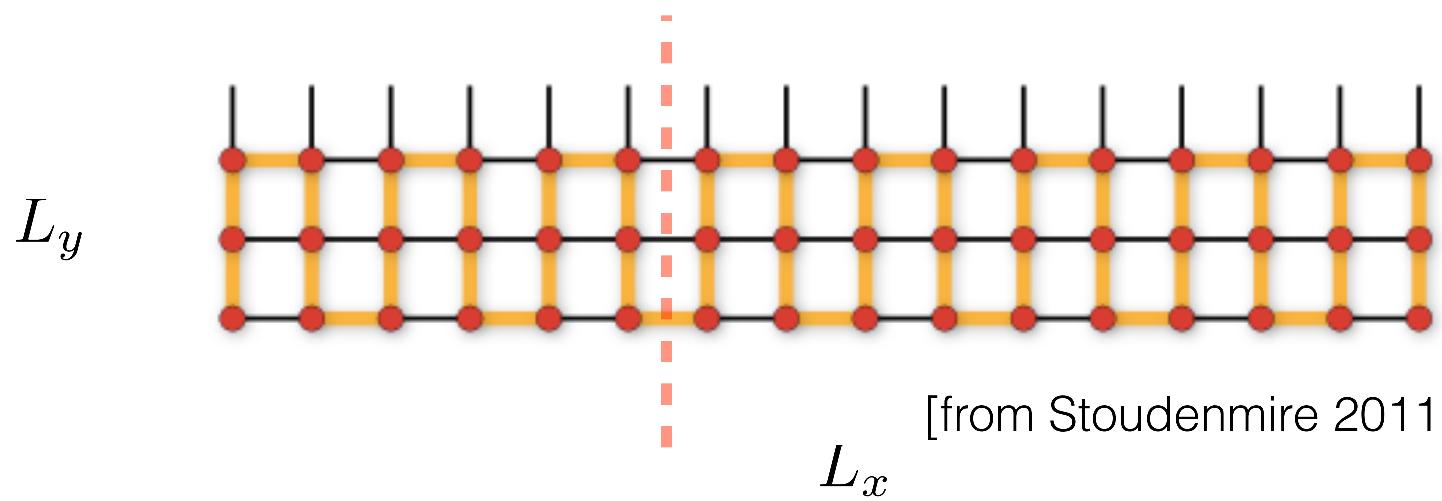
[from Kjäll 2011]

Near QCP: masses of emergent excitations root lattice of E_8

2D DMRG: The Kludge (alias - snake)



Order the 2D lattice into 1D chain with longer-range interactions



[from Stoudenmire 2011]

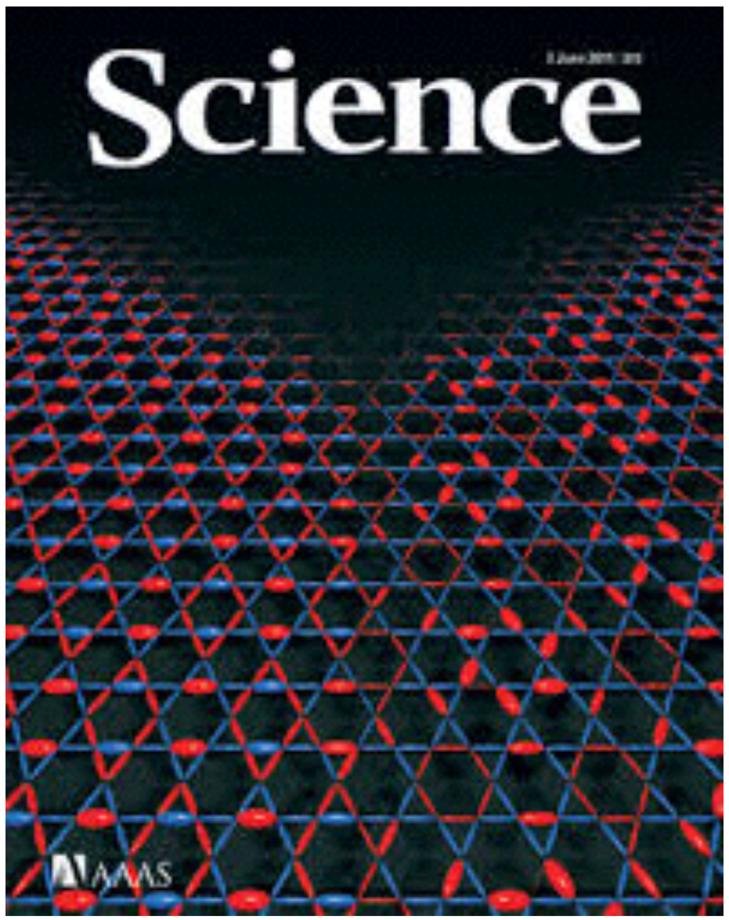
Entanglement scales with circumference: $S_E \sim L_y$

Complexity: $L_x L_y e^{\alpha L_y}$ (DMRG) $e^{\alpha L_x L_y}$ (Exact Diagonalization)

2D DMRG

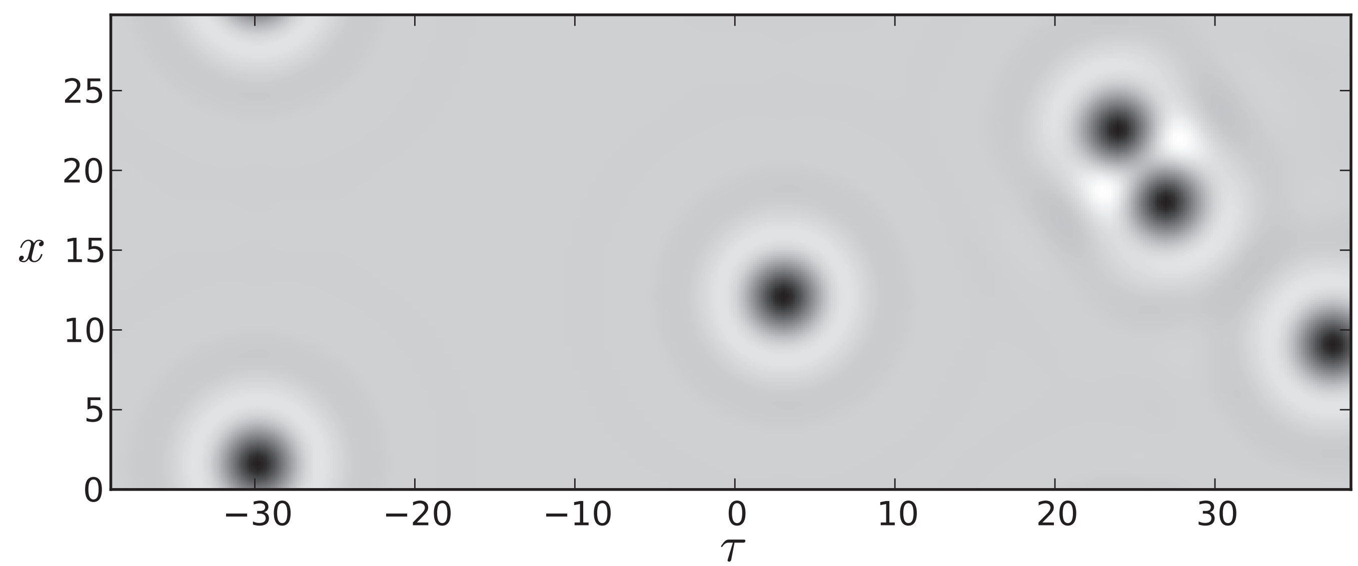
Works if you are lucky
(i.e., near thermodynamic limit on small cylinders)

Frustrated magnetism &
Spin-liquids on cylinders



[Yan, Huse, White 2010]

Fractional quasiparticles in
the fractional quantum hall effect

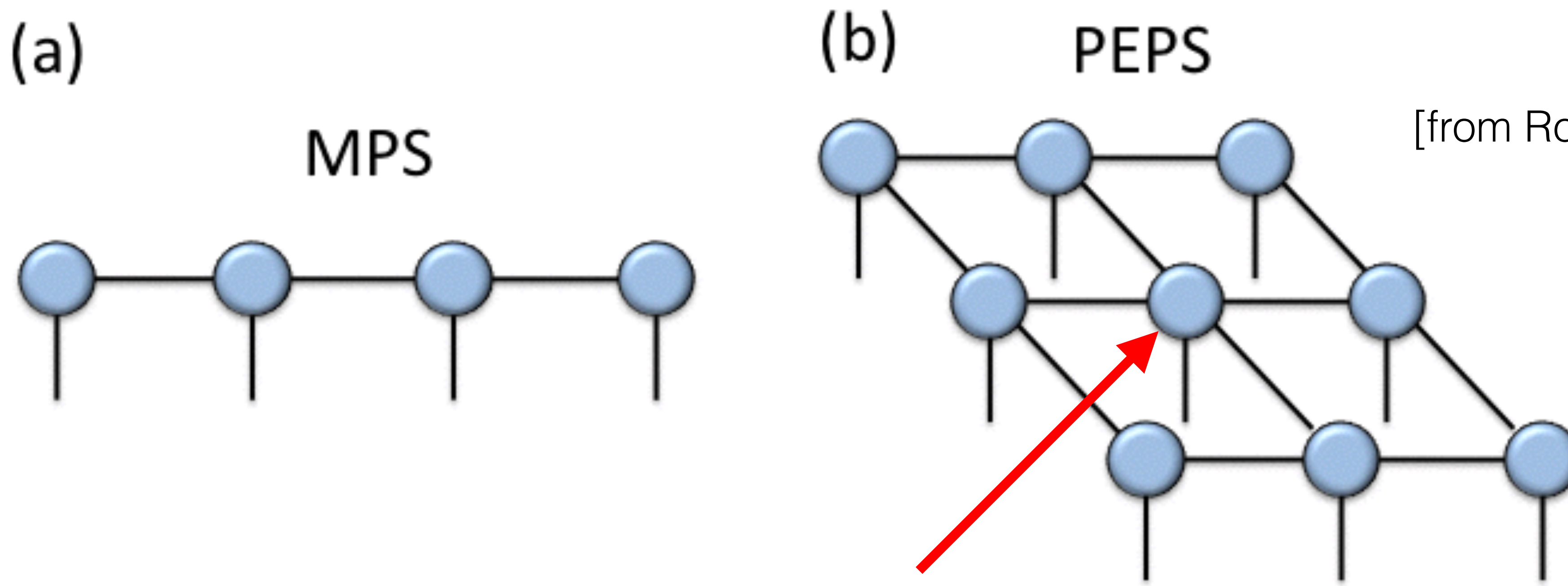


[Zaletel, Mong, Pollmann 2012]

More on measuring topological order
in these studies tomorrow

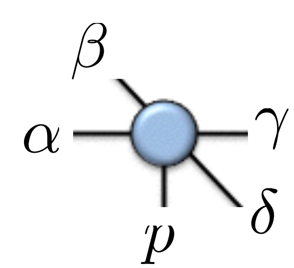
2D Tensor network: The Hope

[Verstraete & Cirac, 2004]



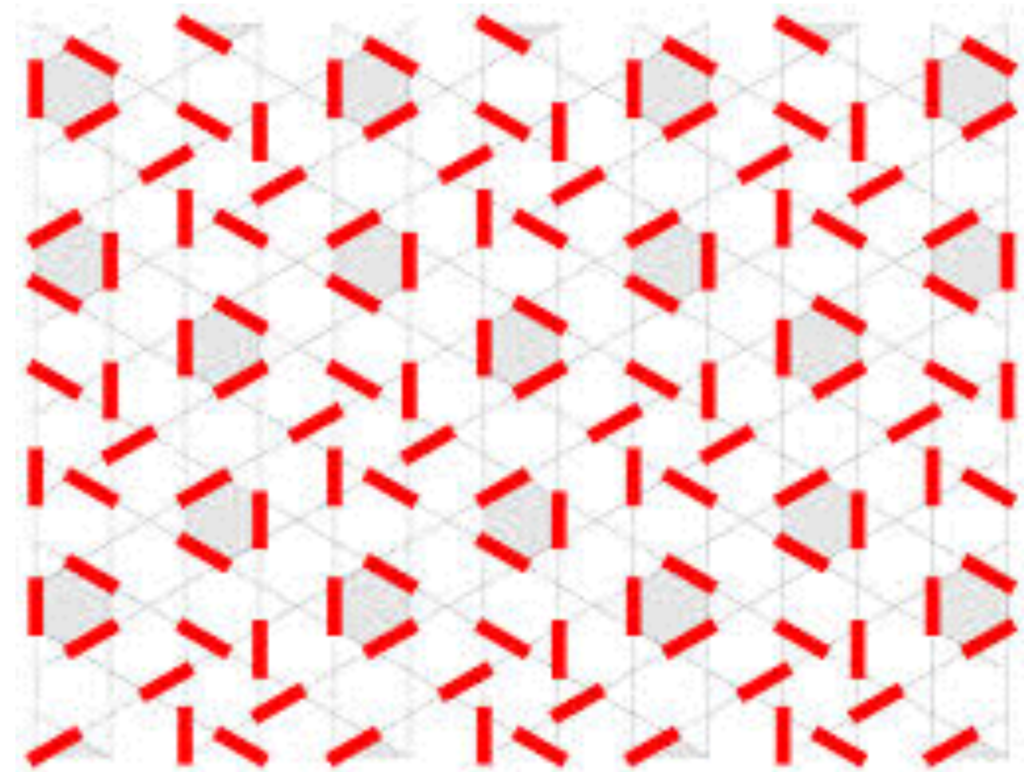
[from Roman Orus, 1306.2164
Great review!]

$$A^{p}_{\alpha\beta\gamma\delta} \quad = $$



Example: dimer model

Kagome NN dimer covering:



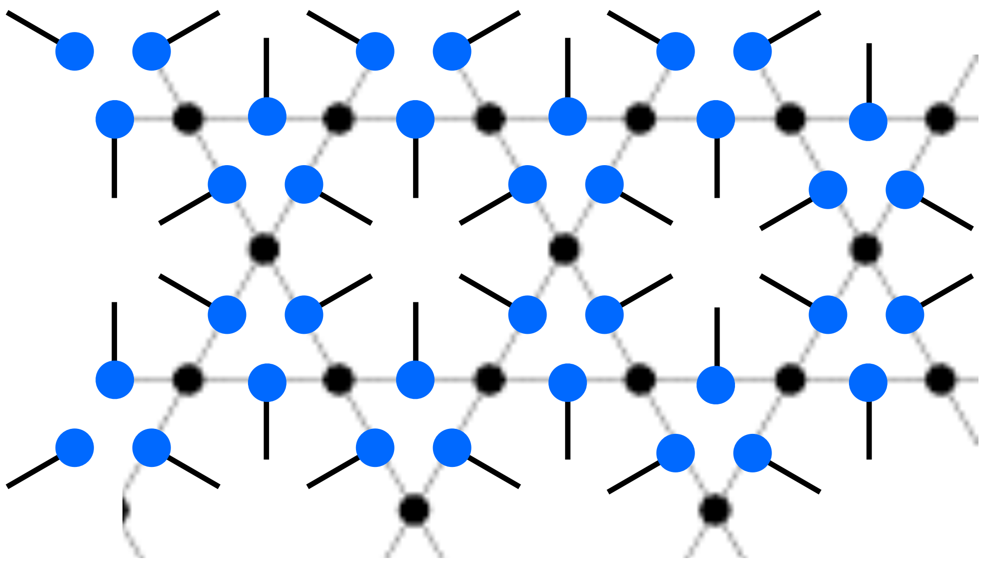
[from Yejin Huh, 2011]

$$|\Psi\rangle = \sum_{\text{dimer coverings}} |\text{hardcore-dimers}\rangle$$

[RVB PEPs examples: Schuch, et al. 2012]

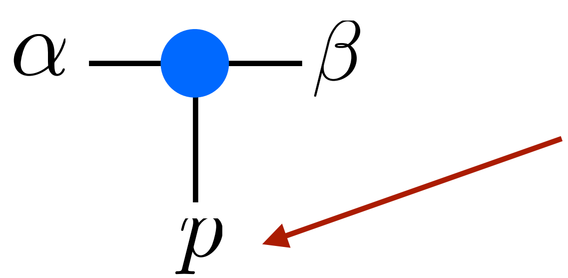
TN for dimers

$$|\Psi\rangle = \sum_{\text{dimer coverings}} |\text{hardcore-dimers}\rangle$$

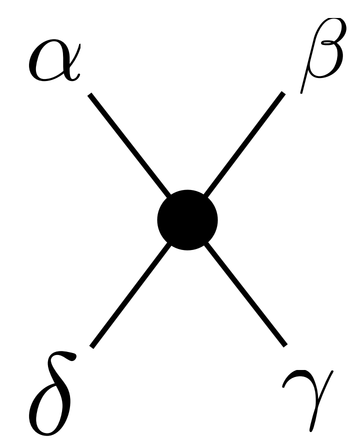


Using different topology than square TN: but you can always regroup things to turn it into “standard” form

$$\chi = 2 : \quad \alpha \in \{0, 1\}$$



p = 0: no dimer
p = 1: dimer



$$A^p_{\alpha\beta} = \begin{cases} 1 & \text{if } p = \alpha = \beta \\ 0 & \text{if else} \end{cases}$$

$$B_{\alpha\beta\gamma\delta} = \begin{cases} 1 & \text{if } \alpha + \beta + \gamma + \delta = 1 \\ 0 & \text{if else} \end{cases}$$

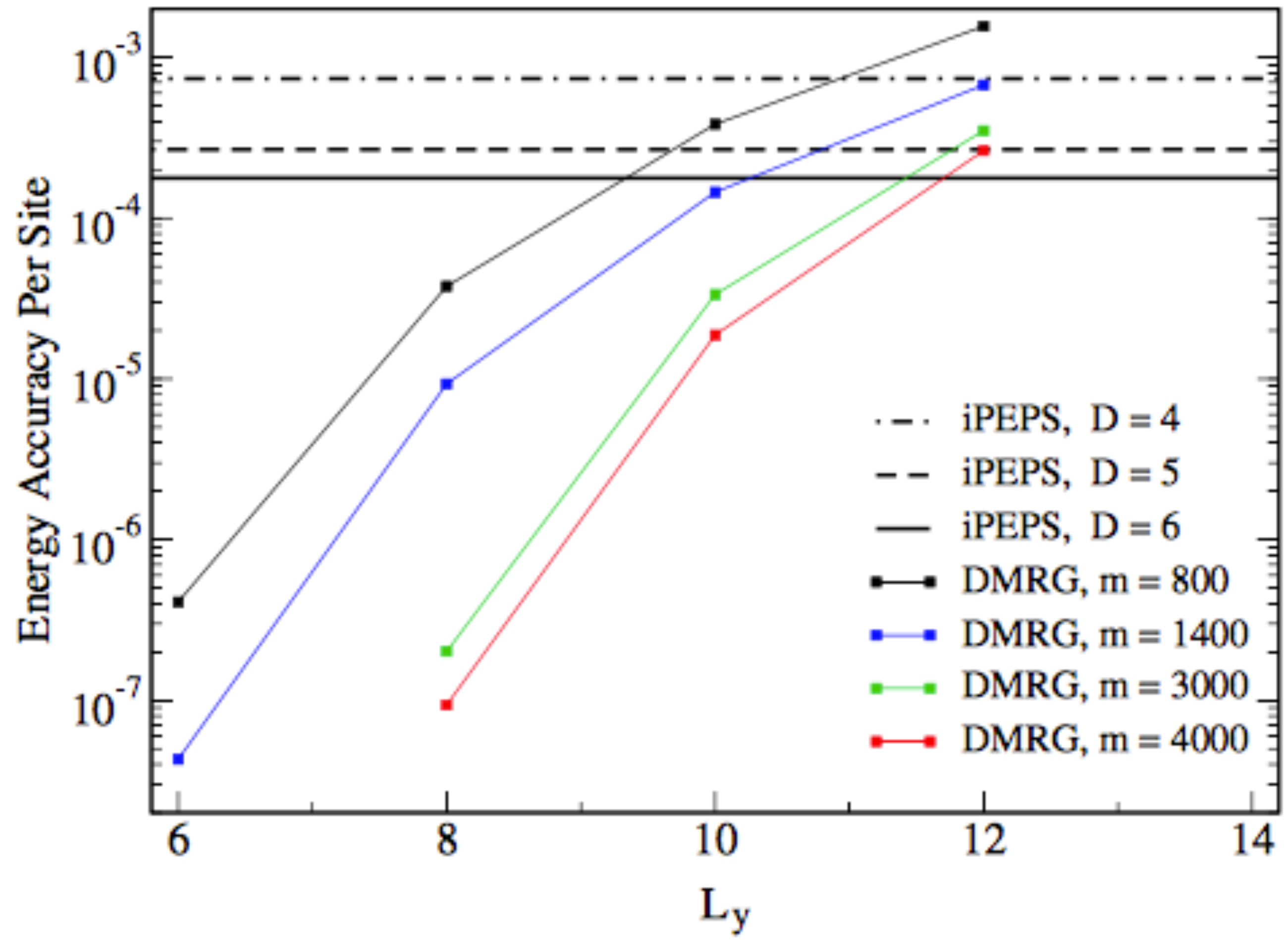
Glue presence of dimer to “virtual” index

The constraint (no physical index)

Why is Kagome still being studied with snakes?

Finding the 2D TN is hard!

Square-lattice
J1



[from Stoudenmire 2011]

Why is Kagome still being studied with snakes?

*** Unsolved problem 0: which phases of matter can be represented by finite dimensional 2D TN? ***

1D MPS: represents gapped states of local H
2D TN: not known (certain things can't be: fermi surface)

*** Unsolved problem 1: what is the *right* way to approximately calculate observables in a 2D TN? ***

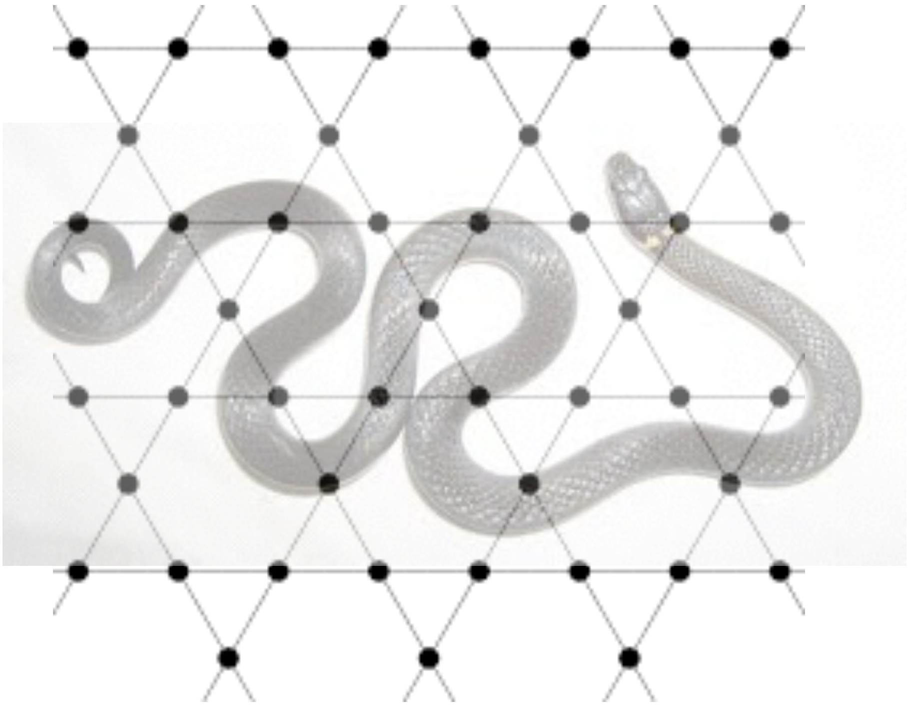
Calculating expectation value in MPS *exactly*: linear complexity in size
Calculating expectation value in 2d TN *exactly*: exponential complexity in size

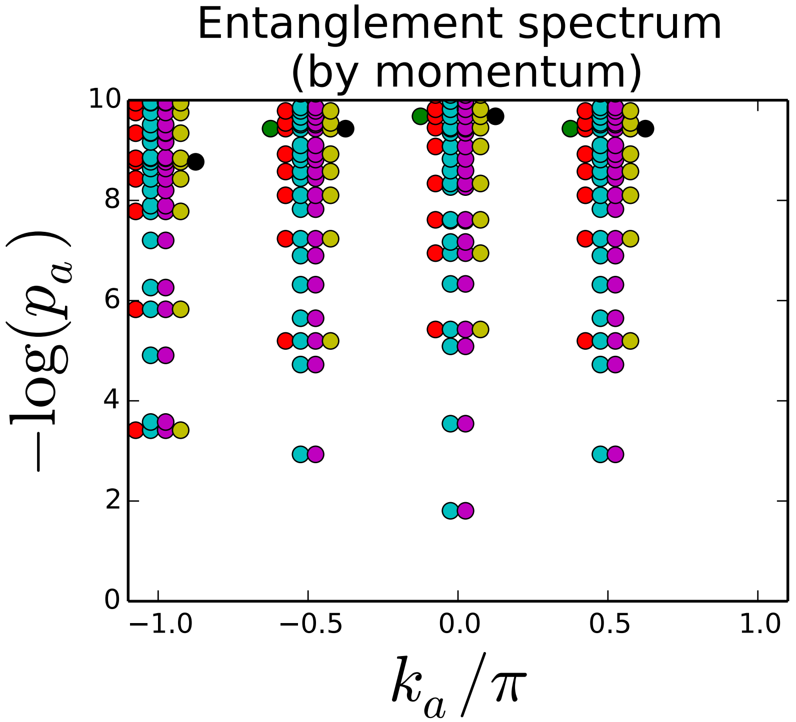
*** Unsolved problem 2: what is the *right* way to find a 2D TN given H? ***

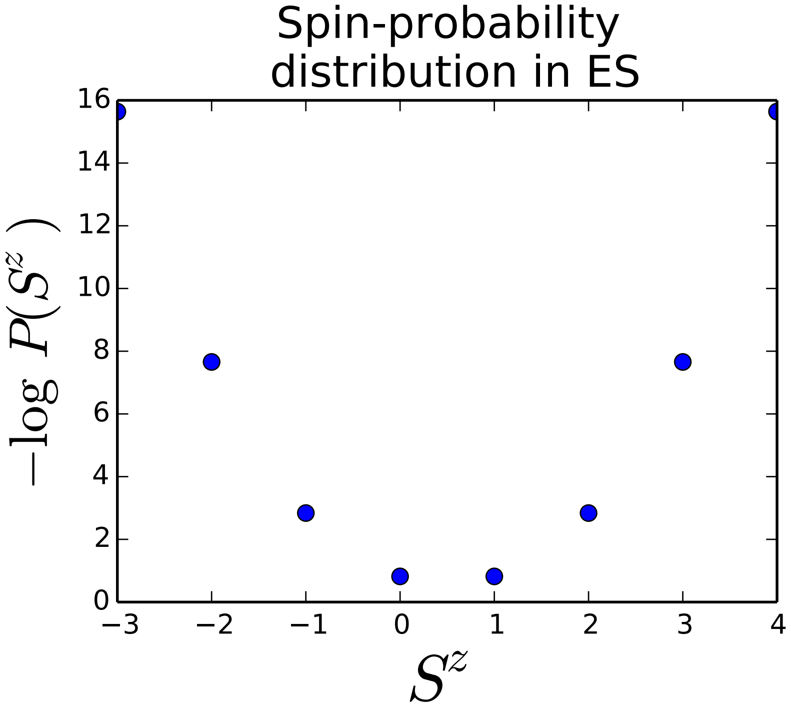
DMRG: it works. complexity χ^3
2D TN: algorithms proposed, but not fully understood what the nature of the approximations is. complexity $\chi^8 - \chi^{10}$

Thanks!

Mike Zaletel
Station Q



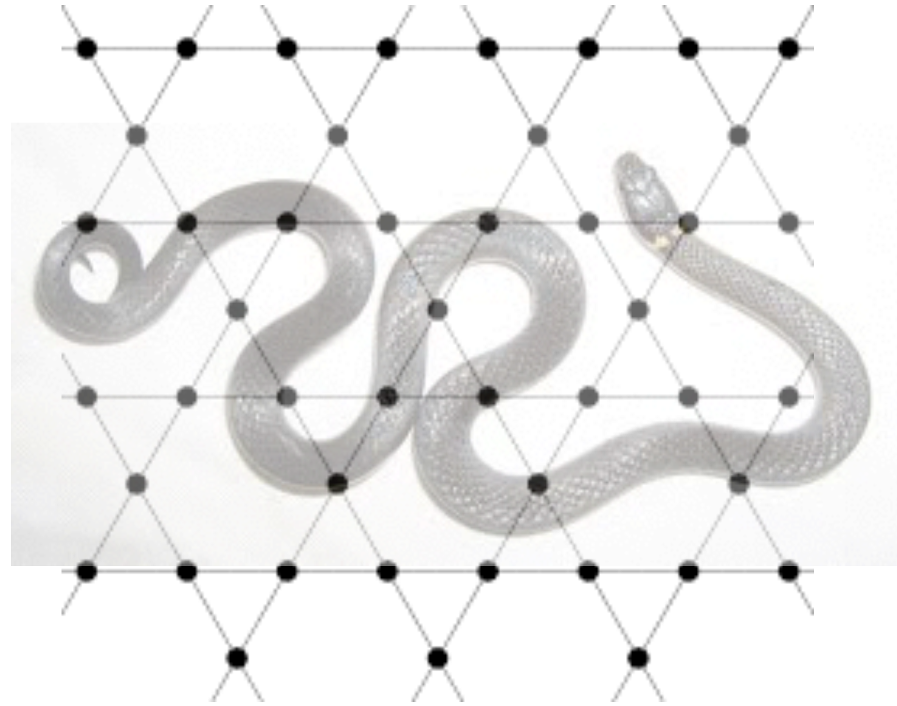


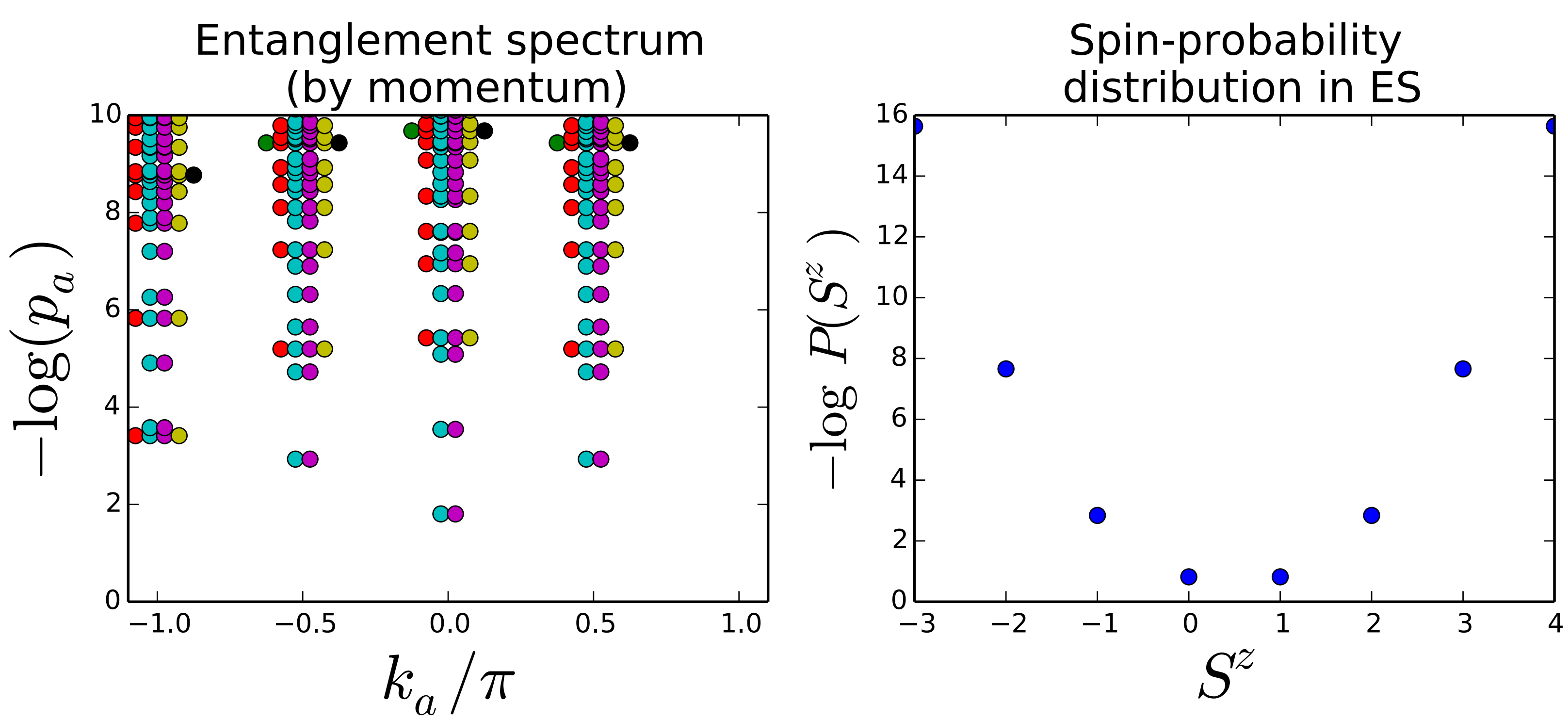


MagLab Theory Winter School 2015

Tensor networks and entanglement spectroscopy

Mike Zaletel
Station Q

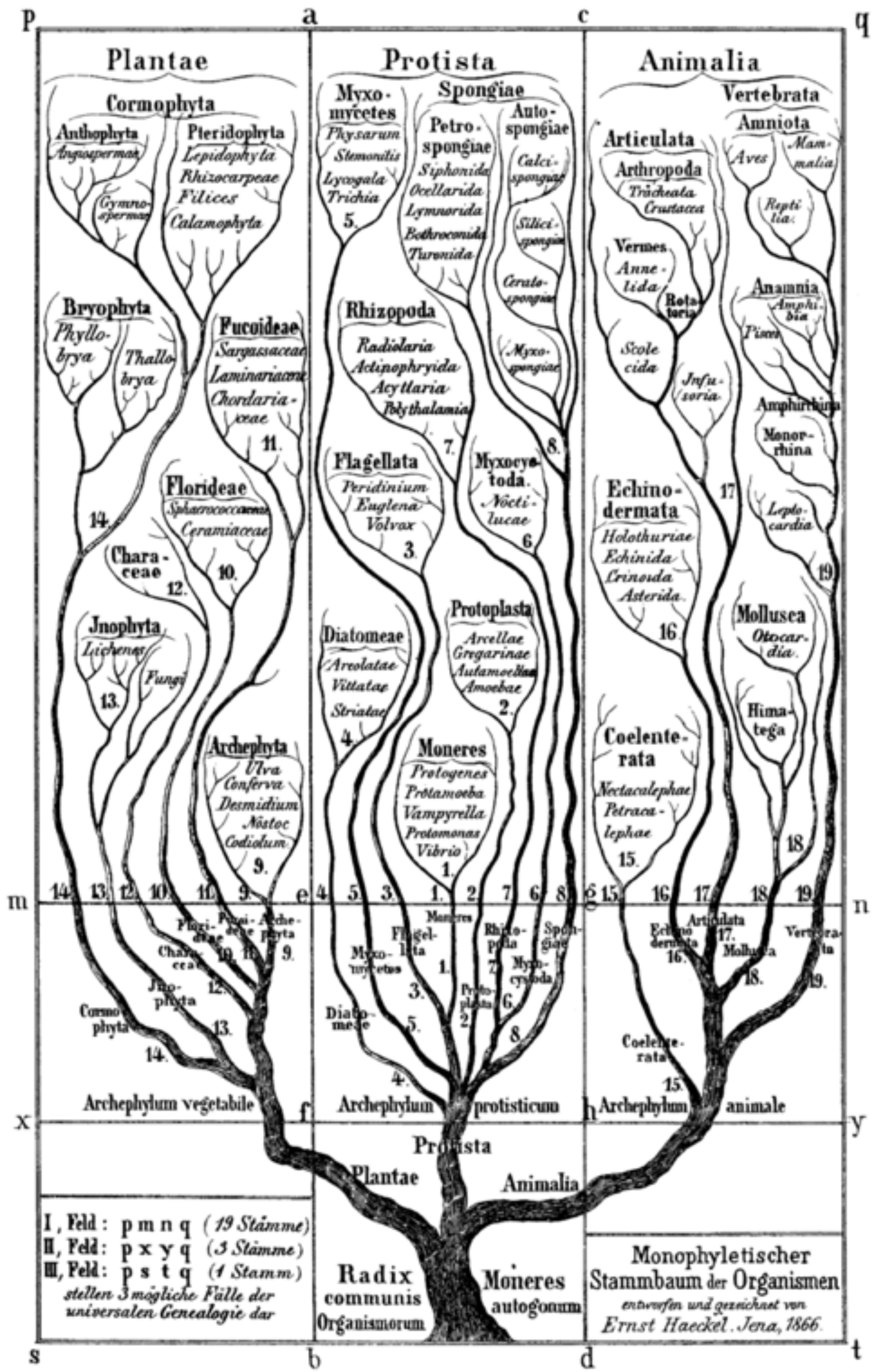


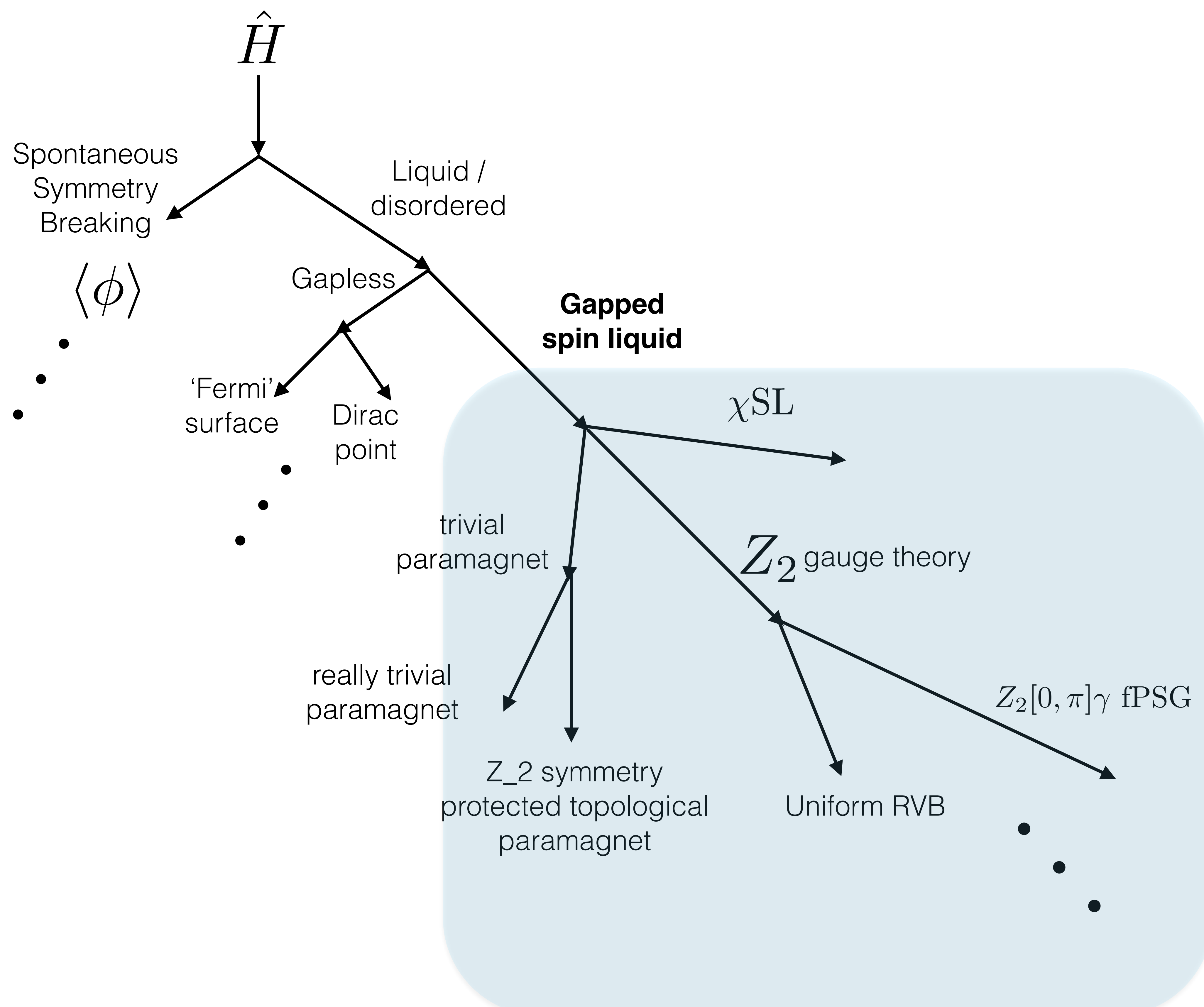


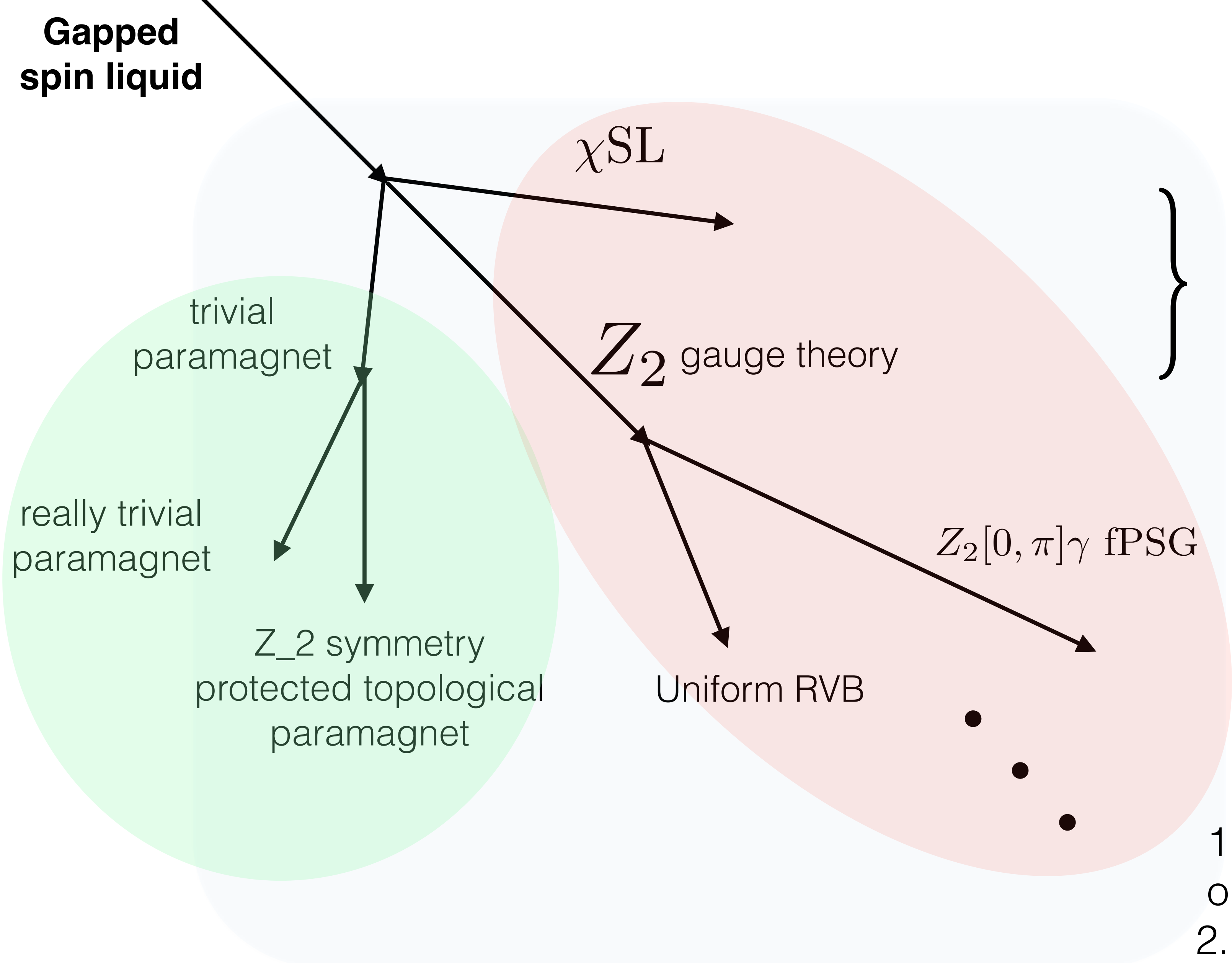
MagLab Theory Winter School 2015

Outline

- Day 1: Introduction to tensor network numerics
- Day 2: Entanglement spectroscopy: detecting emergent anyons in numerics
    - Topological entanglement entropy & quantum dimensions
    - The entanglement spectrum
    - Topological degeneracy of the cylinder
    - Minimally entangled states
    - The Kagome SL
    - Topological Spin & Momentum Polarization


Plantae
Protista
Animalia
Cormophyta
Anthophyta
Angiospermae
Gymnospermae
Pteridophyta
Lepidophyta
Rhizocarpeae
Filices
Calamophyta
Bryophyta
Phyllobrya
Thallobrya
Fucoideae
Sargassaceae
Laminariaceae
Chordariaceae
Florideae
Ceramiaceae
Characeae
Jnophyta
Lichenes
Fungi
Archephyta
Ulva
Conferva
Desmidium
Nostoc
Codiolum
Myxomycetes
Physarum
Stemonitis
Lycogala
Trichia
Spongiae
Petrospongiae
Siphonida
Ocellarida
Lymnorida
Bothroconida
Turonida
Autospongiae
Calcispongiae
Silicispongiae
Ceratospongiae
Myxospongiae
Rhizopoda
Radiolaria
Actinophryida
Acyttaria
Polythalamia
Flagellata
Peridinium
Euglena
Volvox
Myxocystoda
Noctilucae
Protoplasta
Arcellae
Gregarinae
Amoebae
Diatomeae
Areolatae
Vittatae
Striatae
Moneres
Protogenes
Protamoeba
Vampyrella
Protomonas
Vibrio
Vertebrata
Amniota
Aves
Mammalia
Reptilia
Anamnia
Amphibia
Pisces
Amphirhina
Monorrhina
Leptocardia
Articulata
Arthropoda
Tracheata
Crustacea
Vermes
Annelida
Rotatoria
Scolecida
Infusoria
Echinodermata
Holothuriae
Echinida
Crinoida
Asterida
Mollusca
Otocardia
Himatega
Coelenterata
Nectacalephae
Petracalephae
Archephylum vegetabile
Archephylum protisticum
Archephylum animale
Protista
Plantae
Animalia
Radix communis Organismorum
Moneres autogonum
I, Feld: p m n q (19 Stämme)
II, Feld: p x y q (3 Stämme)
III, Feld: p s t q (1 Stamm)
stellen 3 mögliche Fälle der universalen Genealogie dar
Monophyletischer Stammbaum der Organismen
entworfen und gezeichnet von Ernst Haeckel. Jena, 1866.


$\hat{H}$
Spontaneous Symmetry Breaking
$\langle \phi \rangle$
Liquid / disordered
Gapless
'Fermi' surface
Dirac point
Gapped spin liquid
χSL
trivial paramagnet
Z_2 gauge theory
really trivial paramagnet
Z_2 symmetry protected topological paramagnet
Uniform RVB
$Z_2[0,\pi]\gamma$ fPSG



Intrinsic topological order:

1. Emergent quasiparticles with 'anyonic' braiding and statistics
2. e.g. visons & spinons in Z_2 SL

Symmetry enriched topological (SET) order:

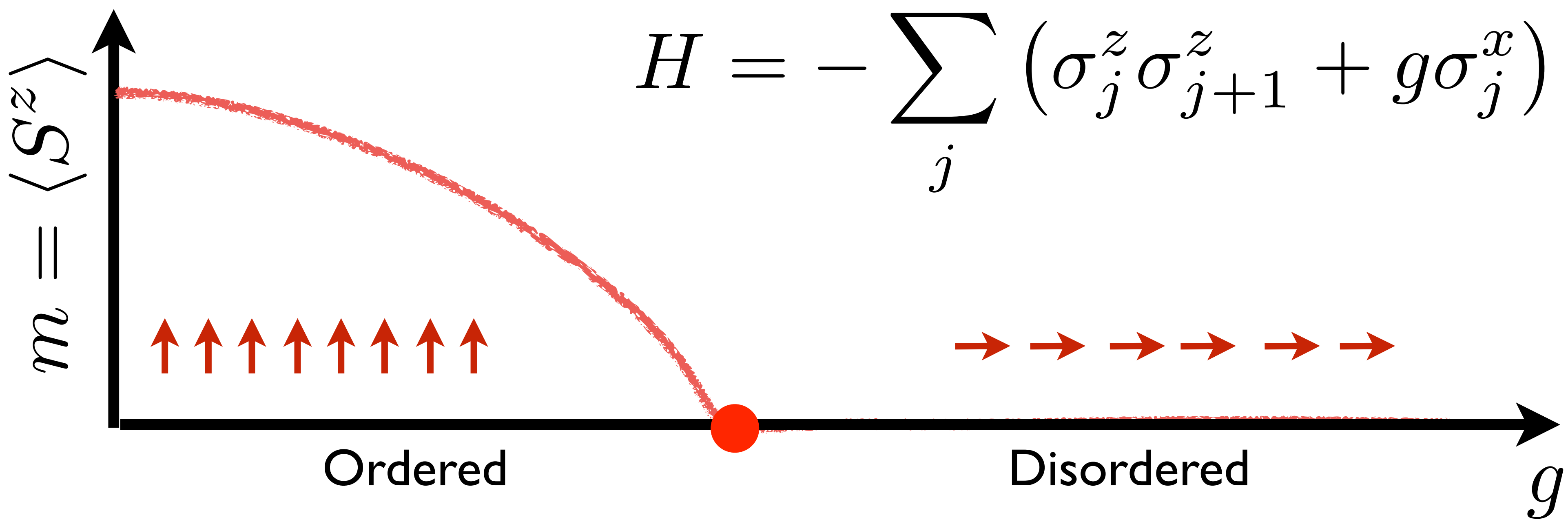
1. Further distinctions between phases based on anyons with "fractional" quantum #s
2. e.g. e / 3 charge of Laughlin quasiparticle

'Symmetry protected' topological (SPT) order:

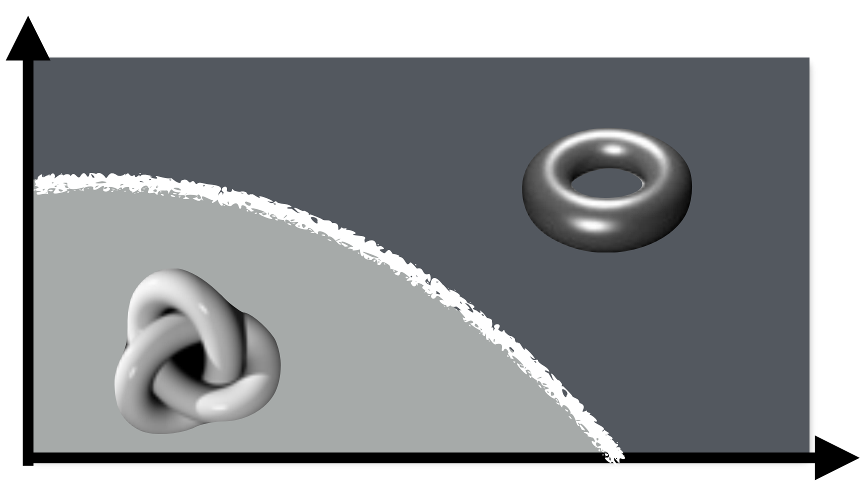
1. No anyonic excitations: distinction requires *symmetries*
2. Quantized responses to flux threading (+generalizations)
3. e.g. 1D spin-1 Haldane chain (AKLT), 2D IQHE, topological insulators

(also chiral order: p+ip superconductor)

SSB.
Detect from order parameter



Topological order.
Need non-local order parameter:
quantum entanglement



[from Pollmann]

Nothing on symmetries today. Just “intrinsic:” TQFT

Goal: given ground state wavefunction,
can we determine the topological order?

$$|\Psi\rangle$$

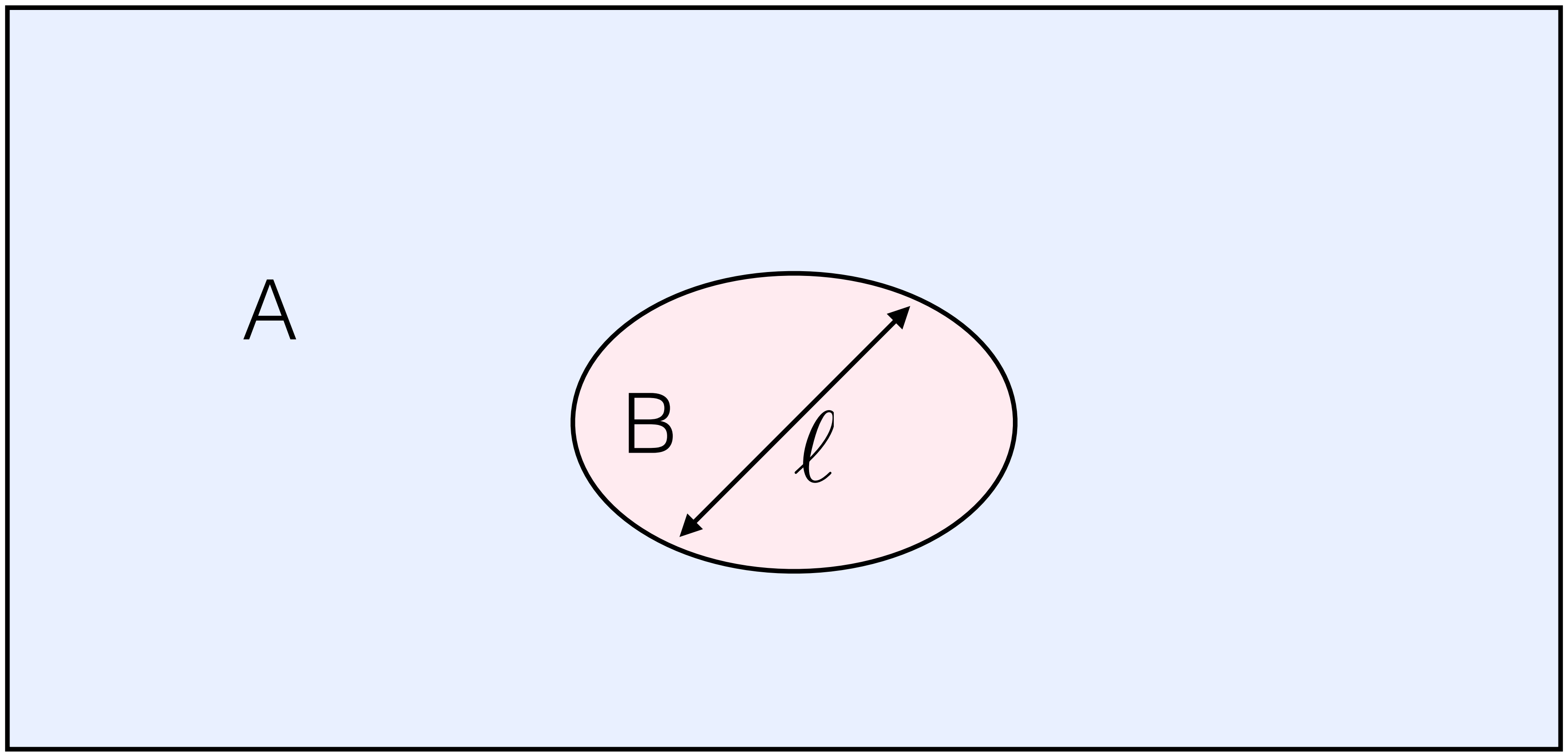
c_-

$\mathcal{S}_{ab}$ TQFT $\mathcal{T}_{ab}$

R^z_{xy}

Yesterday: Entanglement Entropy

$$S = -\mathrm{Tr}\left[\rho_A \log(\rho_A)\right] = -\sum_\alpha \lambda_\alpha^2 \log(\lambda_\alpha^2)$$



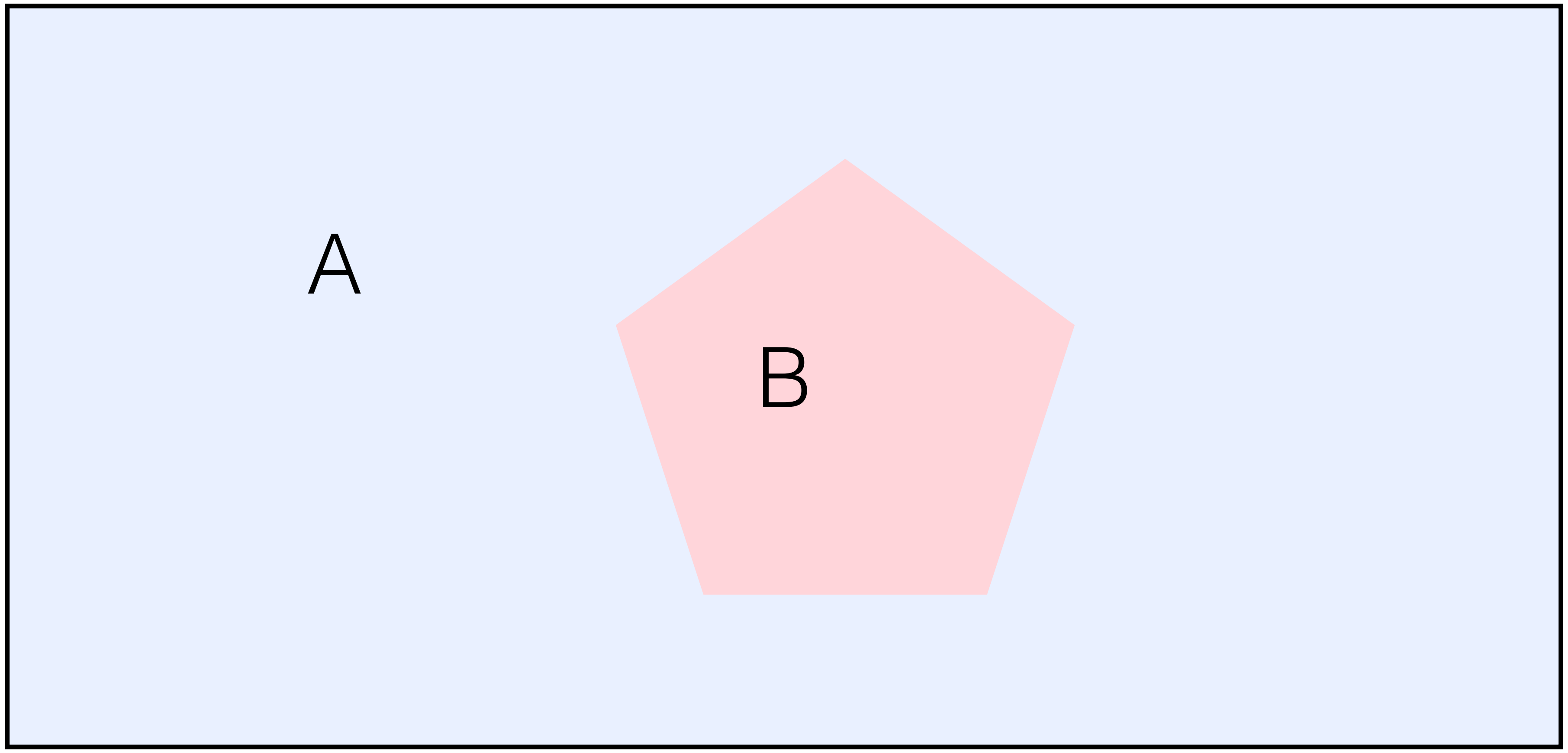
$$S = \alpha\ell - \mathrm{const} + \mathcal{O}(e^{-\ell/\xi})$$

Area law: UV physics

Dimensionless: universal?

Topological Entanglement Entropy

[Kitaev & Preskill, Levin & Wen 2006]



area law corners TEE

$$S_B = \alpha|\partial B| + \sum_{i=1}^{5} c_i - \gamma + \mathcal{O}(e^{-\ell/\xi})$$

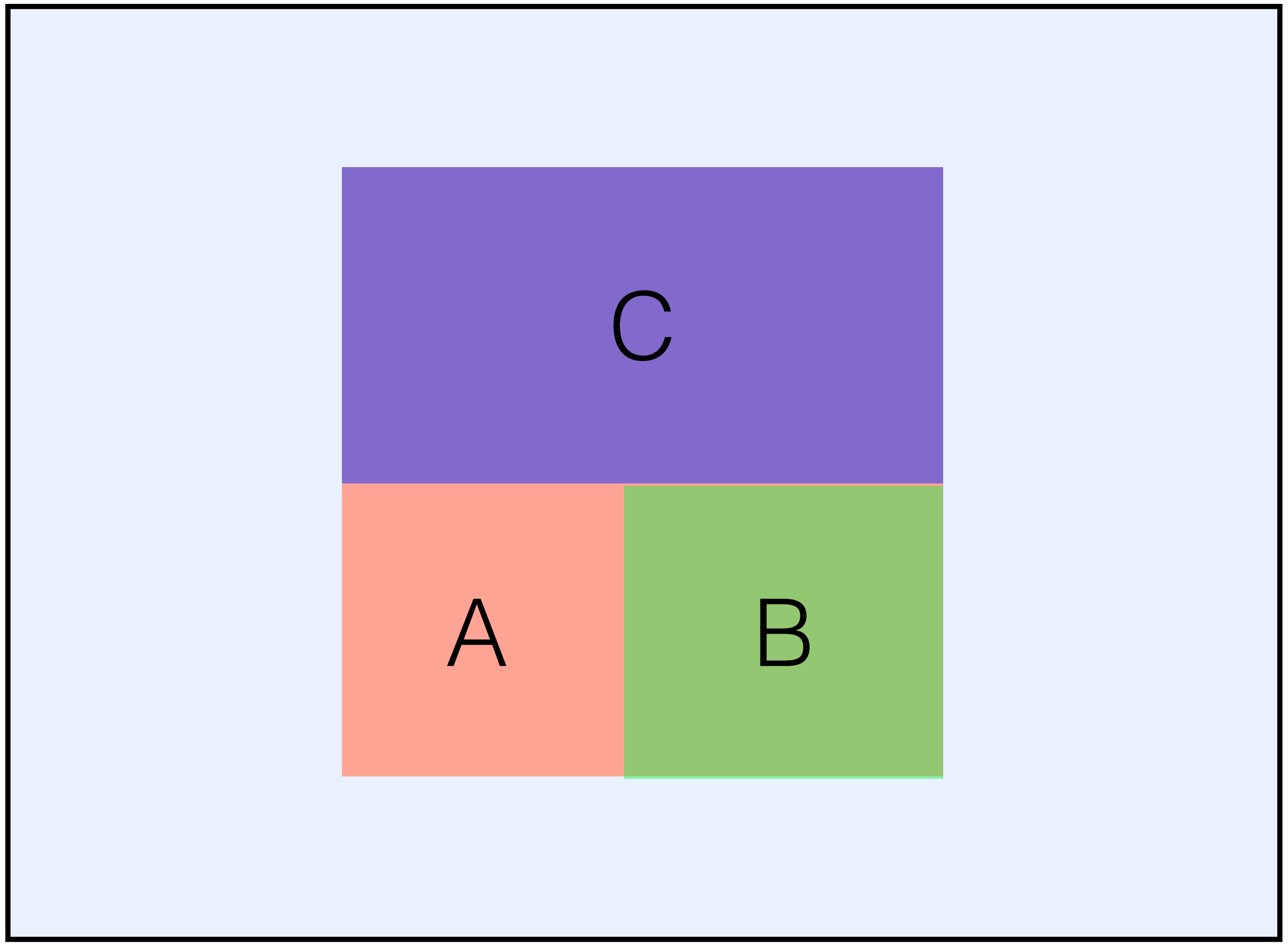
Garbage Gold

Topological Entanglement Entropy

[Kitaev & Preskill, Levin & Wen 2006]

In the ground state:

$$S_B = \alpha|\partial B| + \sum_{i=1}^{5} c_i - \gamma$$

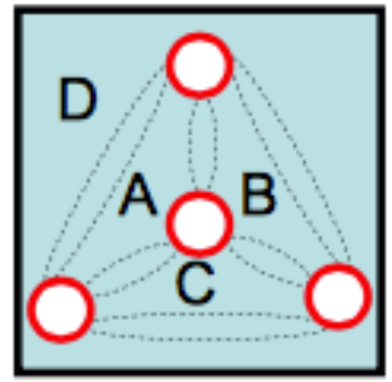


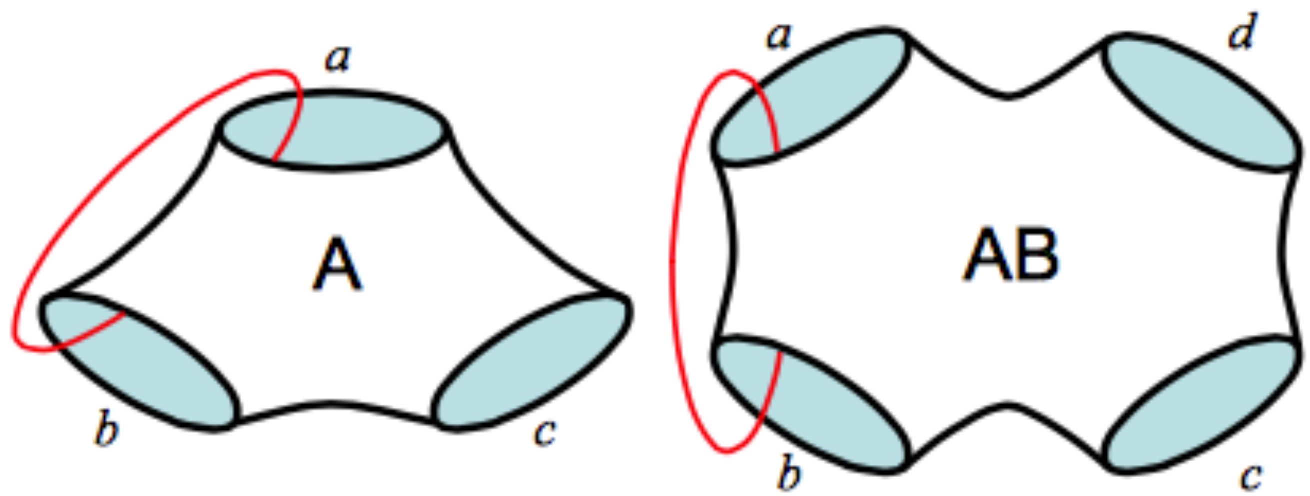
$$+\mathcal{O}(e^{-\ell/\xi})$$

$$-\gamma = S_{ABC} - S_{AB} - S_{BC} - S_{CA} + S_A + S_B + S_C$$

Should be a universal quantity sensitive to ‘non-local’ part of S

TQFT says…





$$S_{\text{topo}} = 2S_3 - \frac{3}{2}S_4 = -\log\mathcal{D} \equiv -\gamma\ . \qquad (15)$$

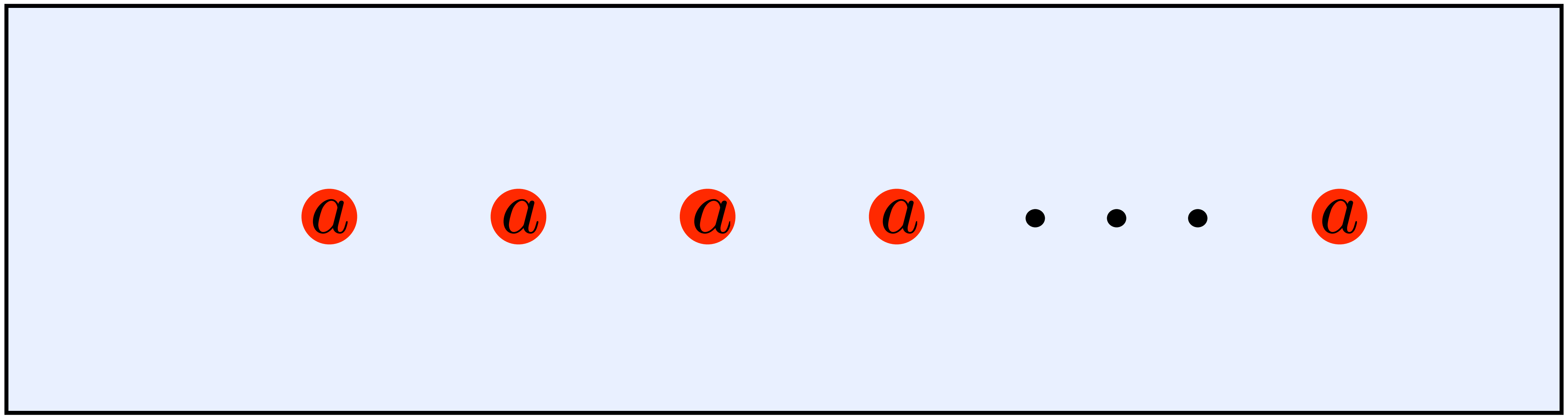
Eq. (15) is our main result. Note that it follows if we

from Kitaev & Preskill 2006

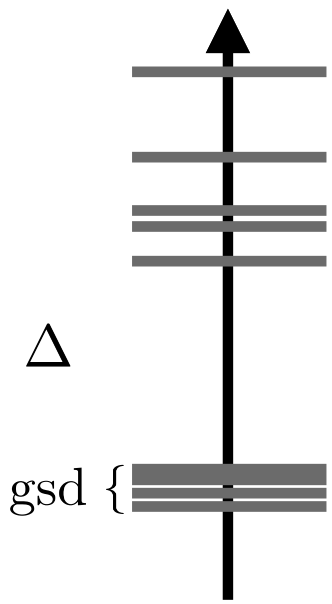
yes but what is $\mathcal{D}$?

Quantum dimensions: a tiny intro

Pin down N anyons on a disc or sphere:



Energy spectrum:



$$\text{gsd} = \begin{cases} 1 & \text{if } N = 0 \\ \sim d_a^N & \text{as } N \gg 1 \end{cases}$$

robust quantum memory! - ??

$d_a \geq 1:$ the ‘quantum dimension’ of anyon a

$d_a = 1:$ ‘abelian’ anyon

$d_a > 1:$ ‘non-abelian’ anyon

TEE: the 'total quantum dimension'

$$-\gamma = S_{ABC} - S_{AB} - S_{BC} - S_{CA} + S_A + S_B + S_C$$

$$\gamma = \log(\mathcal{D}), \qquad \mathcal{D} = \sqrt{\sum_{a=1}^{\#\text{species}} d_a^2}$$

'total quantum dimension'

Z_2 Gauge theory = toric code = Z_2 SL:

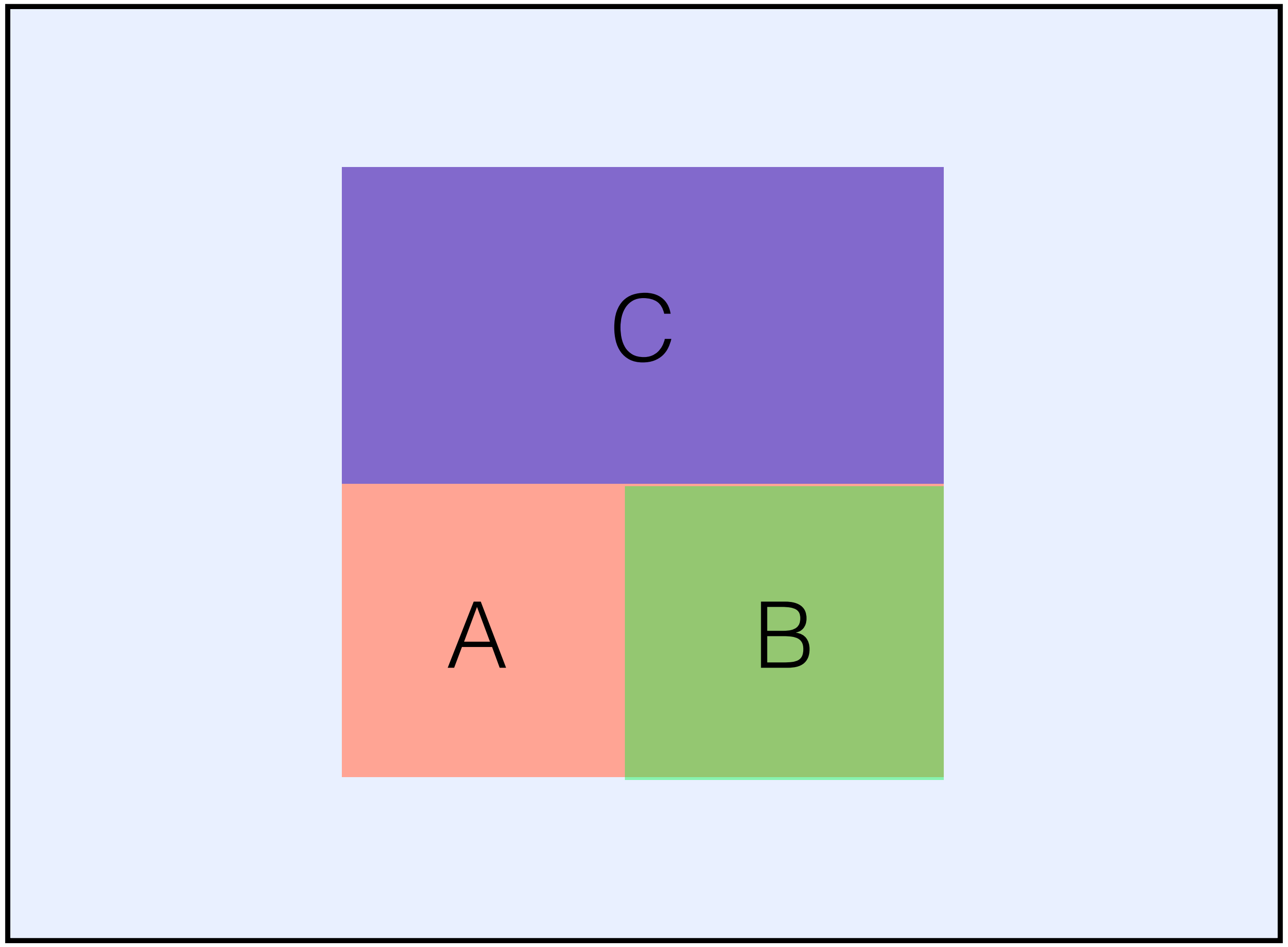
$$a \in \{1, e, m, em\},\ d_a = 1,\ \mathcal{D} = \sqrt{1+1+1+1} = 2,\ \gamma = \log(2)$$

'Trivial' phase: no anyons

$$a \in \{1\},\ d_a = 1,\ \mathcal{D} = \sqrt{1} = 1,\ \gamma = \log(1) = 0$$

$\gamma > 0$: anyons!

This is hard to do in small systems:



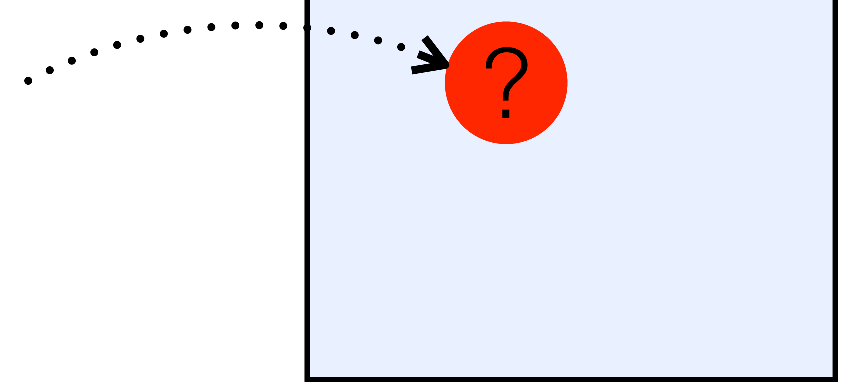
We will return to the practical “cylinder” way shortly:
but we need to figure out some subtleties first!

Using entanglement to detect an anyon

Suppose you've taken note of the properties of a region (density, energy, etc…)

and an excitation wanders in

1



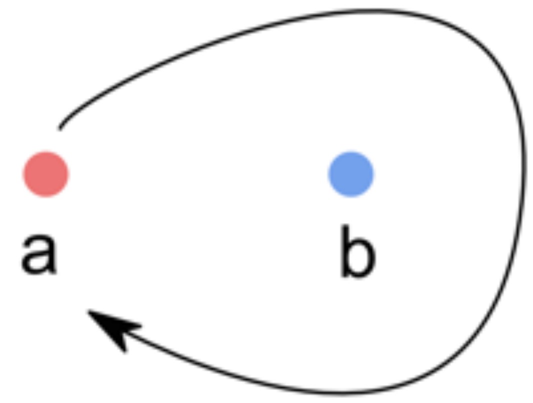
?

Can you tell if the excitation is an anyon?

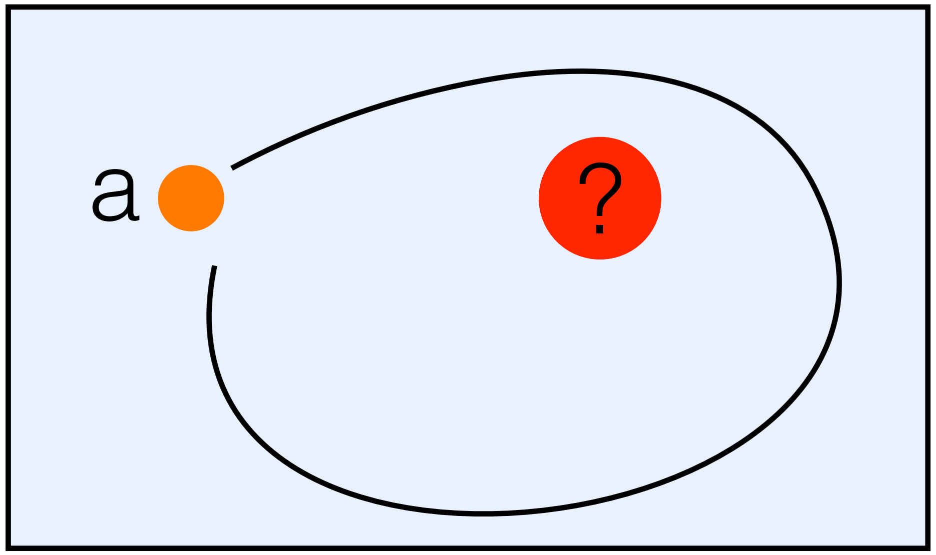
Can you tell what type?

Using entanglement to detect an anyon

Of course! use mutual statistics



$$S_{ab}$$



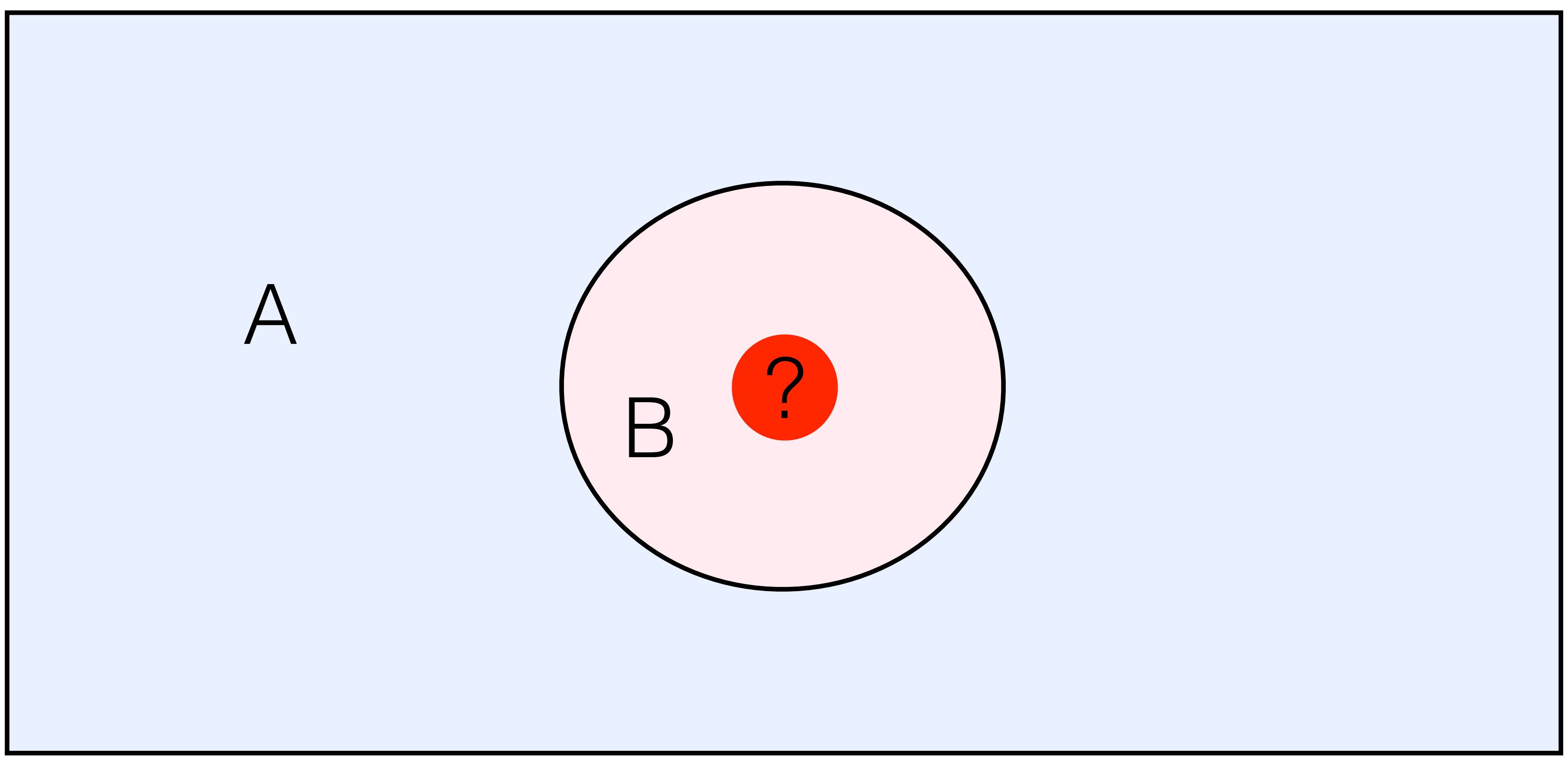
$$= S_{a?}$$

$$S_{a?} = 1 \ \forall a \ \Rightarrow \ ? = 1$$

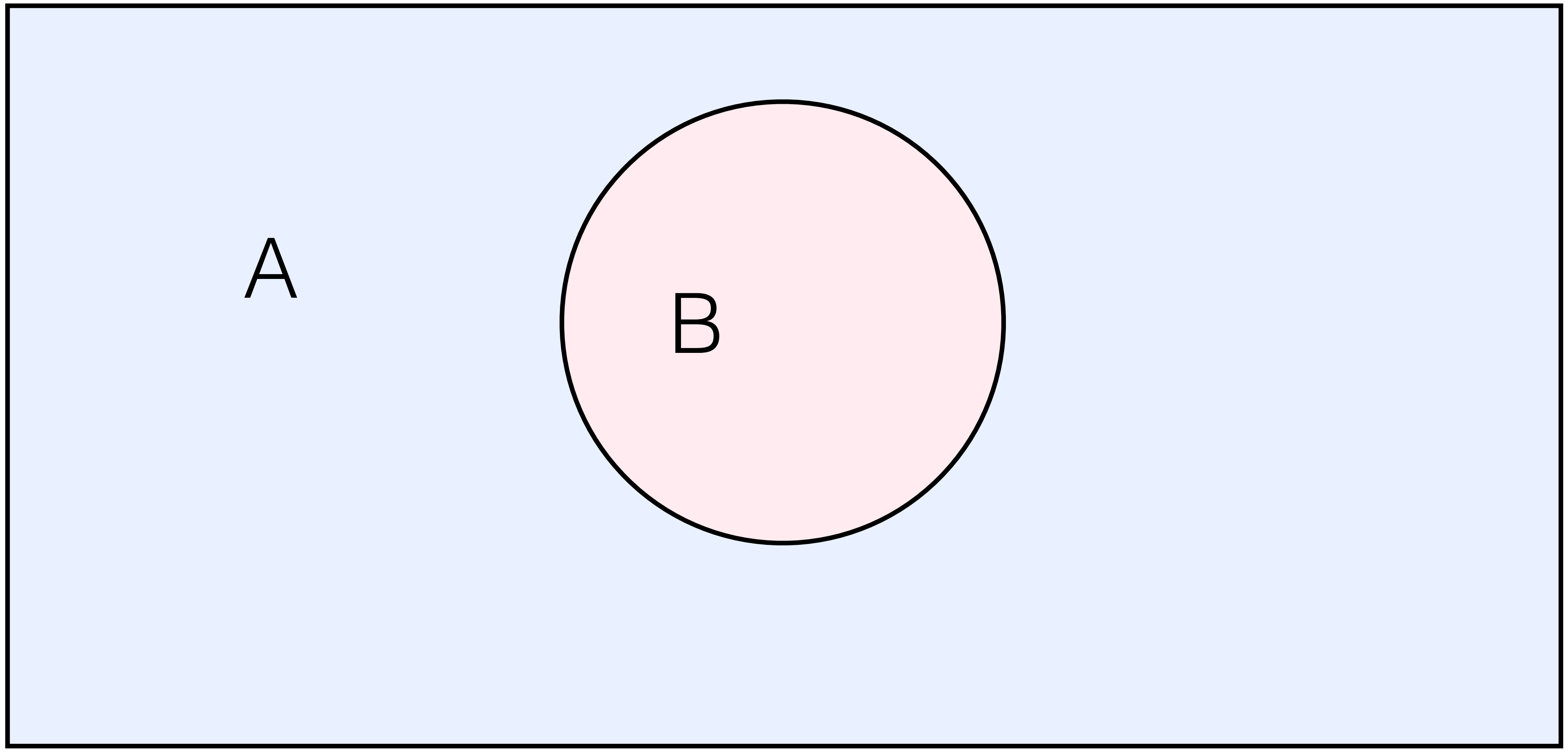
Using entanglement to detect an anyon

Detecting an anyon requires a *loop:*
just like a gaussian surface detects charge

Can an *entanglement cut* serve as
a “gaussian surface” for detecting “anyon charge” ?



Using entanglement to detect an anyon



$$|\Psi\rangle = \sum_\alpha \lambda_\alpha \, |\alpha\rangle_A \, |\alpha\rangle_B$$

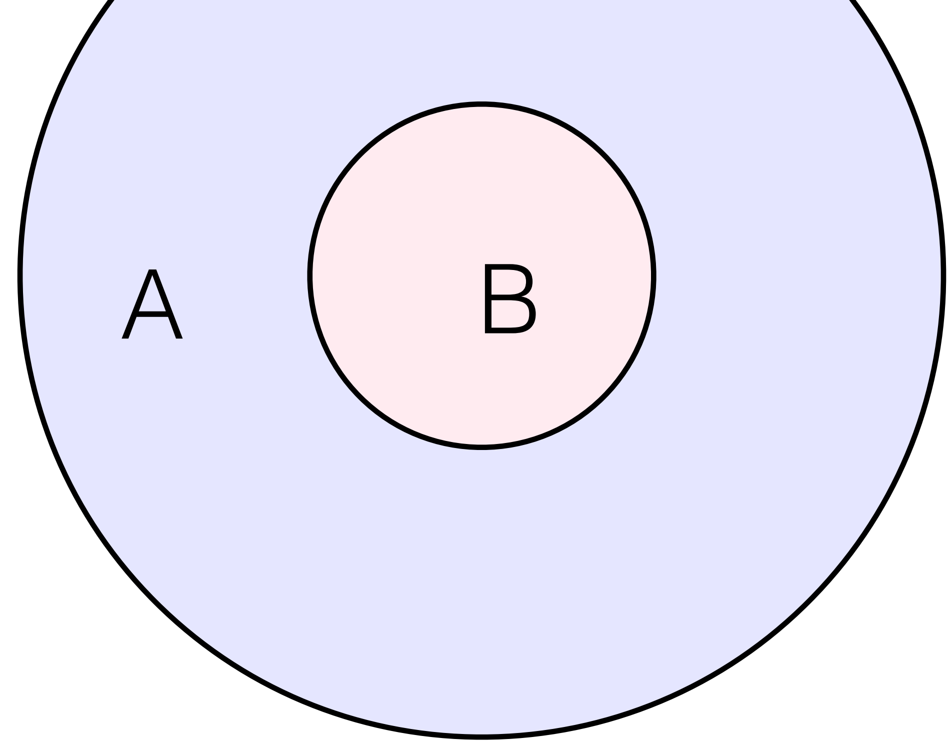
$$\sum_\alpha \lambda_\alpha^2 = 1, \quad S_E = -\sum_\alpha \lambda_\alpha^2 \log(\lambda_\alpha^2)$$

The entanglement spectrum

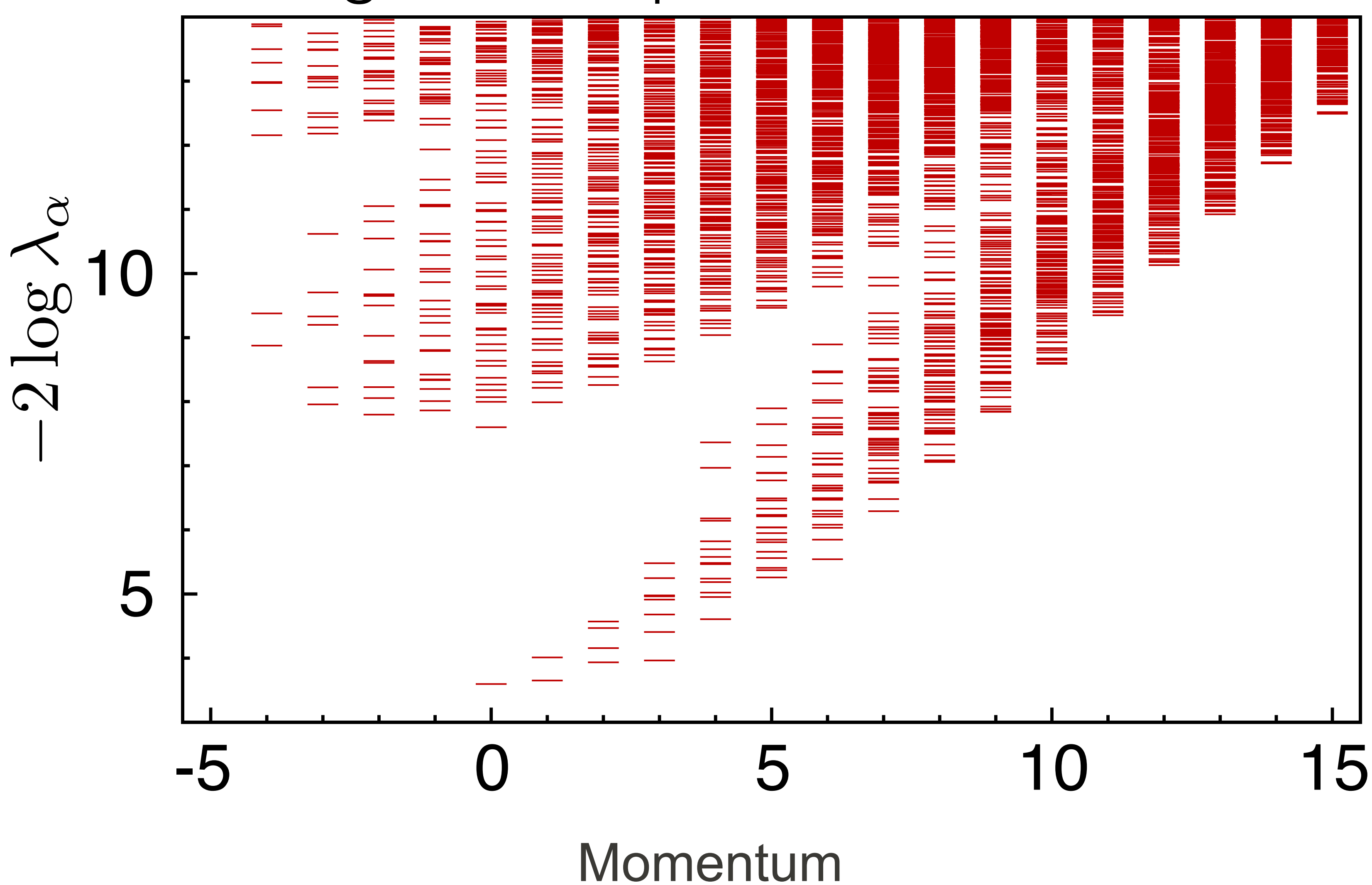
$$|\Psi\rangle = \sum_\alpha \lambda_\alpha |\alpha\rangle_A |\alpha\rangle_B$$

Schmidt states can be assigned good quantum numbers: (here ang. momentum)

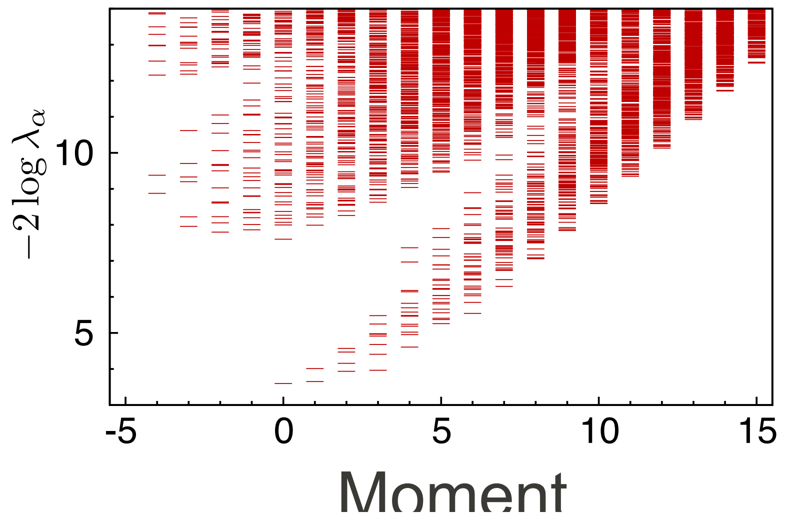
$$\hat{L}^z |\alpha\rangle_B = L_\alpha |\alpha\rangle_B$$



'Entanglement spectrum' vs momentum [Kitaev & Preskill 06; Haldane & Li 08]



from FQHE, Moore-Read state
[Zaletel, Mong & Pollmann 2012]

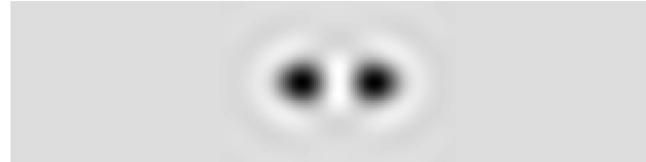


Aside:

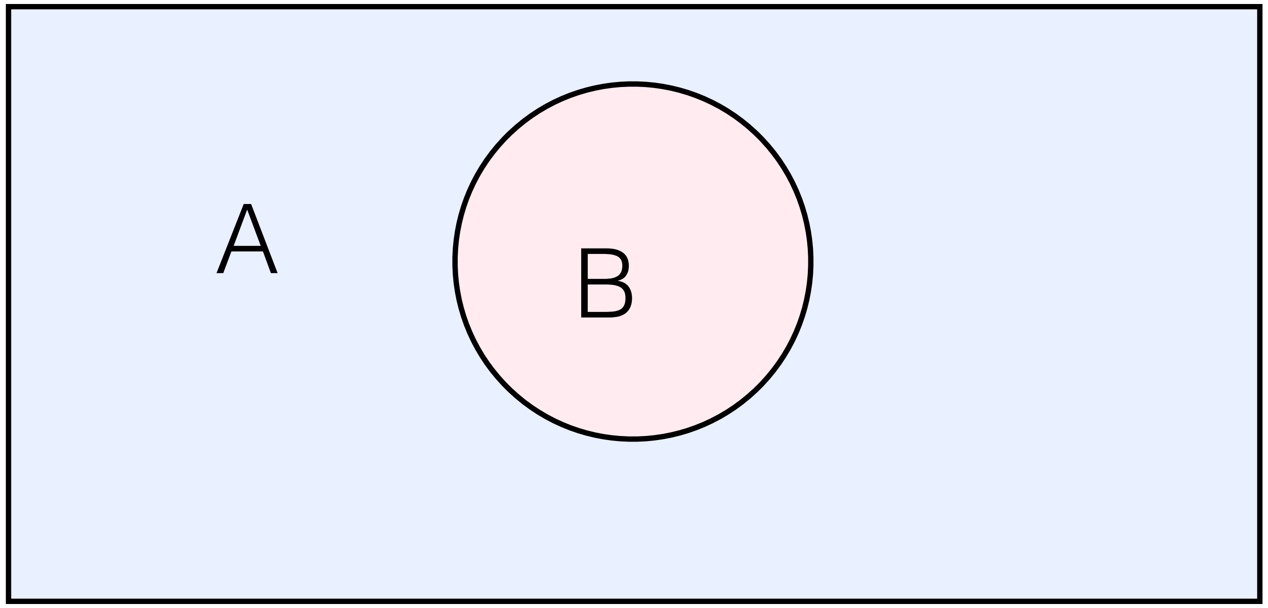
It is not *in general* true that the low - energy part of the entanglement spectrum “is” like the low energy part of the physical edge theory

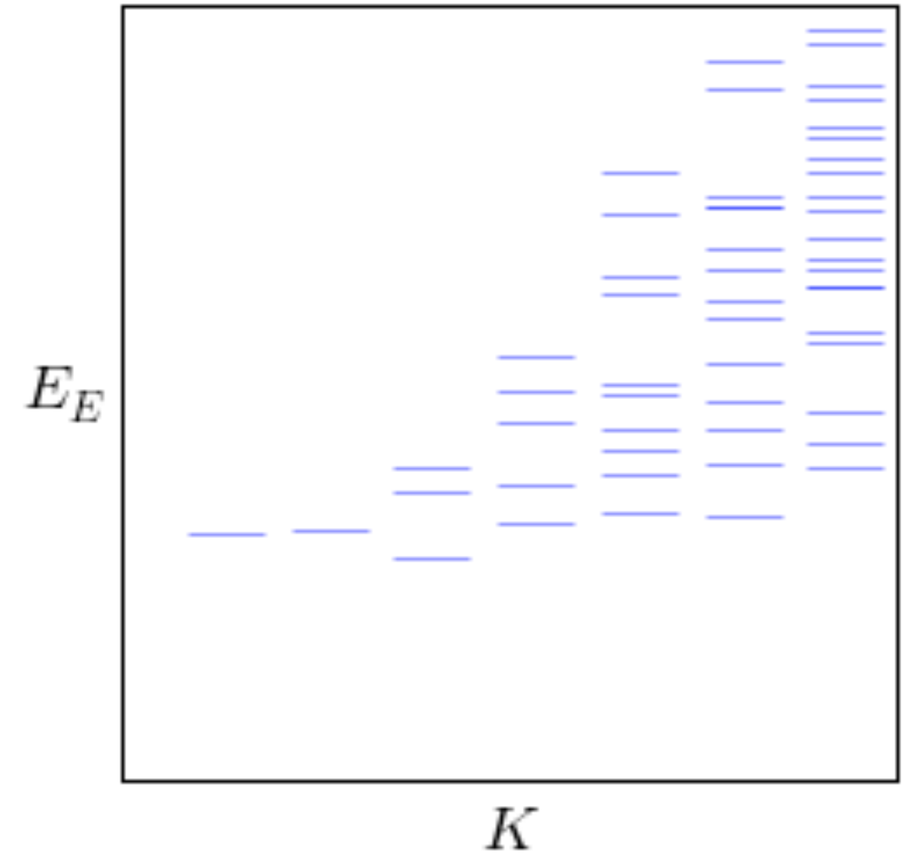
Certain *averaged* properties - like entanglement entropy - are robust. These depend on the *high energy* part of ES.

Anomalies of physical edge = anomalies in entanglement edge
(this is how charge pumping works)

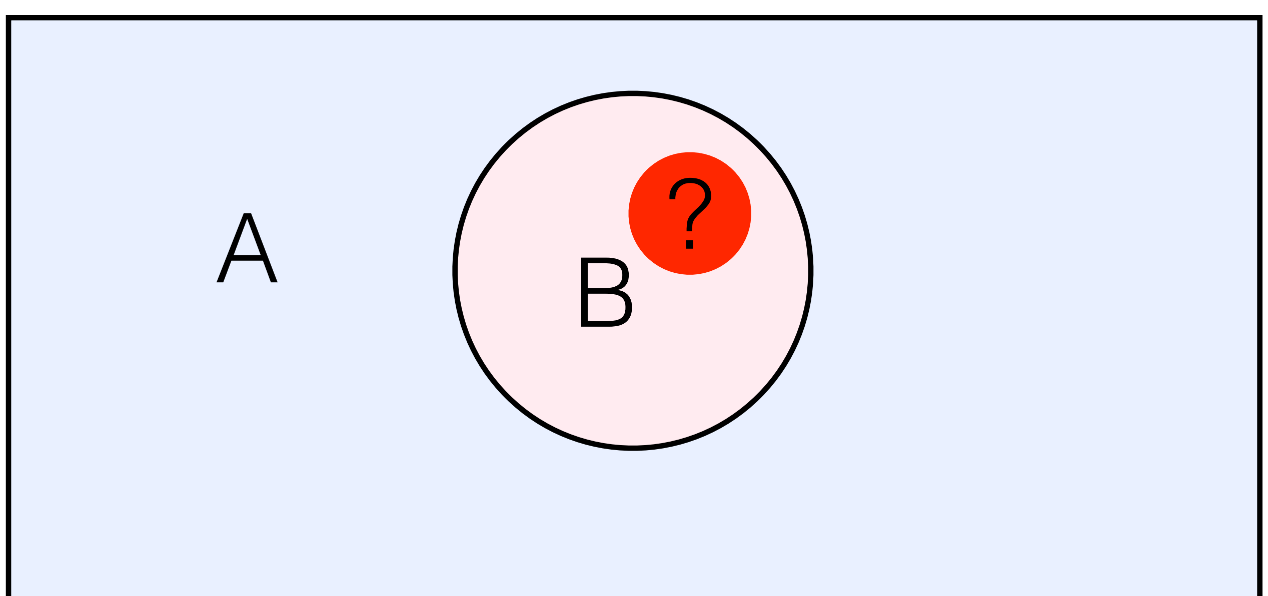
Numerical experiment: 

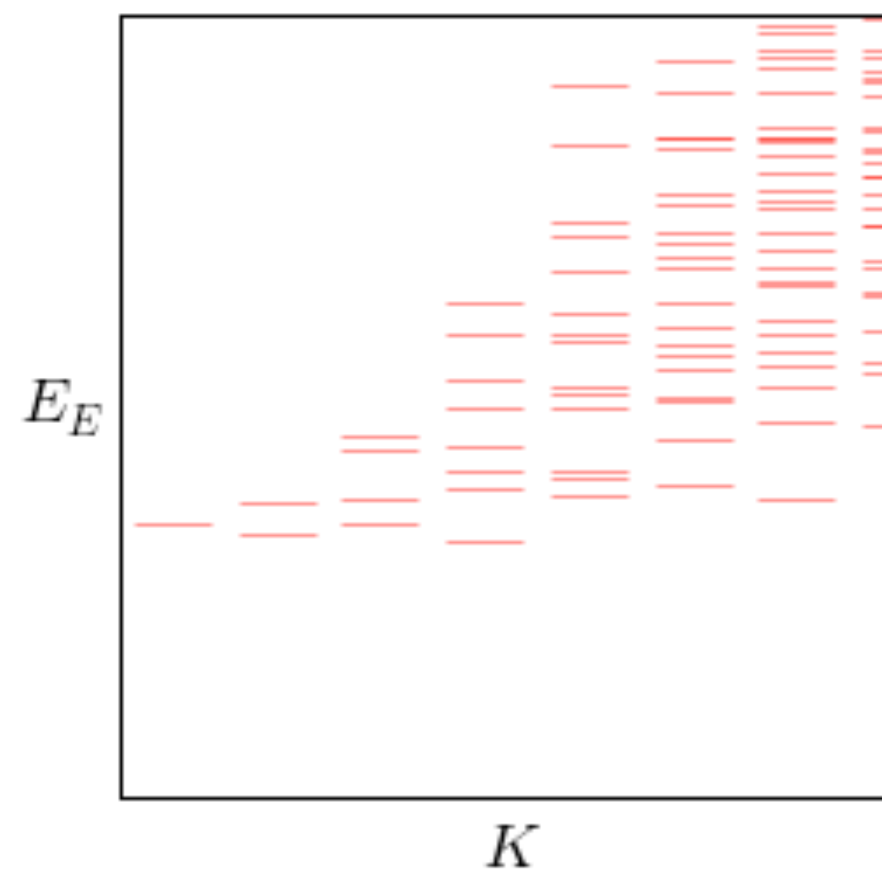
$$E_E = -\log(p)$$





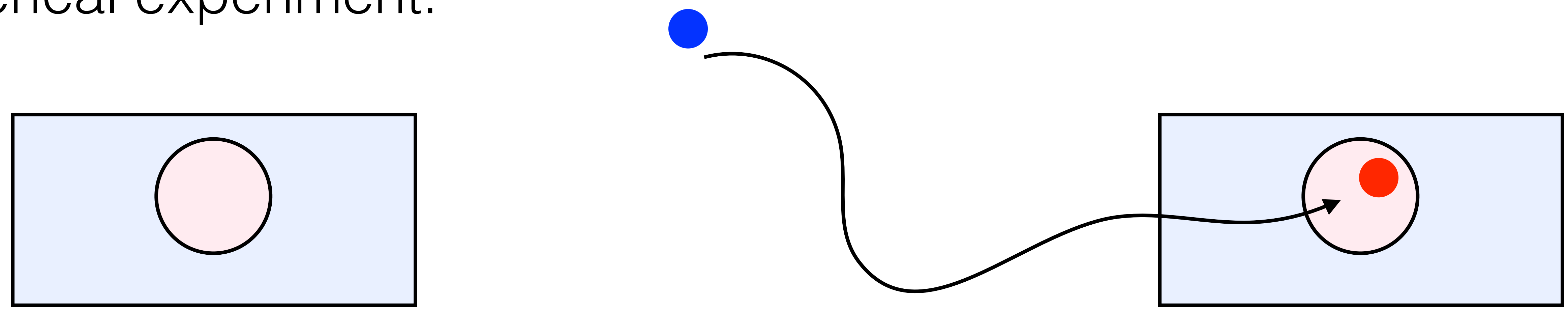
If B is big, ES changes if and *only* if ? is an anyon!



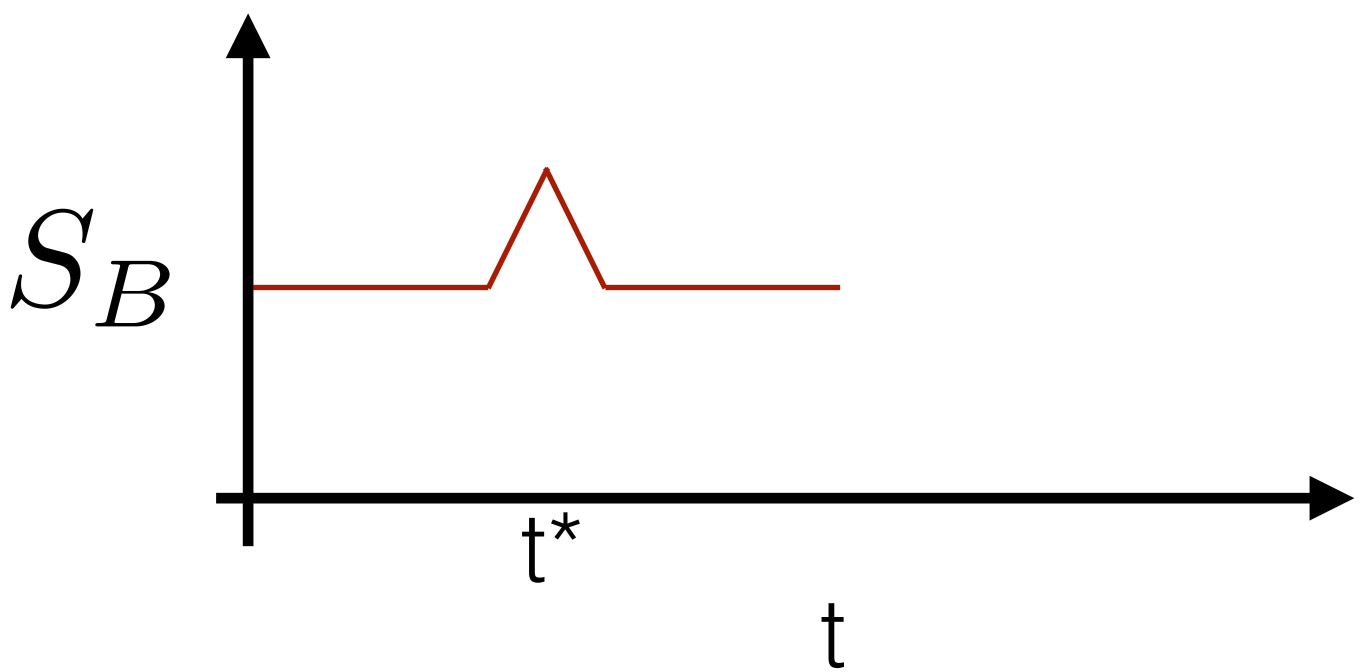


[Kitaev & Preskill 2006; Papic et al. 2010]

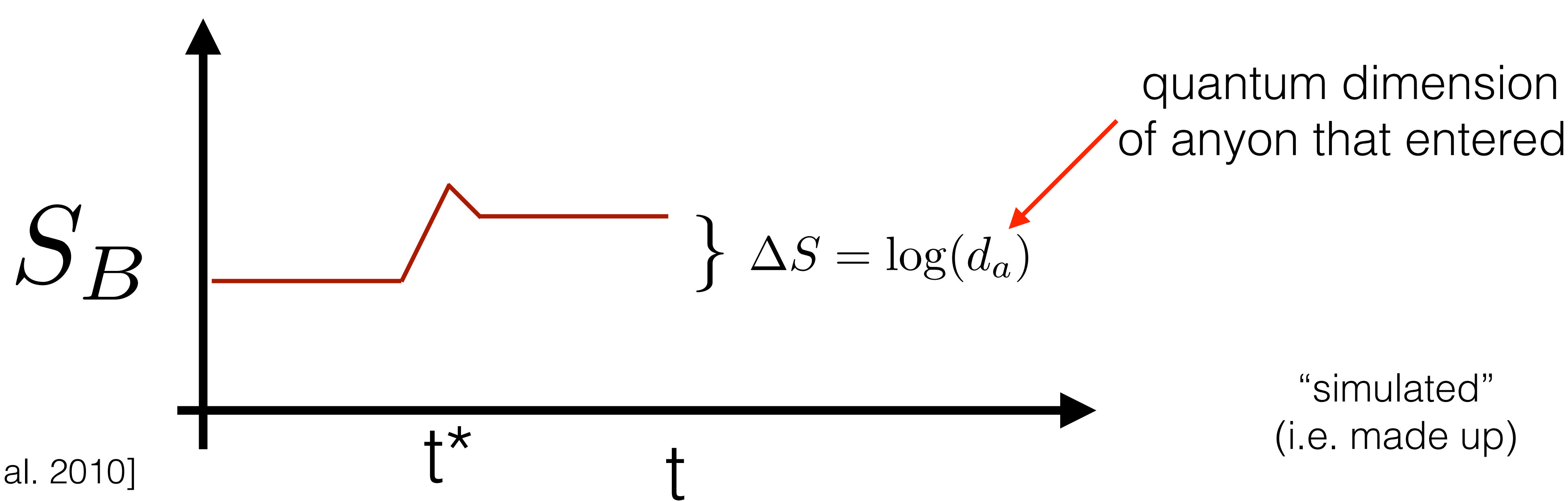
Numerical experiment:



Anyon 1



Anyon 2

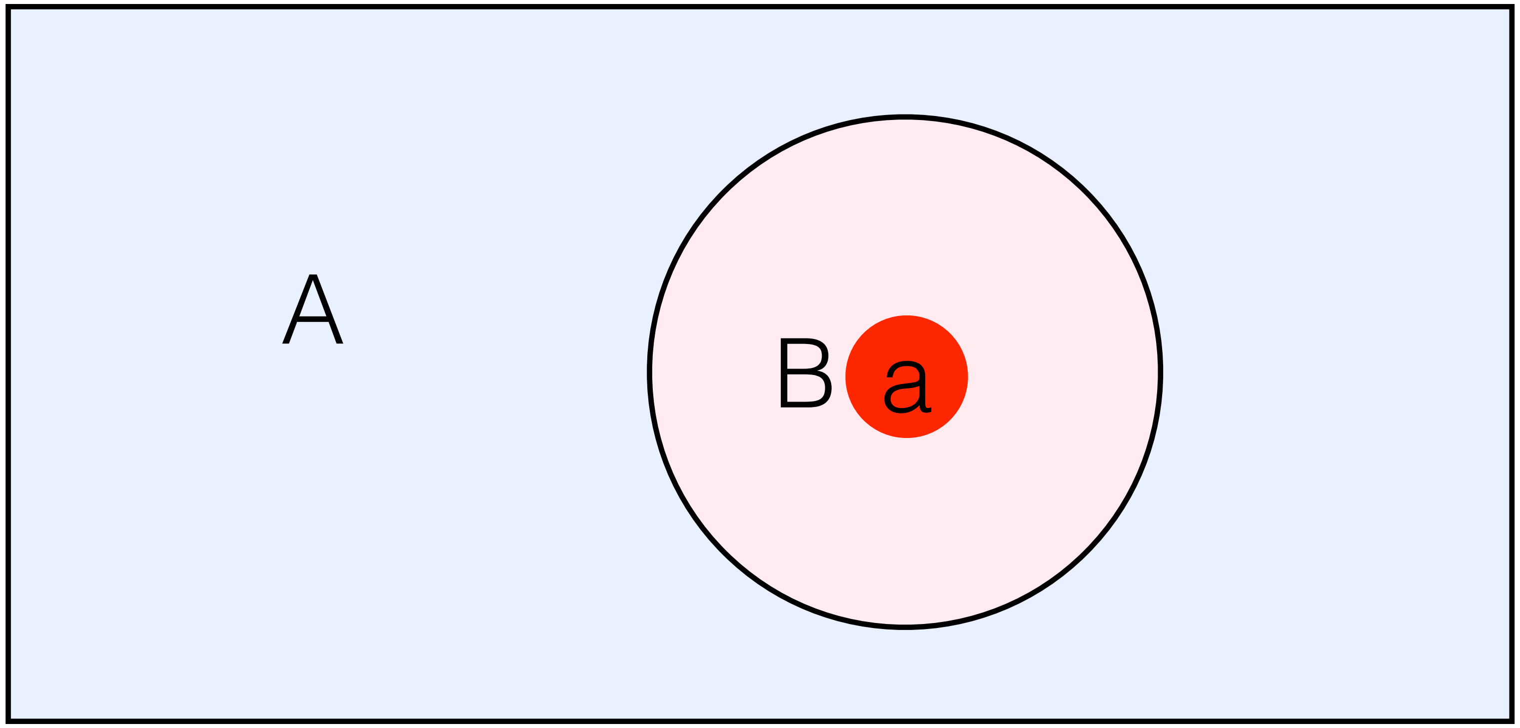


“simulated”
(i.e. made up)

[Kitaev & Preskill 2006; Papic et al. 2010]

TEE: Take II

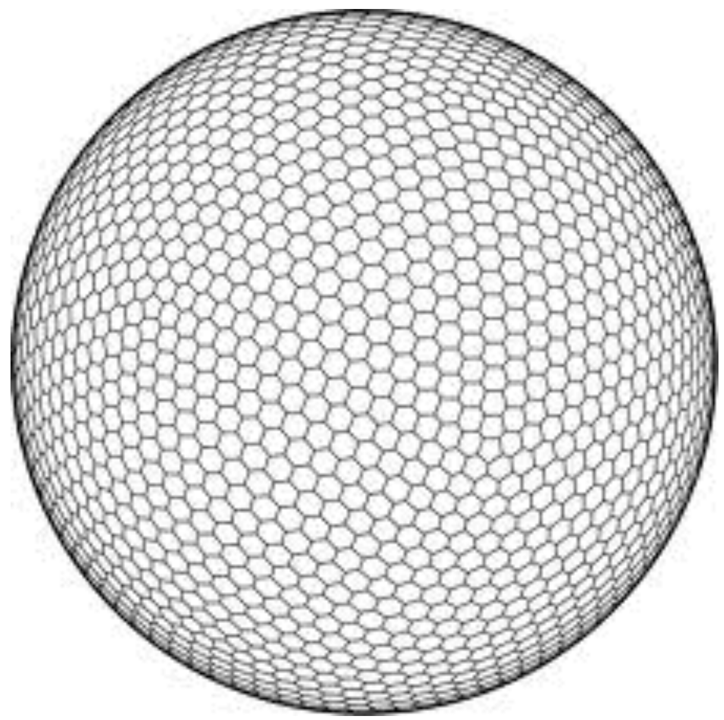
(continuum, smooth cut: no garbage)



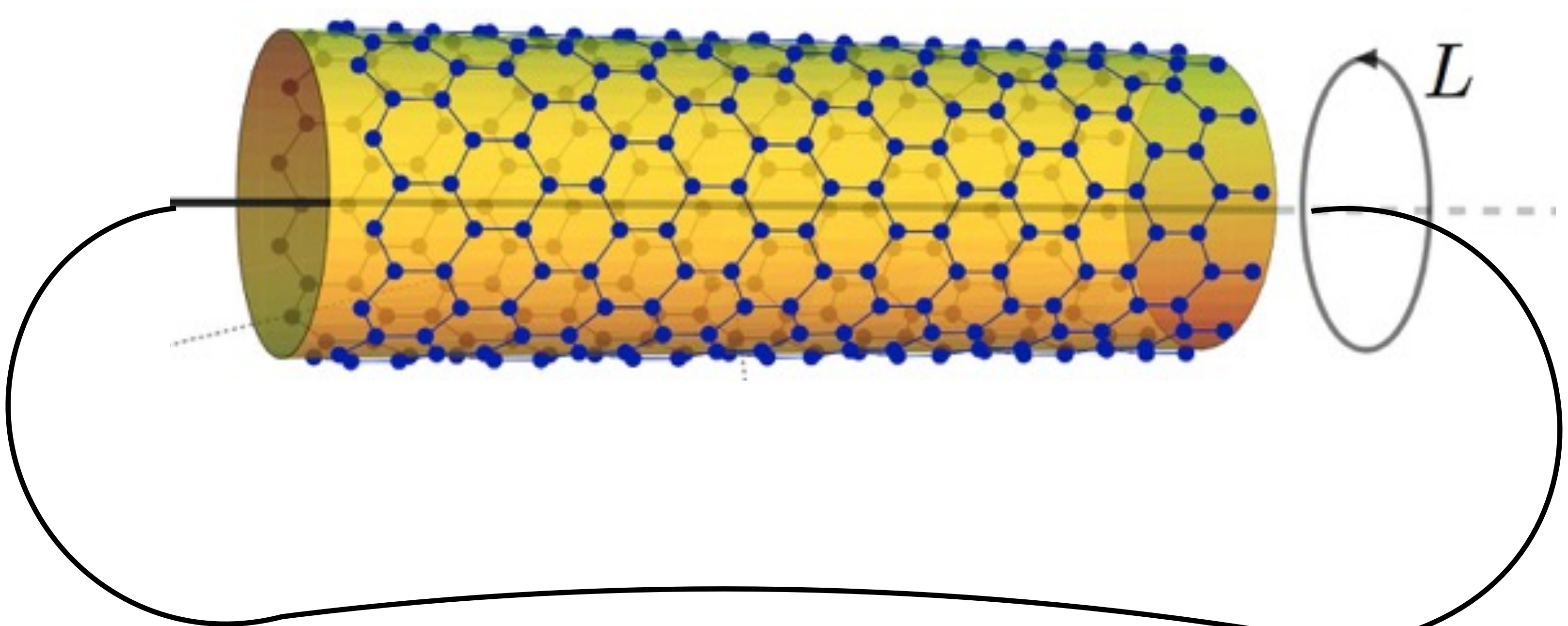
$$S_B = \alpha|\partial B| - \log(\mathcal{D}) + \log(d_a)$$

a = 1 (no anyon) is special case

Topological ground state degeneracy



Sphere:
generically unique ground state



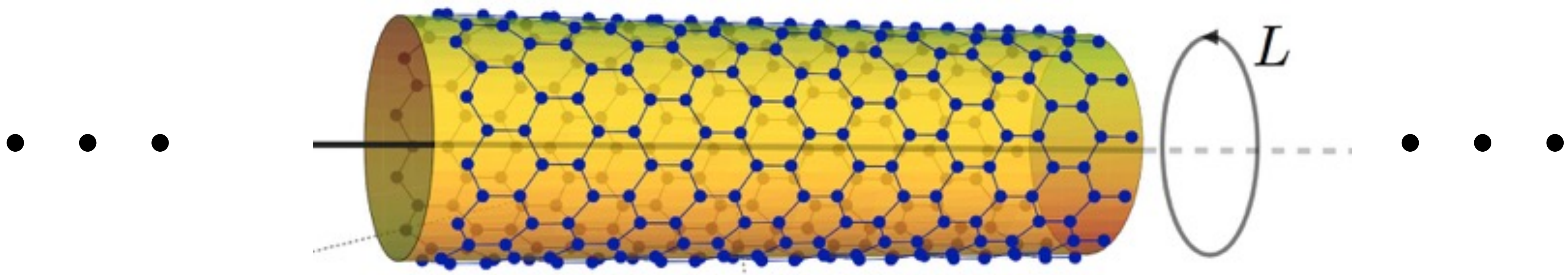
Torus:
$\mathfrak{m}$ degenerate ground states

$\mathfrak{m}$: # of anyon species

The cylinder has genus too!

Infinitely long cylinder: same gsd $\mathfrak{m}$ as torus!

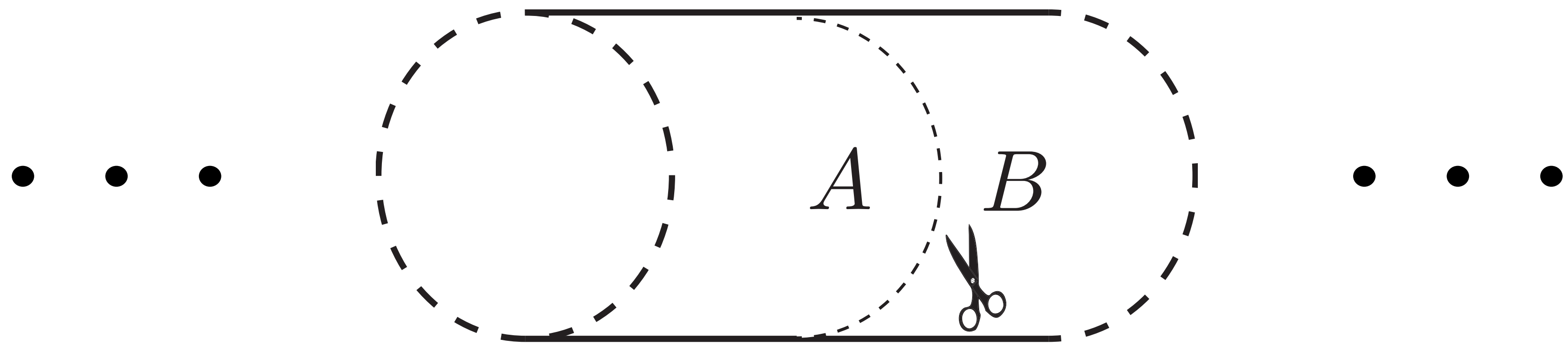
(we'll remind ourselves why shortly)



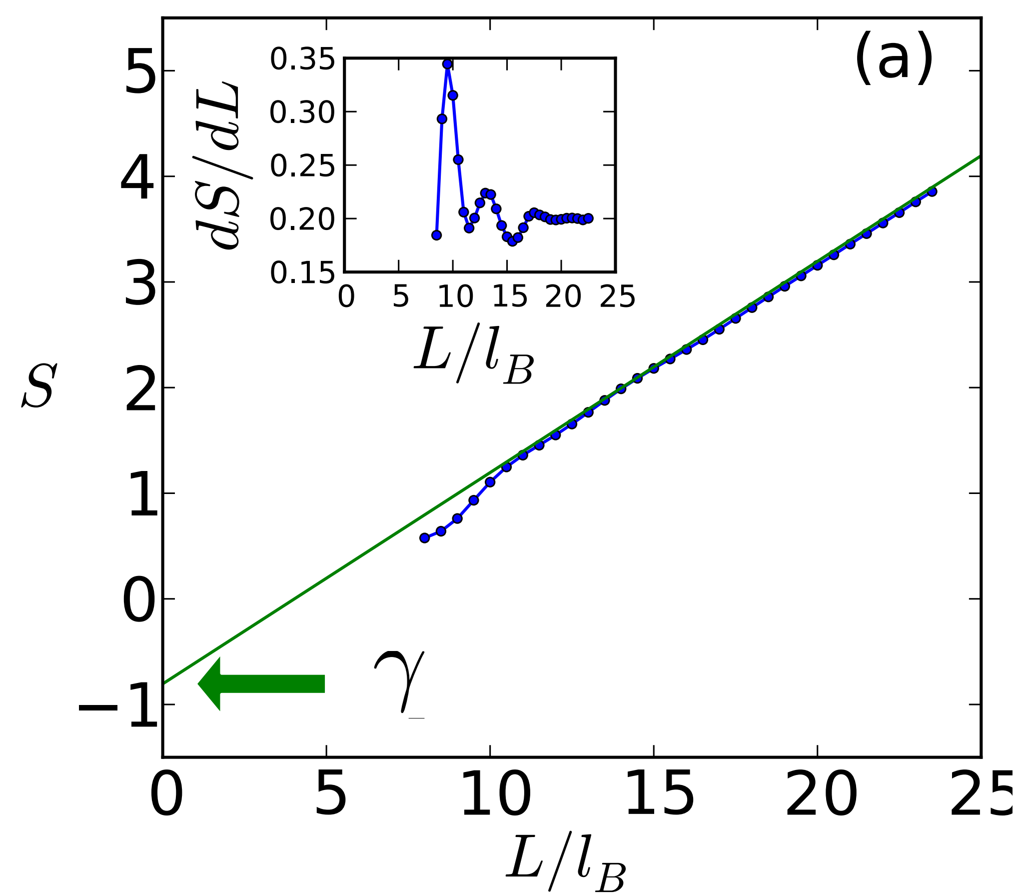
Great for numerics! *No* edge effects,
uses full translation invariance, and still has gsd

[Cincio & Vidal 2012; Zaletel, Pollmann & Mong 2012]

Sorting out cylinders



Entanglement entropy of infinite cylinder vs circumference

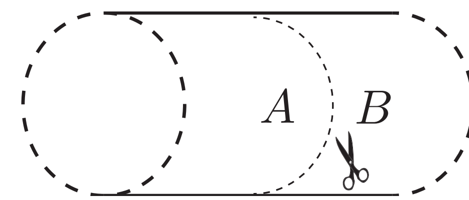


[Zaletel, Mong, Pollmann 2013]

$$S = \alpha L - \gamma$$

Minimally entangled states

[Zhang, Grover, Turner, Oshikawa & Vishwanath]



Infinite cylinder: $S = \alpha L - \gamma$

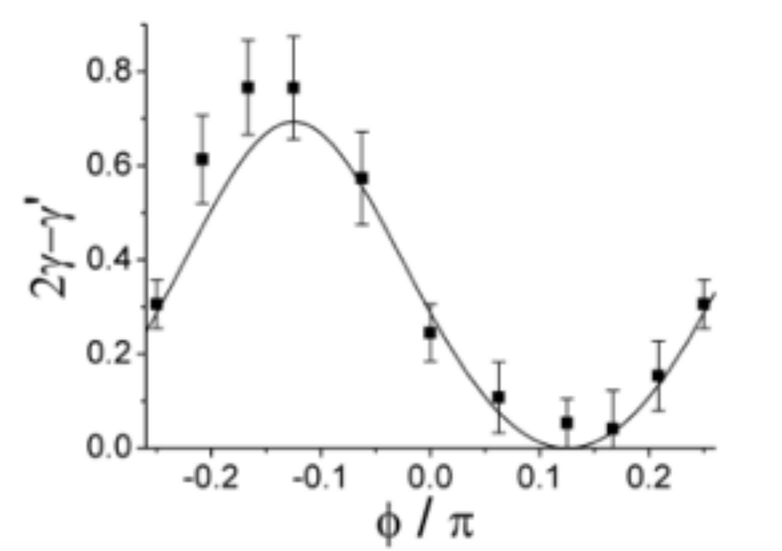
What's γ ?

Naive answer: like a disc, $\log(\mathcal{D})$

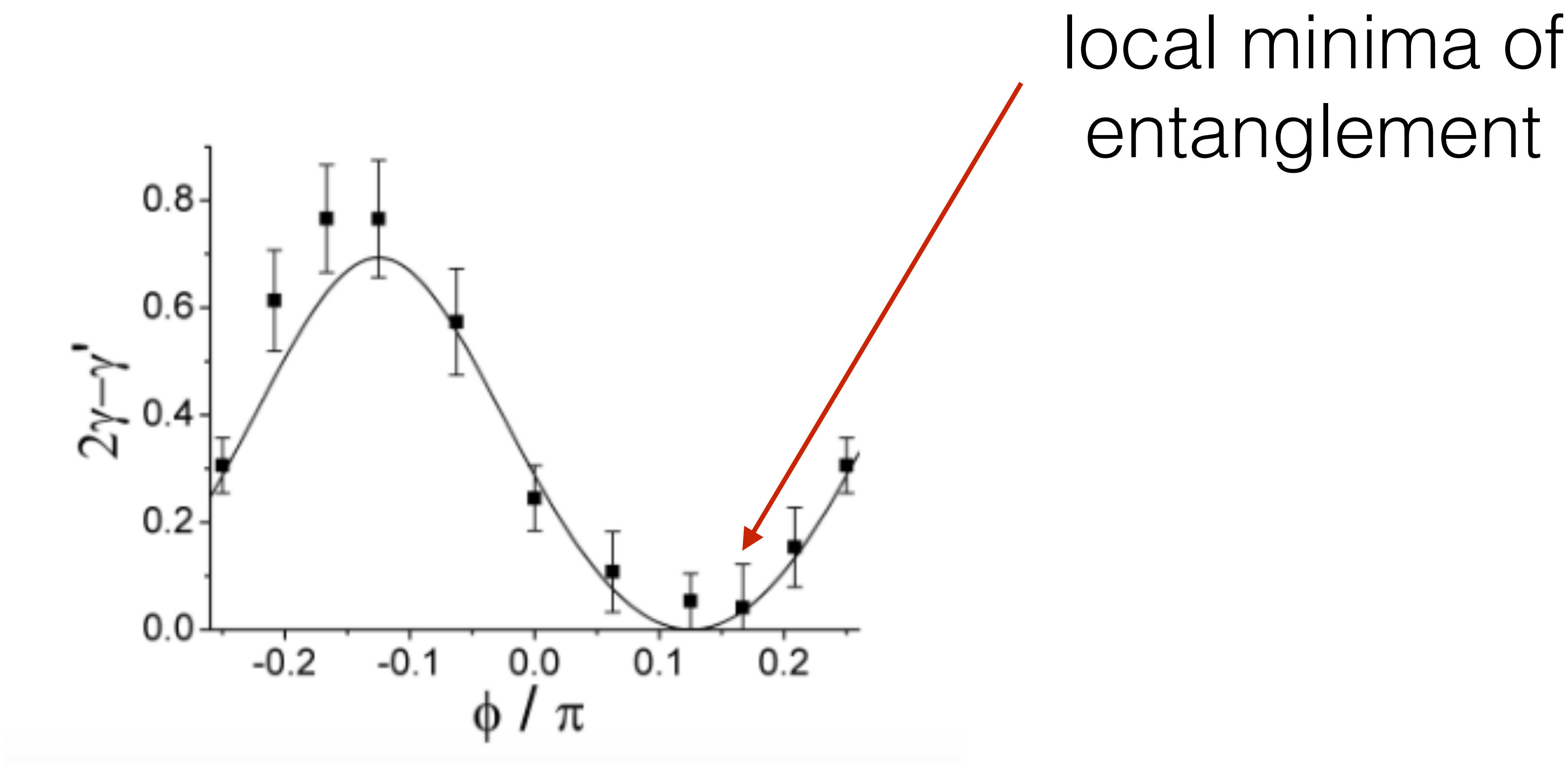
Answer: it depends!

There is an $\mathfrak{m}$ dimensional manifold of ground states: the E.E. depends continuously on the state

$$S(c_i) = S[\sum_{a=1}^{\mathfrak{m}} c_a \left|a\right\rangle]$$



Minimally entangled states



There is a special basis $\{|a\rangle\}$

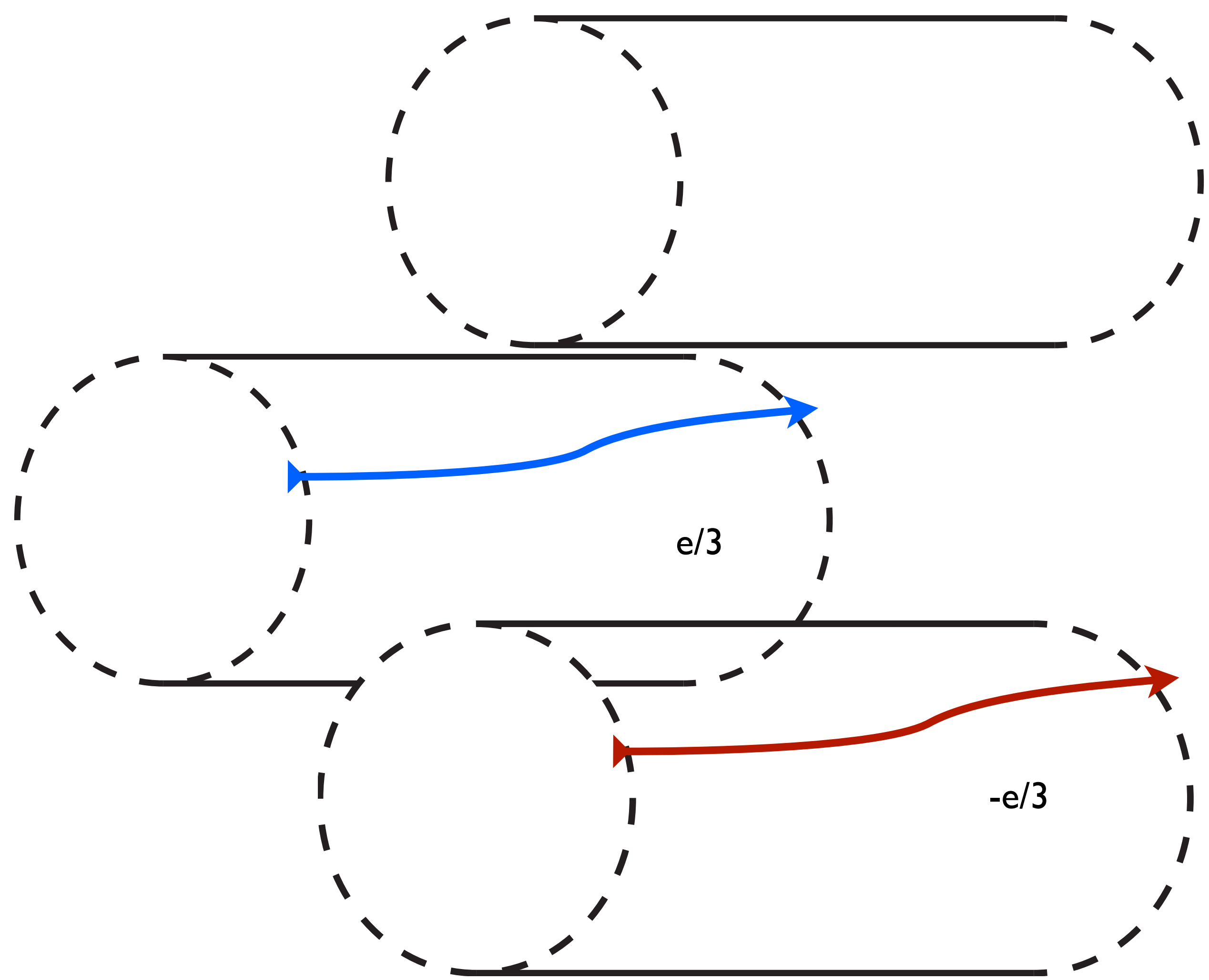
which are local minima of entanglement; in this basis

$$S_a = \alpha L - \log(\mathcal{D}) + \log(d_a)$$

$|a\rangle \leftrightarrow$ anyon types

[experts: subtle when [Tx,Ty] PSGs]

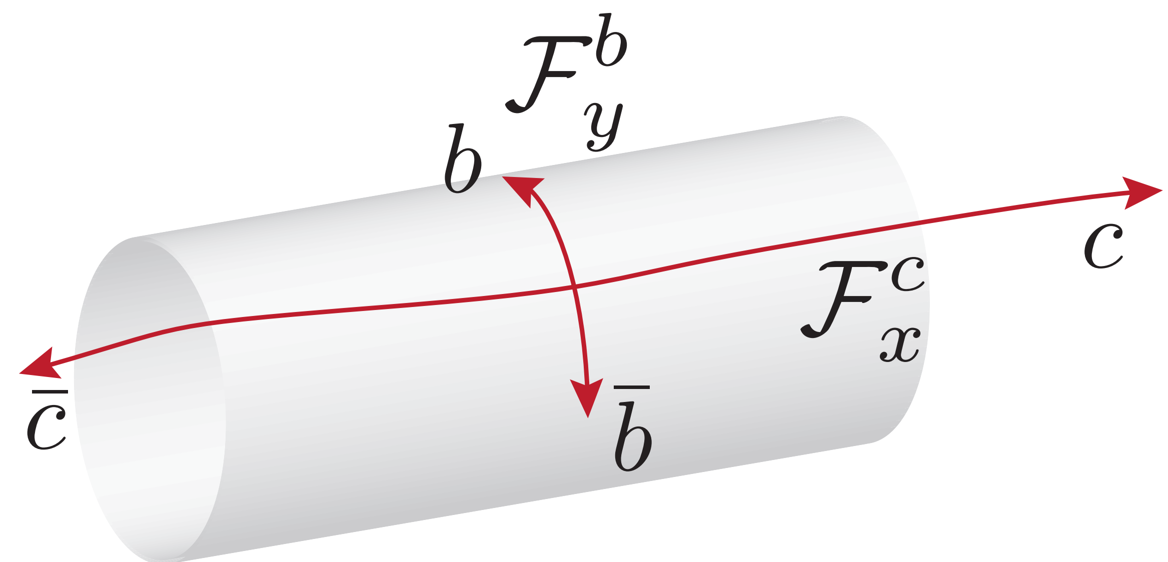
The minimally entangled states have
“definite topological flux threading the cylinder”



What does this mean?

A more physical picture

a) Standard story:
(like strings in toric code)



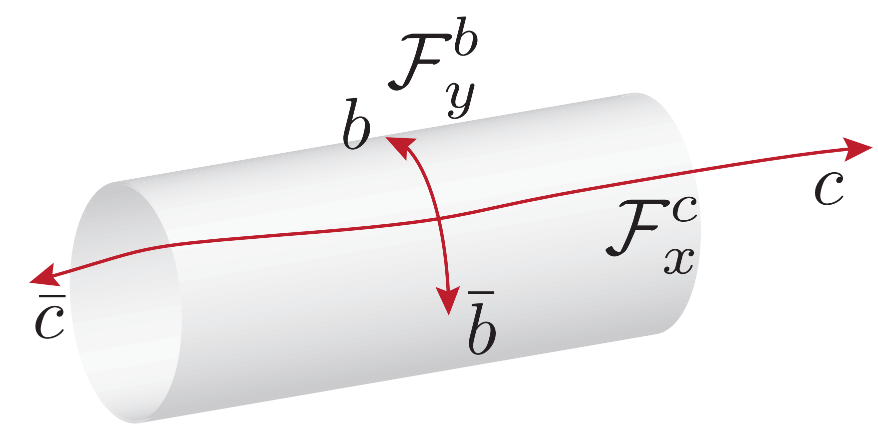
$\mathcal{F}_y^b$ separate $b\bar{b}$ pair around cylinder

$\mathcal{F}_x^c$ separate $c\bar{c}$ pair out to infinity

(unitary operators in ground state manifold)

A more physical picture of MES

a)

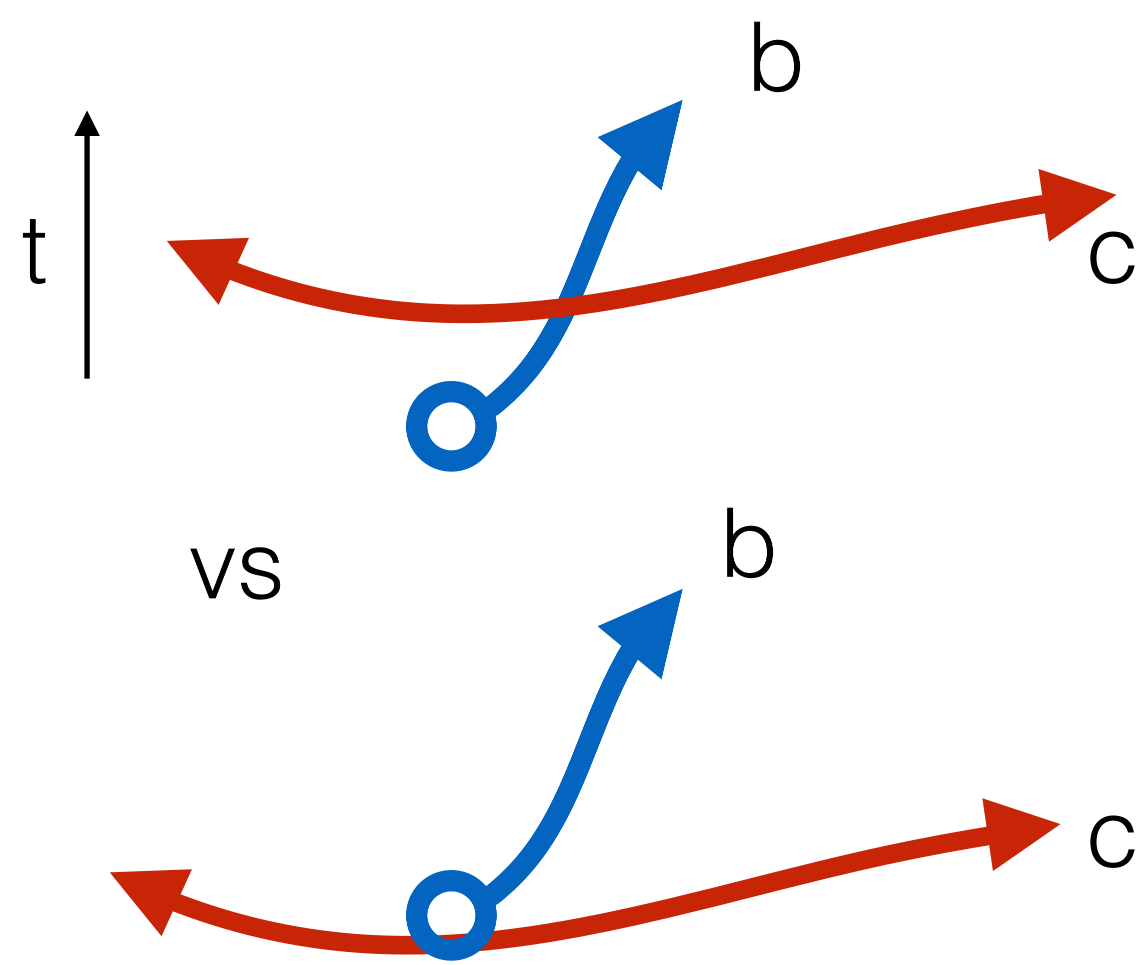


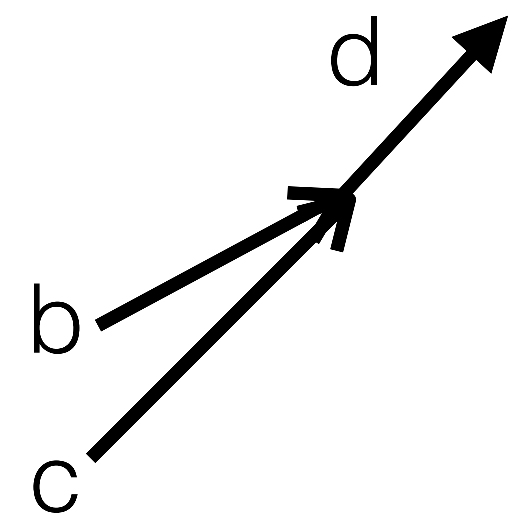
Statistics

$$\mathcal{F}_y^b \mathcal{F}_x^c = \frac{S_{bc}}{S_{b\mathbb{1}}} \mathcal{F}_x^c \mathcal{F}_y^b :$$

Fusion

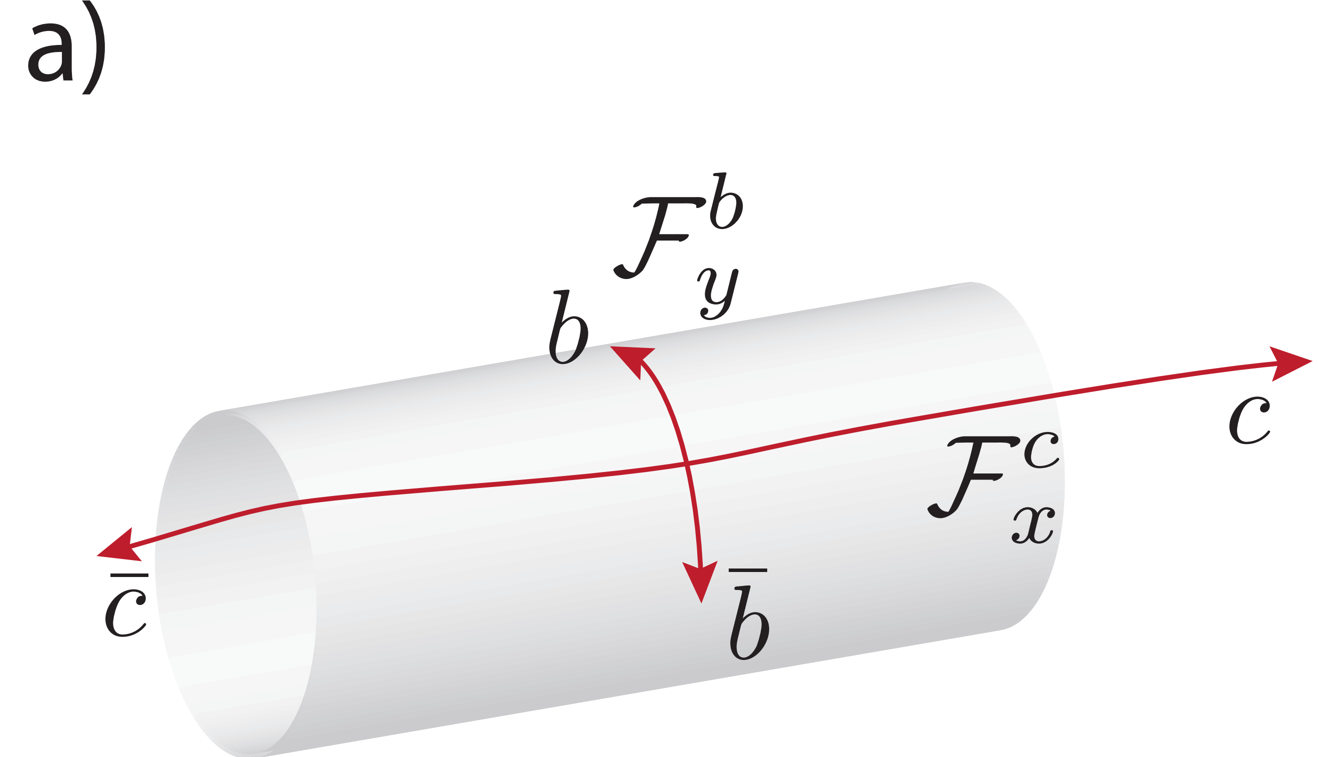
$$\mathcal{F}^b_{x/y}\mathcal{F}^c_{x/y} \propto \mathcal{F}^{b\cdot c}_{x/y}, \quad b\cdot c = \sum_d N^d_{bc}\, \mathrm{d}$$





This algebra ***requires*** an $\mathfrak{m}$ dimensional representation: **GSD**

A more physical picture of MES



$$\mathcal{F}_x^b \, |a\rangle = |b \cdot a\rangle$$

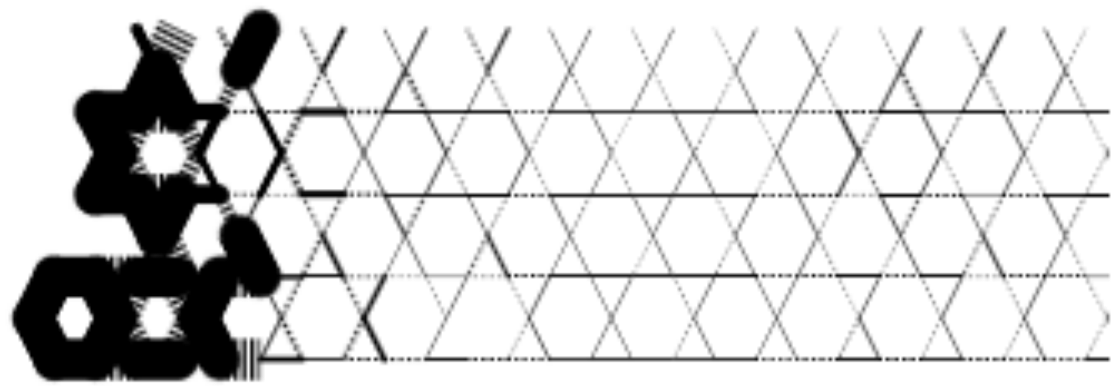
Permutes MES

$$\mathcal{F}_y^b \, |a\rangle = |a\rangle \cdot \frac{S_{ba}}{S_{b\mathbb{1}}}$$

Diagonal in MES

Unless you try really hard DMRG always produces MES basis (for two reasons…)

Example: MES of the Kagome Heisenberg Model



[from Yan, Hus & White 2010]

S = 3/2 per unit cell: H.O.L.S.M.A. says either

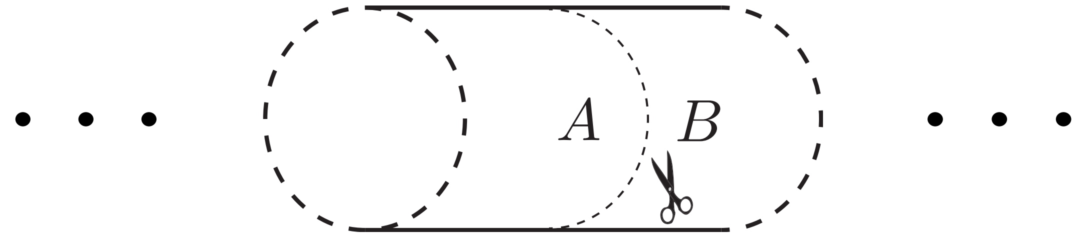
[Lieb, Schultz & Mattis 1961; Oshikawa 2000; Hastings 2003]

1. SSB (not seen in dmrg)
2. Gapless (DMRG gap = 0.05 J)
3. Topologically ordered

(no 'trivial' paramagnet)

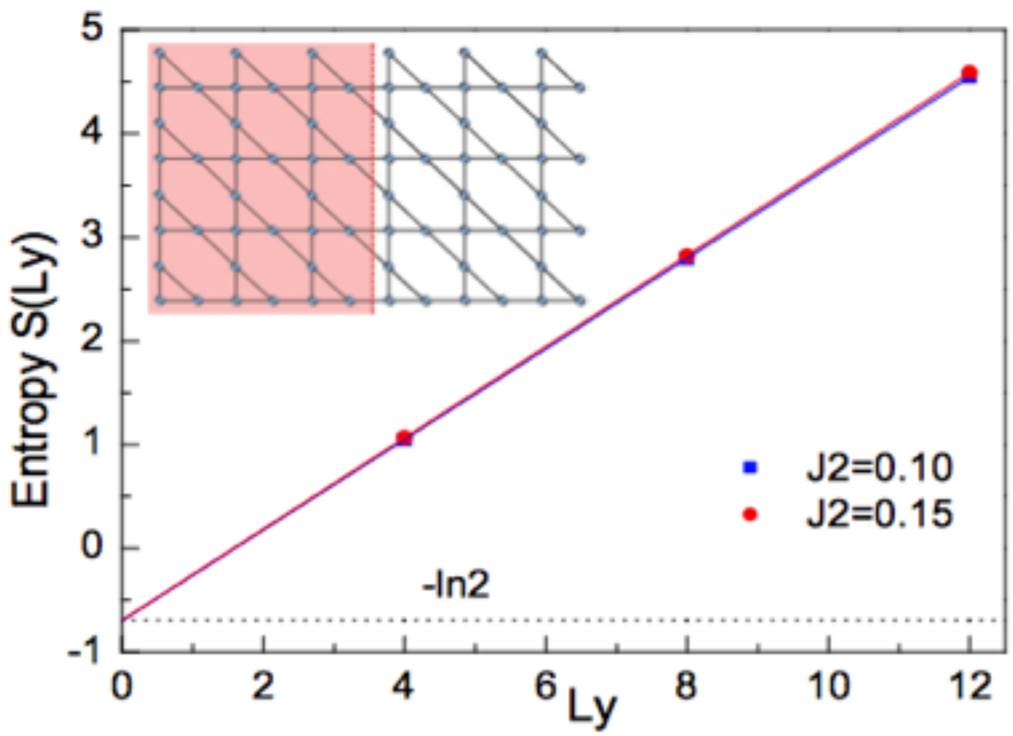
What type of topological order?

Kagome TEE



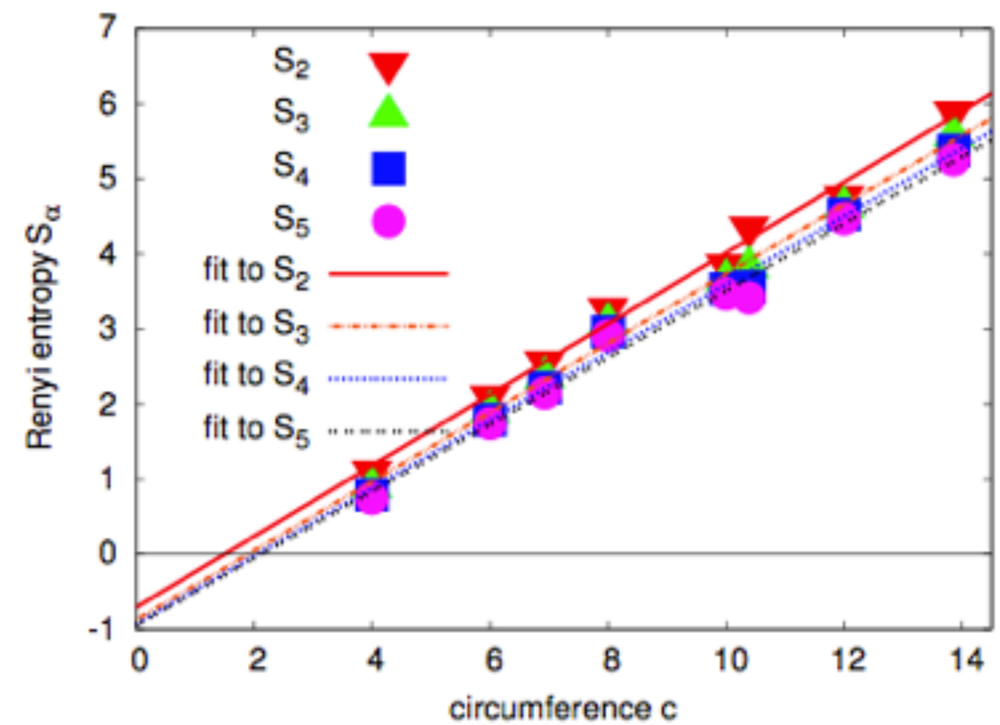
$$S_E = \alpha L - \log(\mathcal{D})$$

With J2



Jiang, et al. Nature 2012

J2 = 0 (shakier)



Depenbrock, et al. PRL 2012

Enumerate all TQFTs with T-reversal & $\mathcal{D} \leq \sqrt{5}$

Two contenders: **Z2 gauge theory** or **double semion**

Both have $\gamma = \log(2)$; close to numerics

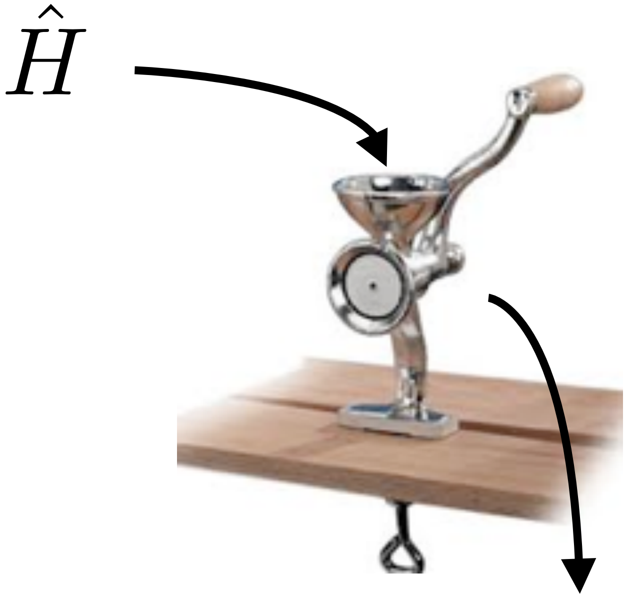
Double semion impossible for subtle symmetry reasons

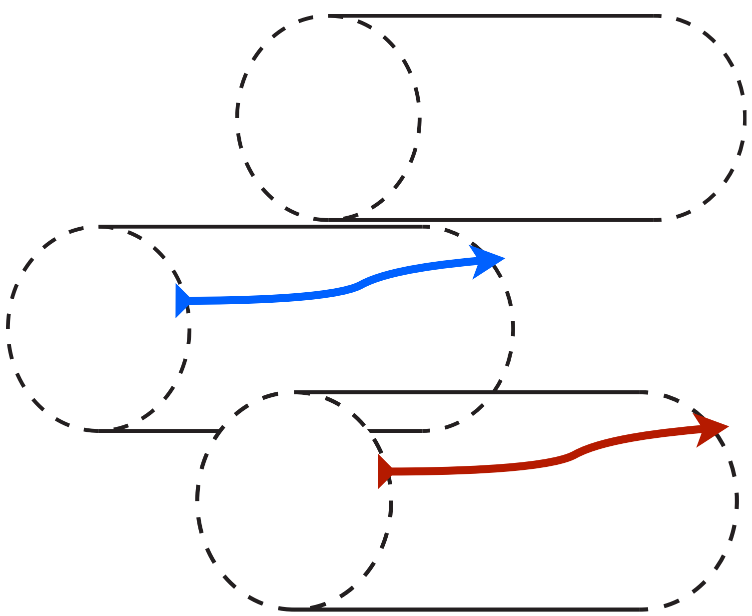
[Zaletel & Vishwanath 2014]

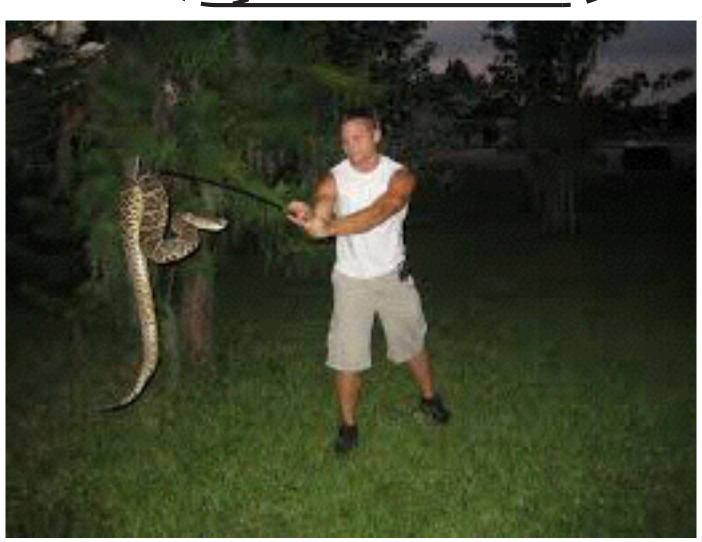
Seems like Z_2 Gauge-Theory (toric code)
- just like we've heard about this week!

$$\{1, e, m, em\} \Leftrightarrow \{1, b, v, f\}$$

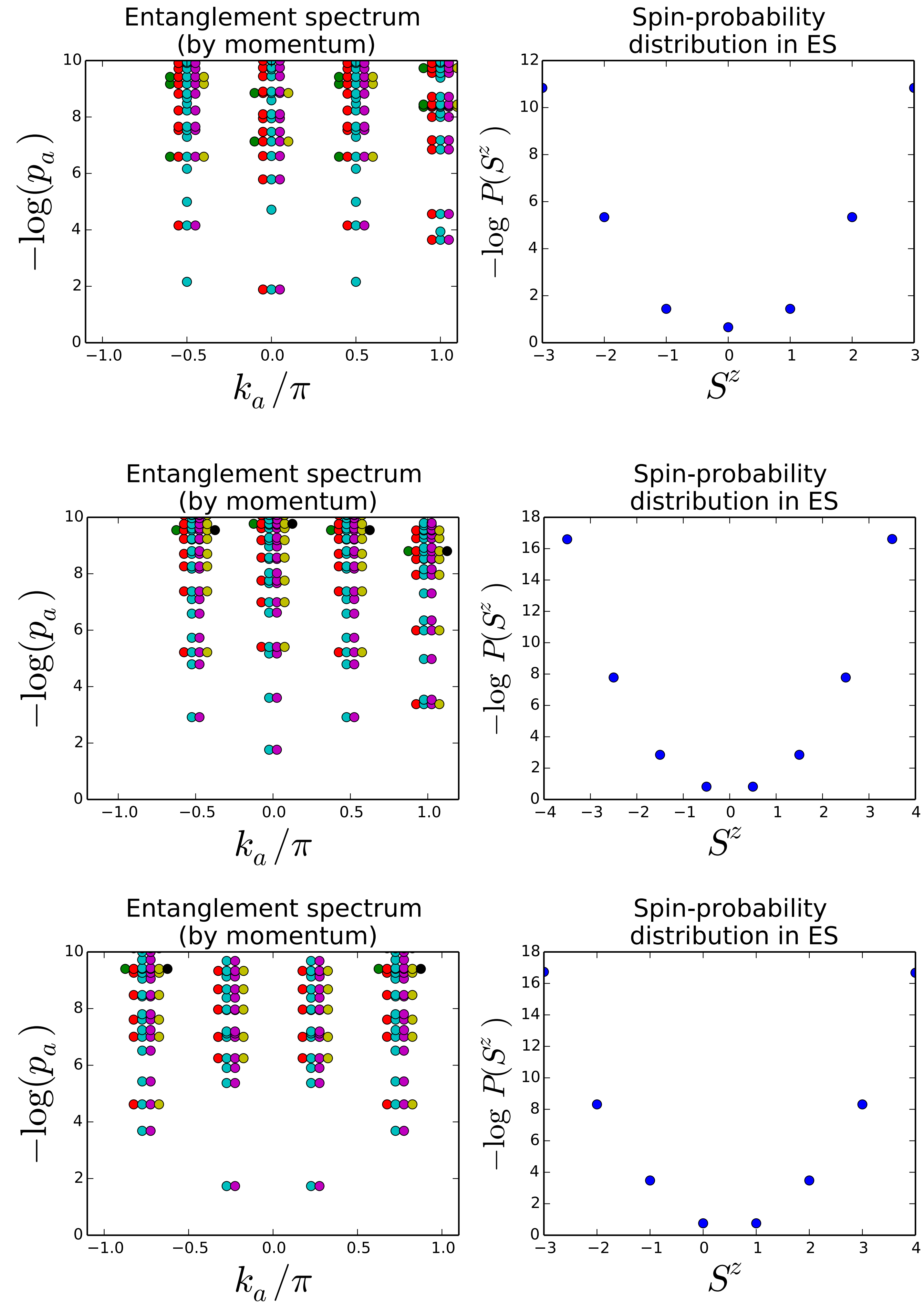
Bosonic spinon (b)
Vison (v)
Fermionic spinon (f = v b)

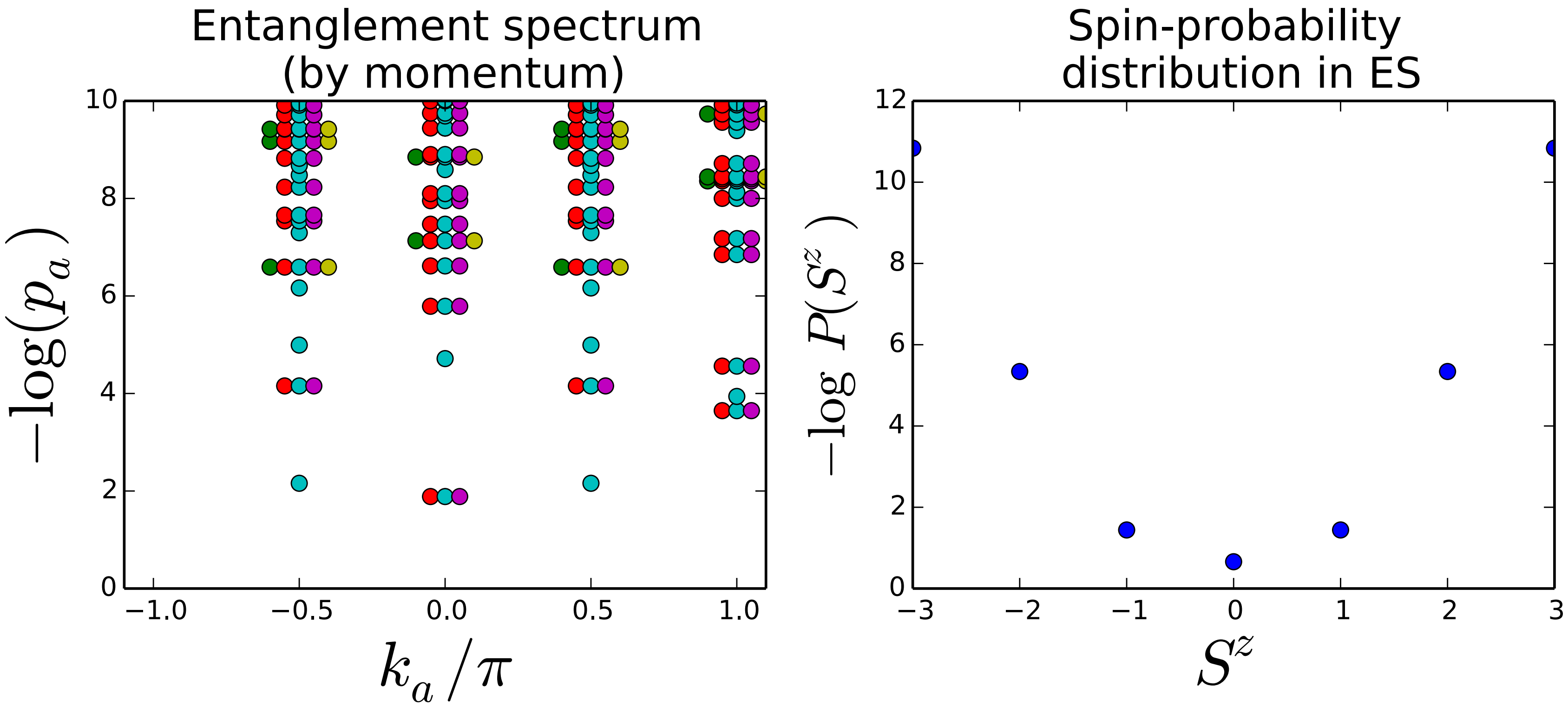




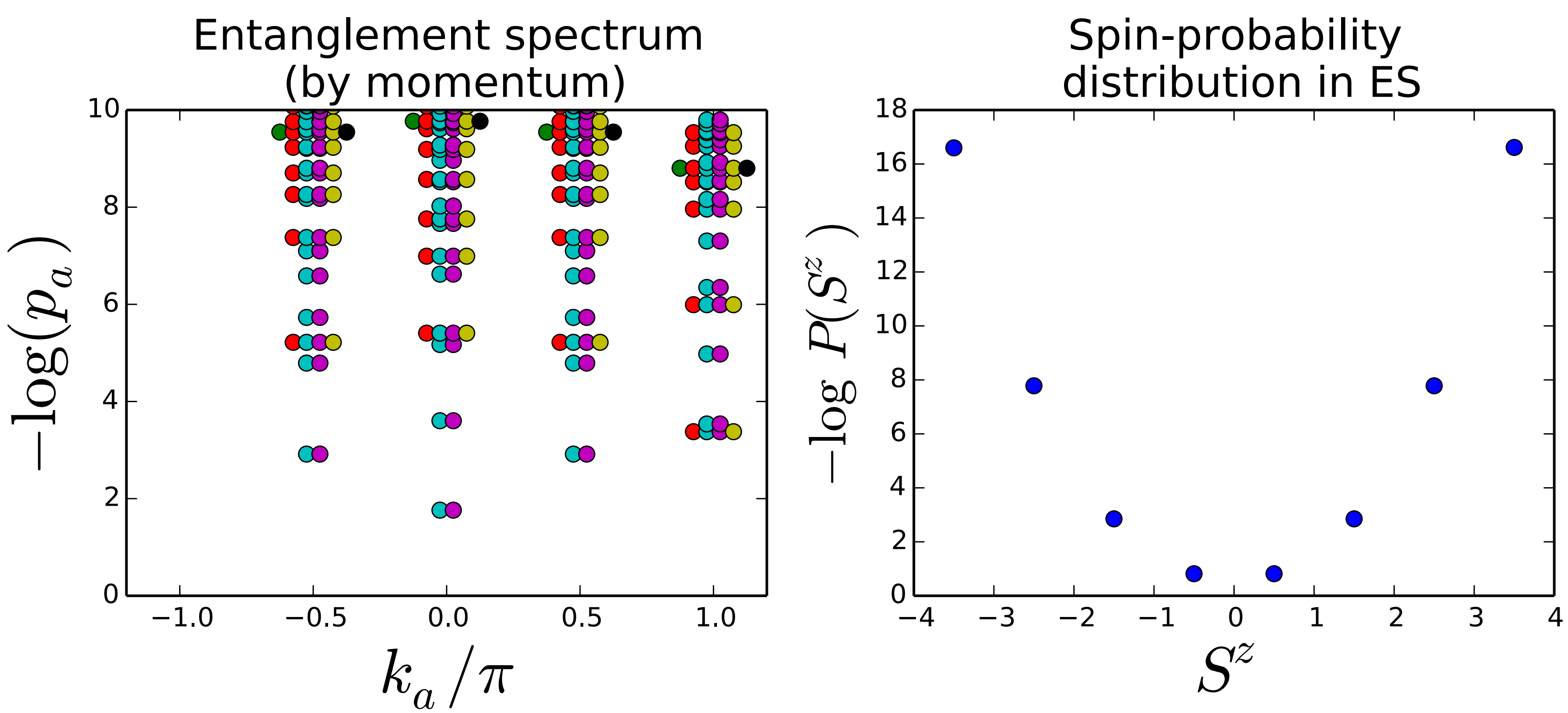


In collaboration with
Zhenyue Zhu, Yuan-Ming Lu,
Steve White, David Huse & Ashvin Vishwanath

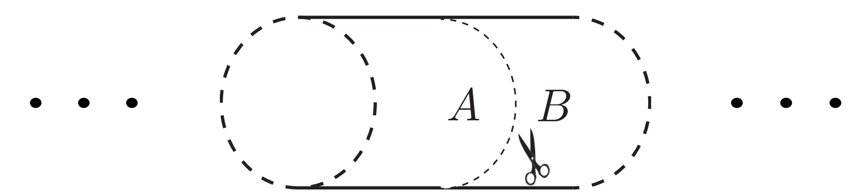




Vacuum or vison (S = 0)



bosonic or fermionic spinon (S = 1/2)

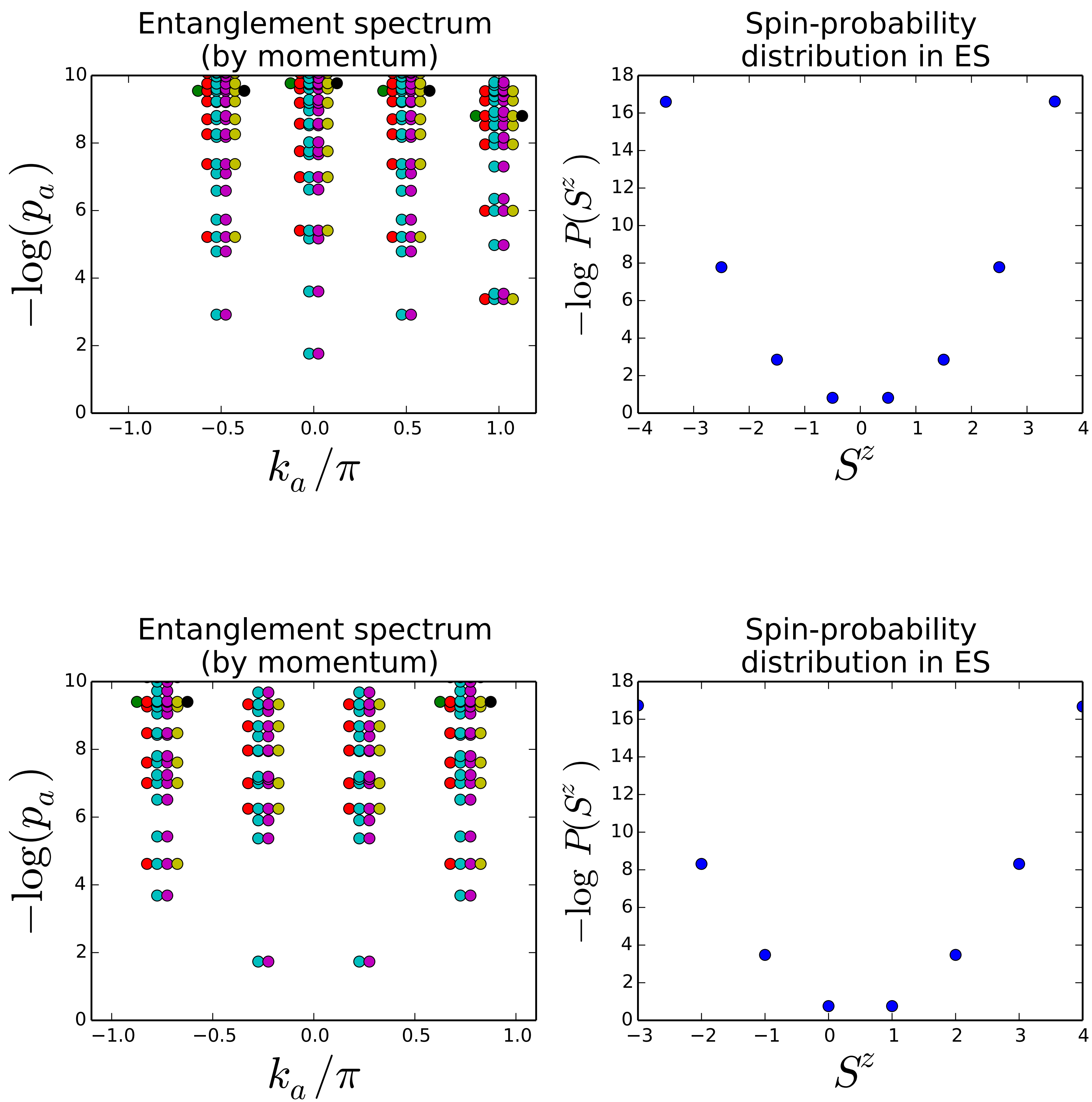


$$|\Psi\rangle = \sum_\alpha \lambda_\alpha |\alpha\rangle_A |\alpha\rangle_B$$

$$\hat{S}^z |\alpha\rangle_A = S^z_\alpha |\alpha\rangle_A$$

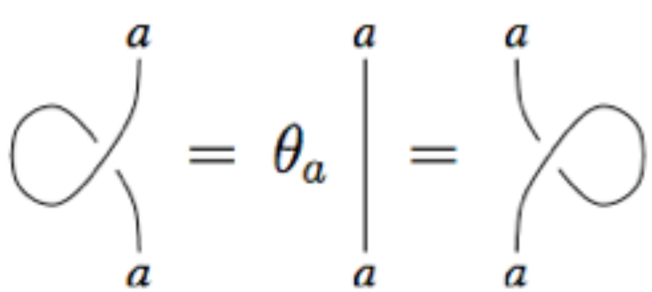
$$\hat{T}^y |\alpha\rangle_A = e^{ik_\alpha} |\alpha\rangle_A$$

Which is bosonic spinon, and which is fermionic?

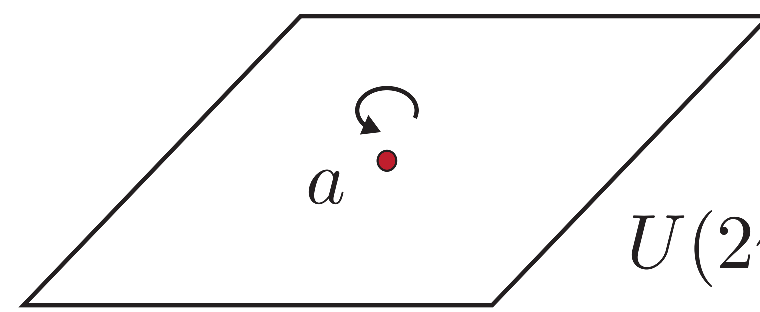


The “topological spin” $e^{2\pi i h_a}$

Mutual statistics of b / f the same. Need to examine:



$$\theta_a = e^{2\pi i h_a}$$



$$U(2\pi) = e^{2\pi i (h_a + \mathbb{Z})}$$

Bosonic spinon: $1 = e^{i \cdot 0}$

Fermionic spinon: $-1 = e^{i \cdot \pi}$

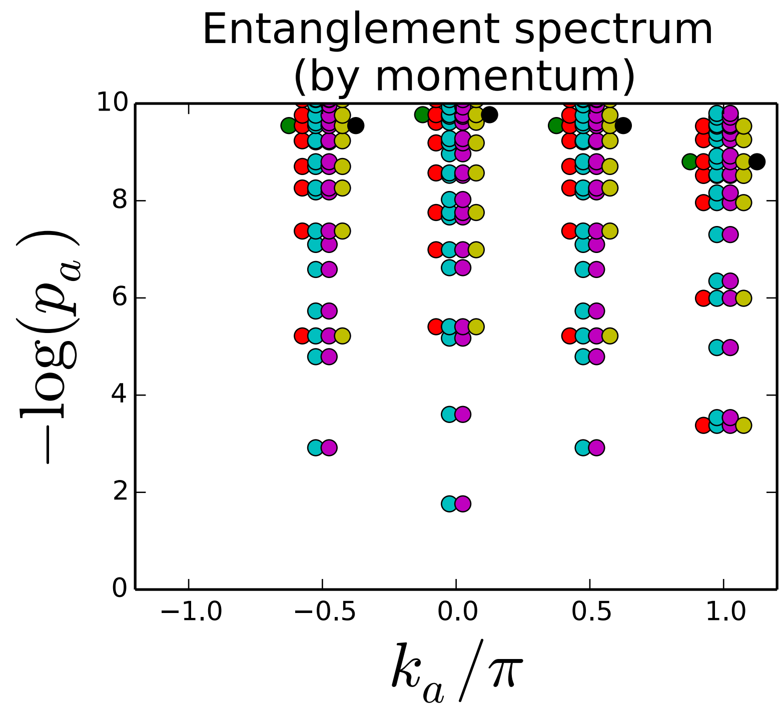
"Momentum polarization"

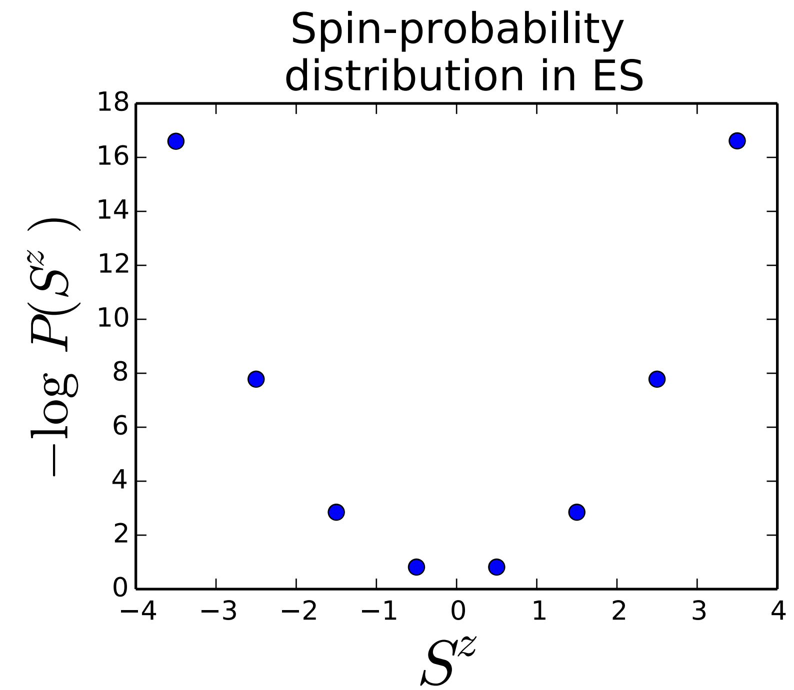
[Zaletel, Mong & Pollmann 2012;   Tu, Zhang, Qi 2012]

$$|\Psi\rangle = \sum_\alpha \lambda_\alpha \, |\alpha\rangle_A \, |\alpha\rangle_B$$

$$\hat{T}^y \, |\alpha\rangle_A = e^{ik_\alpha} \, |\alpha\rangle_A$$

$$k_\alpha \in \frac{2\pi}{L_y}\mathbb{Z}$$



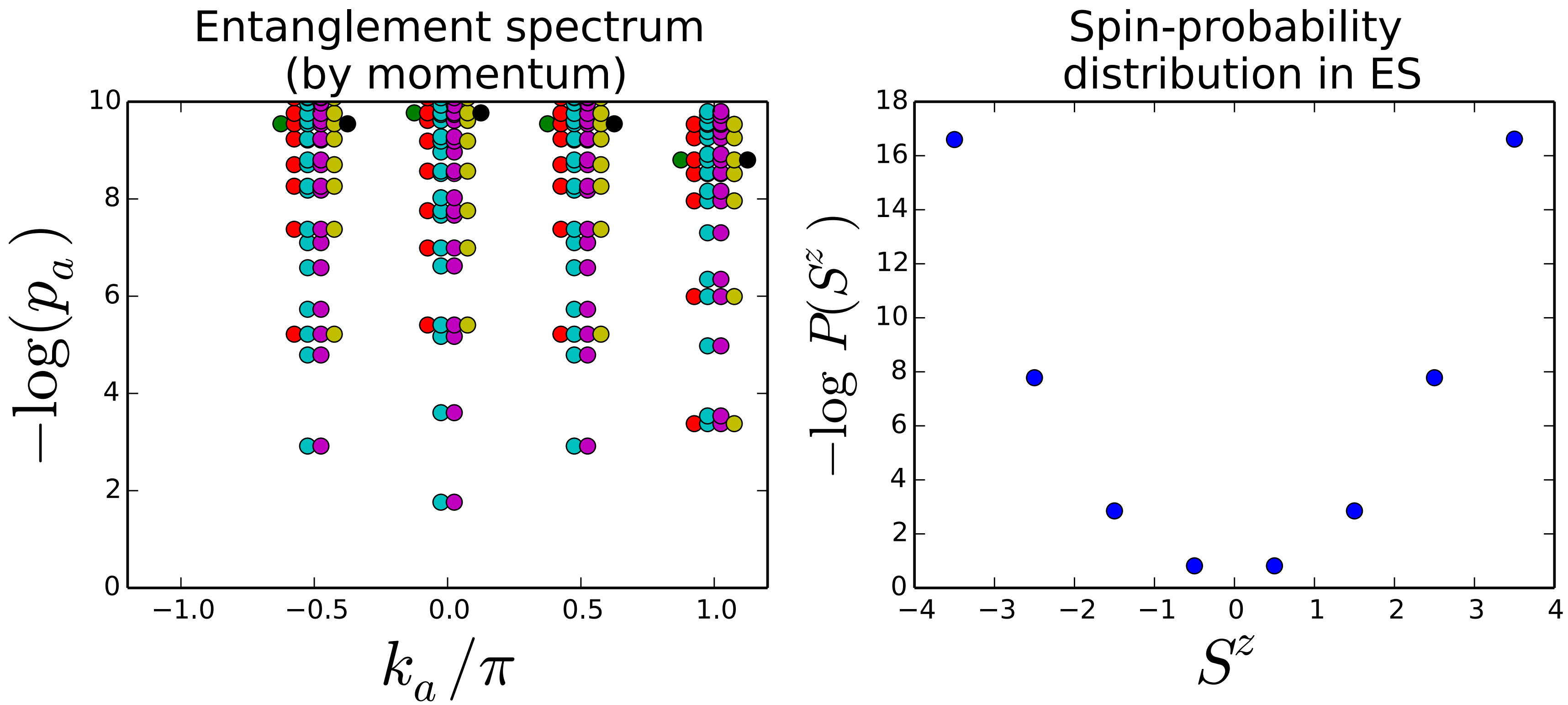


Take "thermal average" of the momentum quantum #s:

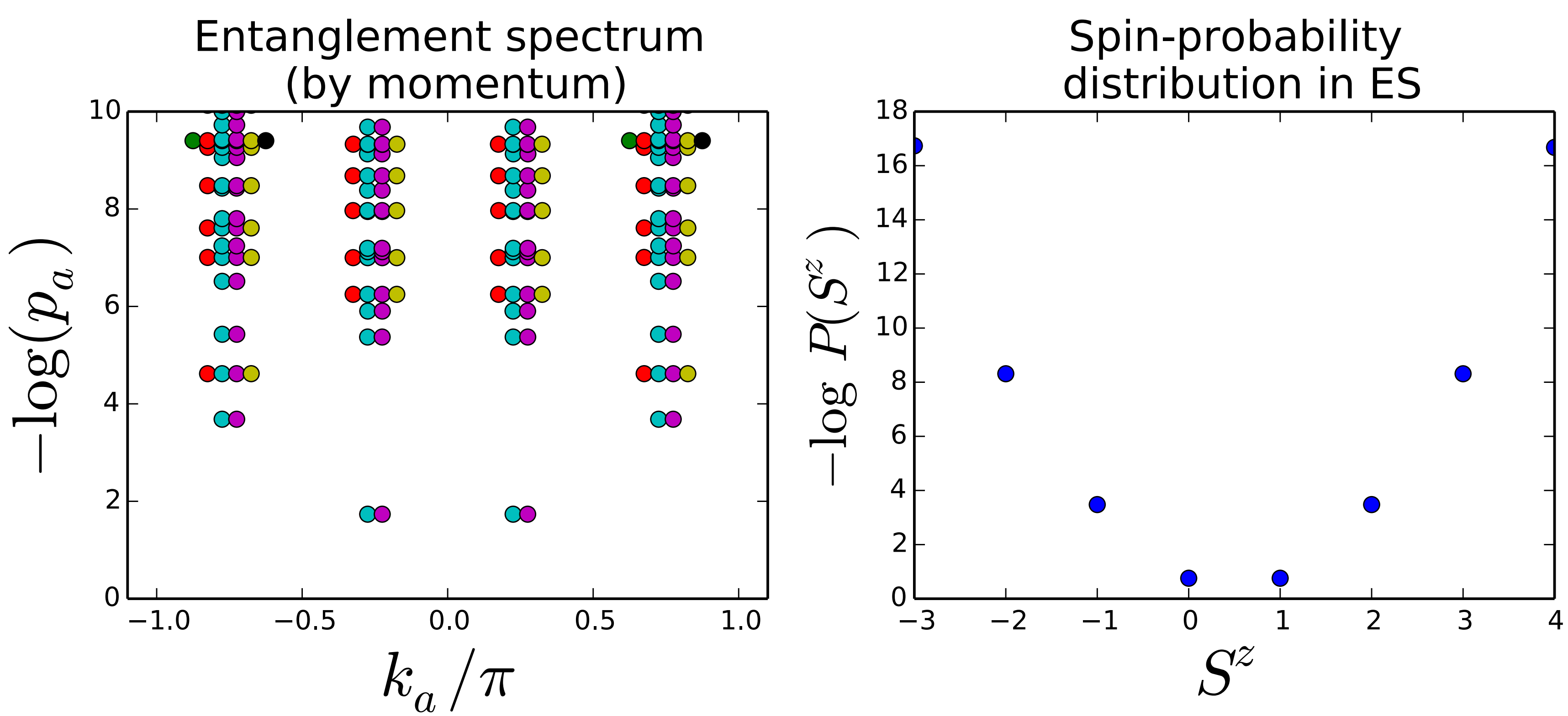
$$e^{2\pi i h_a - \eta L_y^2} = \left[\sum_\alpha \lambda_\alpha^2 e^{ik_\alpha} \right]^{L_y}$$

The topological spin!

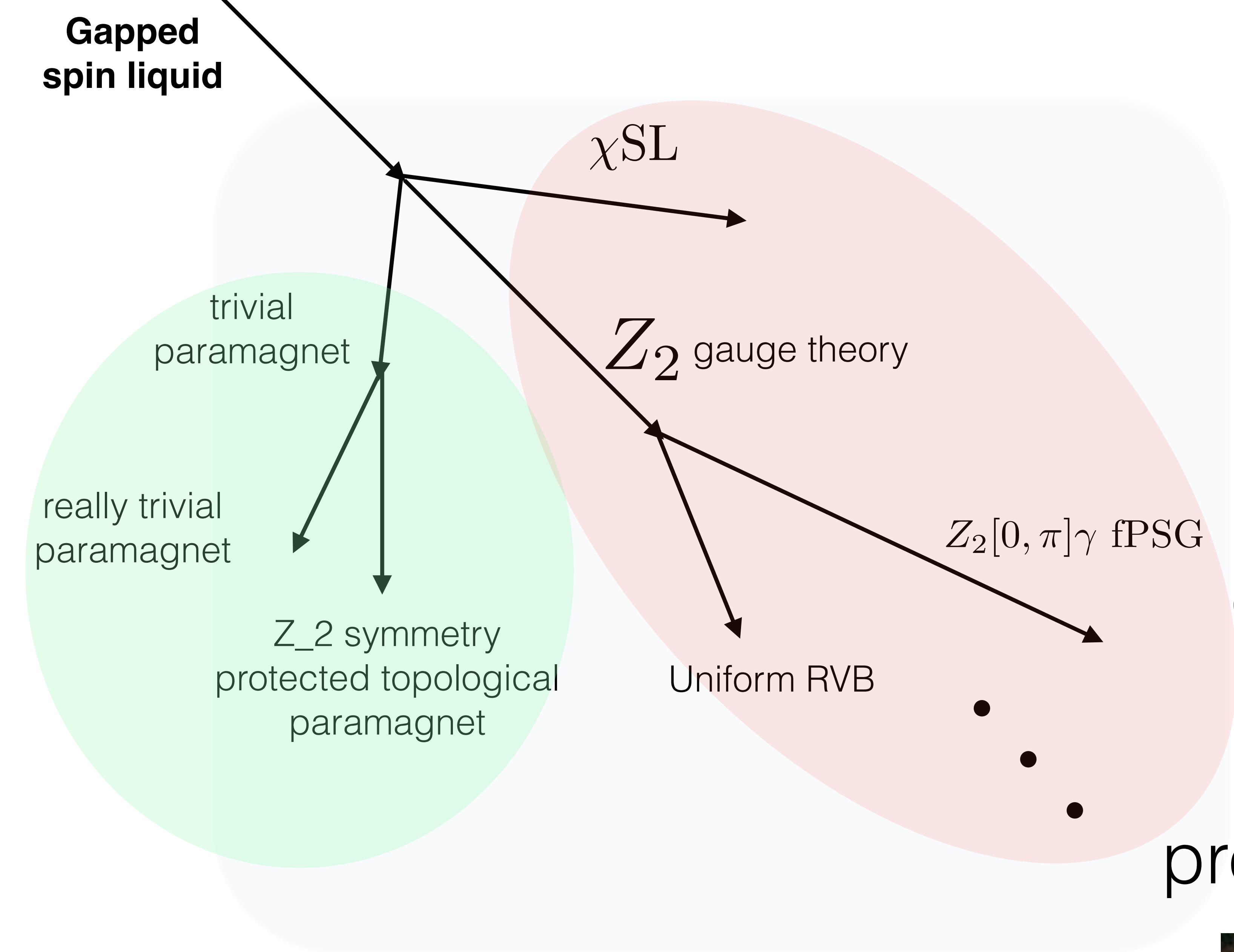
[Details for Z_2 spin liquid in Zaletel, Lu & Vishwanath]



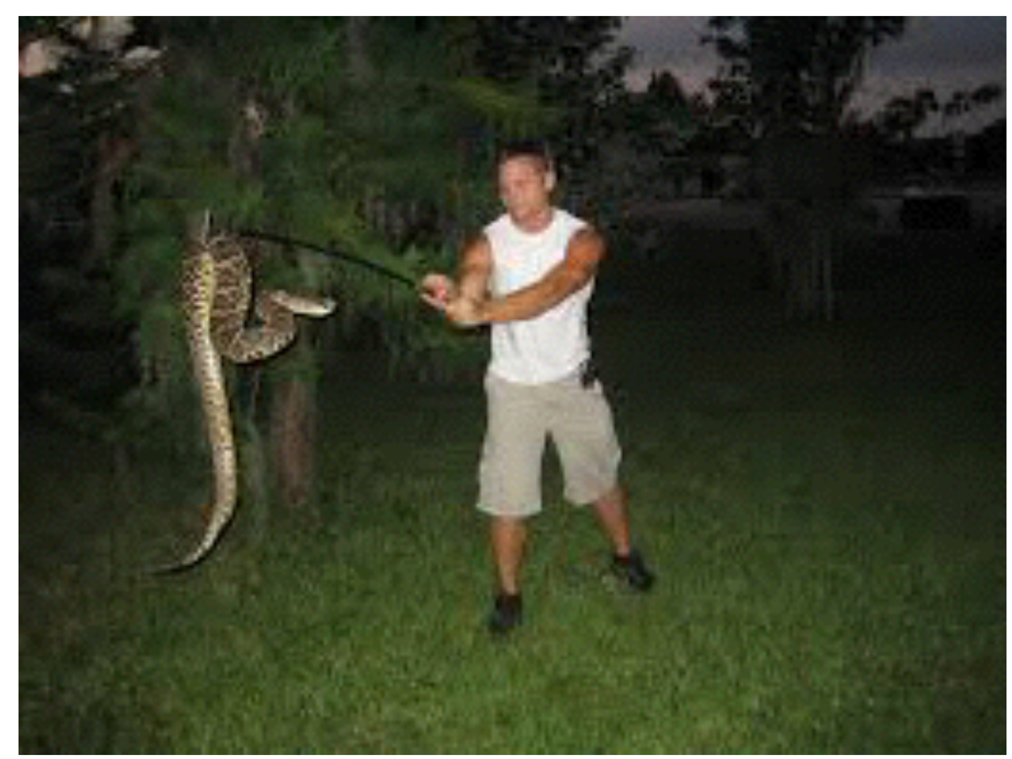
Bosonic Spinon



Fermionic Spinon



More to say…
once you know MES &
topo order you can
find “SET” order: the
projective symmetry group



Mike Zaletel
Station Q

Thanks!