Theoretical approaches to the Fractional Quantum Hall effect

Ady Stern (Weizmann)

1. The Quantum Hall effect

2. Topological superconductivity

3. Topological universe on a graphene sheet

4. Topological insulators

5. Topological classification

6. Gapless topological phases

7. Material prediction

8. States of topological Order

9. Experimental tools

On EDX, CampusIL & Youtube

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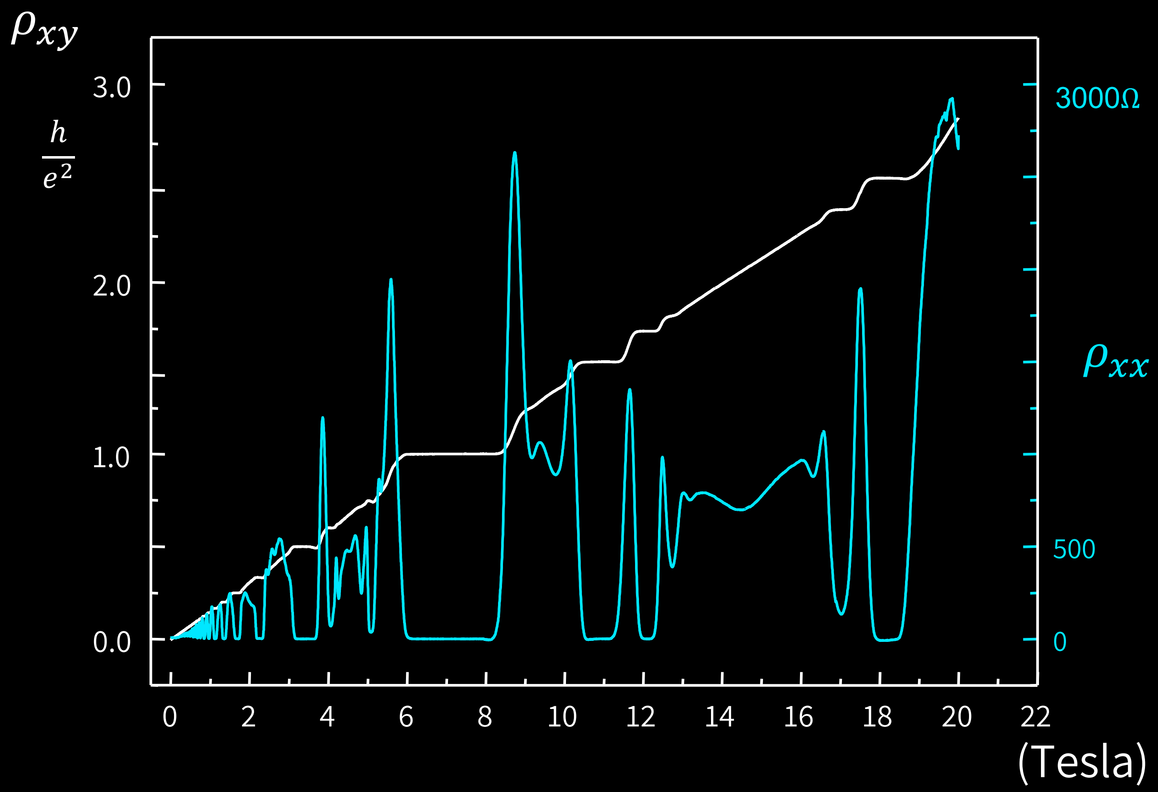
Search for *Topological quantum matter Weizmann online*

We know the problem

$$H = \sum_i \frac{1}{2m}\left(p_i - \frac{cB \times r_i}{c}\right)^2 + \frac{1}{2}\sum_{i \neq j} V\left(\left|r_i - r_j\right|\right) +$$

$$\sum_i V_{disorder}(r_i)$$

We know the answer…



Magnetic Field B ∝ $1/\nu$

What's there to do?

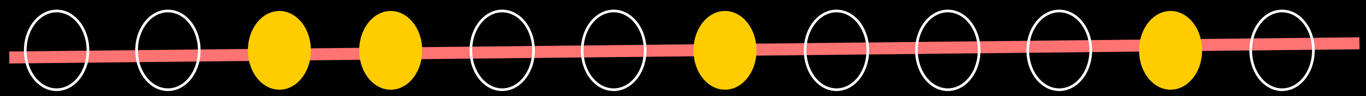
We would like to understand:

- Why gaps open at some fractional filling factors, and do not open at others?
- What determines those filling factors?
- How are fractional charges realized? Other properties of quasi-particles?
- Edge structure
- Magnitude of the gaps, dependence on type of interaction
- Response functions
- …

Difficulties:

$$H = \sum_i \frac{1}{2m}\left(p_i - \frac{cB \times r}{2c}\right)^2 + \frac{1}{2}\sum_{i \neq j} \frac{e^2}{|r_i - r_j|}$$

Neglecting the interaction is not a good starting point for fractional fillings!



Two dimensionless numbers in the problem

1. Ratio of interaction energy to cyclotron energy $\frac{e^2/l_B}{\hbar\omega_c}$
2. Filling fraction

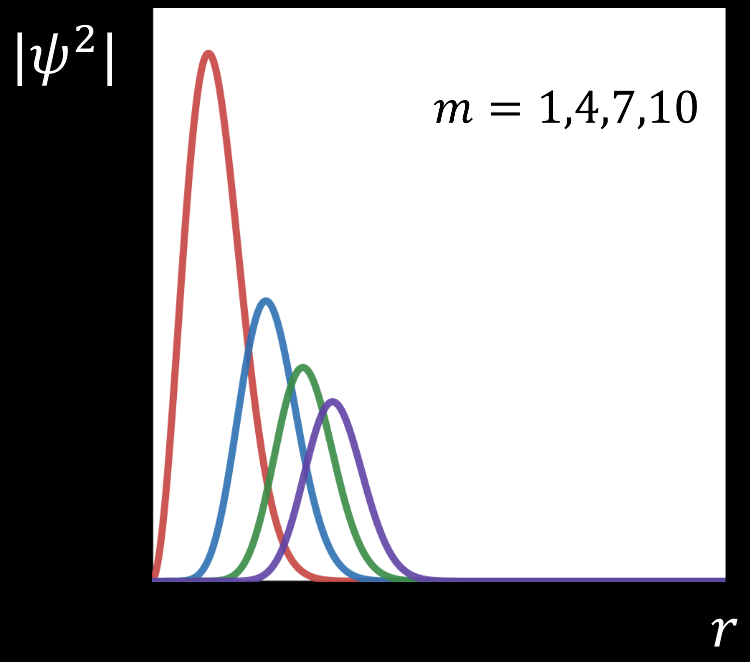
Theoretical approaches:

- Guessing wave functions – Laughlin wave function
- Exact numerical diagonalization
- Wire constructions
- Flux attachment – Composite Fermion Theory

$\nu = \frac{1}{3}$: The Laughlin wave function in five easy steps

1. Lowest Landau level in the symmetric gauge

$$\psi_m(z, z^*) \propto z^m e^{-\frac{|z|^2}{4l_H^2}} \qquad z = x + iy$$



$\nu = 1$:

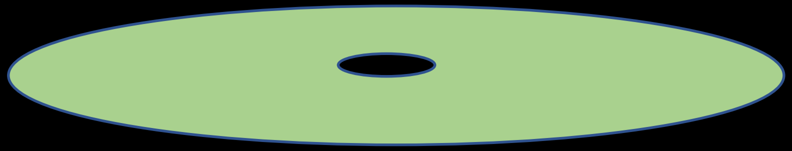
2. Filling all states with $0 < m < N$. A Slater determinant wave function
   - A droplet of a full Landau level.
   - The Slater determinant is van der Munde's

$$\Psi_{LLL} = \prod_{i \neq j} (z_i - z_j)\, e^{-\frac{\sum_i |z_i|^2}{4 l_H^2}}$$

   - The highest power is N, the size of the droplet $\sim N l_B^2$

3. Adding a flux quantum at the center $m \to m+1$

$$\prod_i z_i \, \Psi_{LLL} = \prod_i z_i \prod_{i \neq j} (z_i - z_j) \, e^{-\frac{\sum_i |z_i|^2}{4 l_H^2}}$$

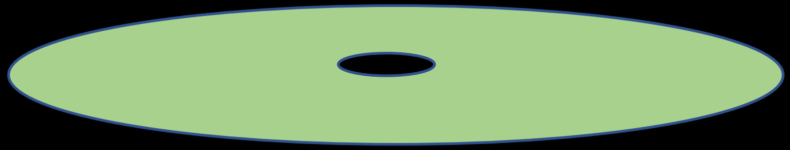


4. The Laughlin wave function – a droplet with three times the same size

$$\psi_{\nu=\frac{1}{3}} = \prod_{i<j} \left(z_i - z_j\right)^3 e^{-\frac{\sum_i |z_i|^2}{4 l_H^2}}$$

5. The Laughlin quasi-hole

$$\Psi^{\mathrm{qh}}_{\nu=\frac{1}{3}} = \prod_i z_i \prod_{i<j} \left(z_i - z_j\right)^3 e^{-\frac{\sum_i |z_i|^2}{4 l_H^2}}$$



As expected, 1/3 of the electron charge

Fractional statistics

What is the statistics of a quasi-particle?

- Construct a Hamiltonian with two quasi-particles at the ground state

$$H(\{r_i\}; R_1, R_2)$$

- Interchange R_1 and R_2 adiabatically
- The ground state acquires a geometric phase

$$\int d\ell \, \langle g.s.(R_1, R_2) | \nabla g.s.(R_1, R_2) \rangle$$

A geometric phase for a single quasi-hole

$$\Psi^{\mathrm{qh},R}_{\nu=\frac{1}{3}} \propto \prod_i (z_i - R) \prod_{i<j} (z_i - z_j)^3 \, e^{-\frac{|z|^2}{4l_H^2}}$$

$$Im \int d\ell \cdot \langle \Psi^{R_1} \left| \nabla_{R_1} \Psi^{R_1} \right\rangle = 2\pi \int da \cdot n$$

Quasi-hole winding another – an extra $2\pi/3$

Anyons

In summary,

The virtues of Laughlin's wave function:

1. For a small number of electrons – remarkable overlap with the exact ground state
2. Exact ground state of a designer-made Hamiltonian
3. Gets the topological properties right

Estimate of an energy gap

Flux attachment and Composite Fermion Theory

Halperin, Lee, Read and many other works (1993)

Composite fermion theory

$$H = \sum_i \frac{1}{2m}\left(p_i - \frac{eB \times r_i}{2c}\right)^2 + \frac{1}{2}\sum_{i \neq j} \frac{e^2}{|r_i - r_j|}$$

$$H\Psi = E\Psi$$

$$\Psi(\ldots r_i \ldots r_j) = -\Psi(\ldots r_j \ldots r_i)$$

Flux attachment

$$\chi(\{r_i\}) = \Psi(\{r_i\})\ e^{-i\alpha \sum_{i<j} \arg(r_i - r_j)}$$

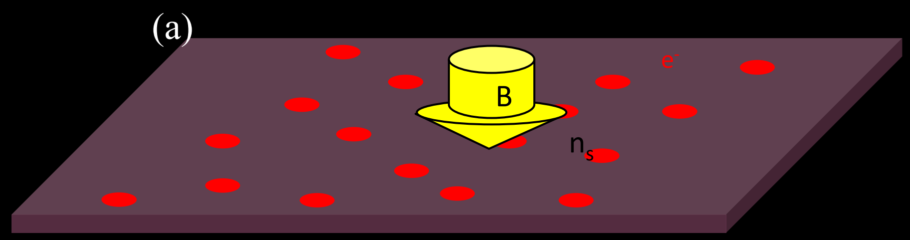
χ satisfies a Schrodinger equation with a different Hamiltonian,

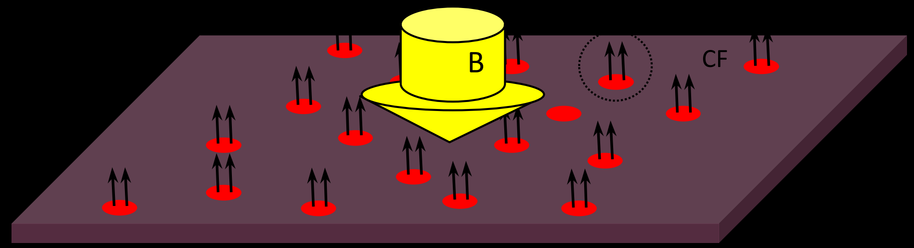
where the kinetic energy changes to:

$$\sum_i \frac{1}{2m}\left(p_i - \frac{eB \times r_i}{2c} + a(r_i)\right)^2$$

where $\nabla \times a(r) = \alpha \phi_0 \sum_j \delta(r - r_j)$

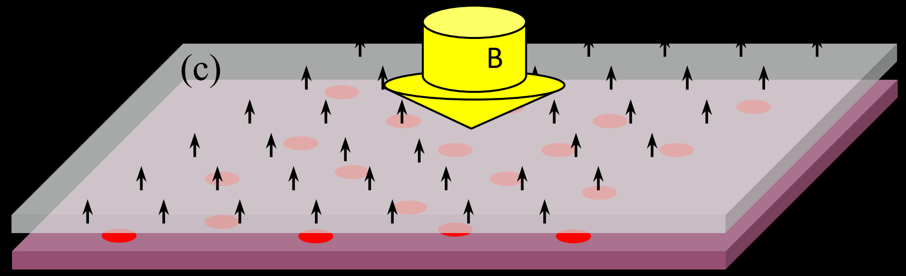
α even - (composite) fermions





$$\Delta B = B - \alpha\phi_0 n$$

Mean field theory: $\nabla \times a(r) = \alpha\phi_0\langle n\rangle$



$\alpha = 2$

At the mean field Hartree level, we map interacting electrons

$$(n, B) \Rightarrow (n, \Delta B)$$

where

$$\Delta B = B - 2\phi_0 n.$$

This changes the filling factor $\nu \Rightarrow \nu_{CF}$

where $$\frac{1}{\nu_{CF}} = \frac{1}{\nu} - 2$$

Fractions are mapped onto integers!

$$\frac{p}{2p+1} \Rightarrow p$$

Gaps appear “naturally”, but with wrong energy scale $E_g = \hbar\Delta\omega_c = \hbar e \Delta \mathrm{B}/mc.$

Large B – Quantum Hall States

Fractional charge

Insert a composite fermion into a FQHE system:

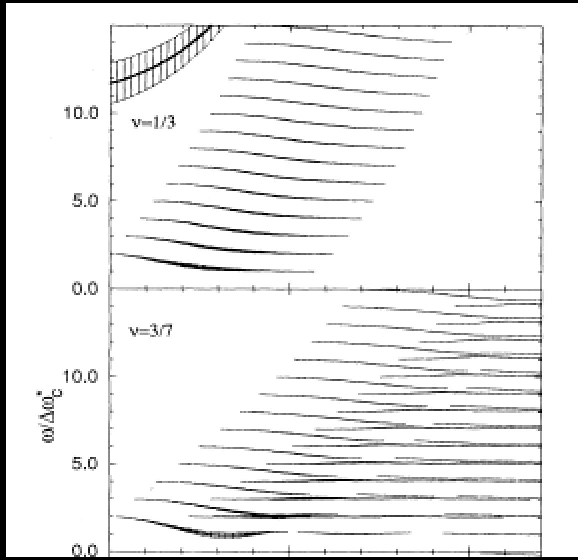
- A charge e is inserted
- α flux quanta are turned on, generating an azimuthal electric field, leading to a radial current, reducing the local charge. The net charge:

$$e - 2\phi_0\sigma_{xy} = e\left(1 - 2\frac{p}{2p+1}\right) = \frac{e}{2p+1}$$

Excitation modes

$$\rho_e = \rho_{CF} + \begin{pmatrix} 0 & 2h/e^2 \\ -2h/e^2 & 0 \end{pmatrix}$$

An excitation mode - $\det \rho(q,\omega) = 0$ (Remember that $E = \rho J$).



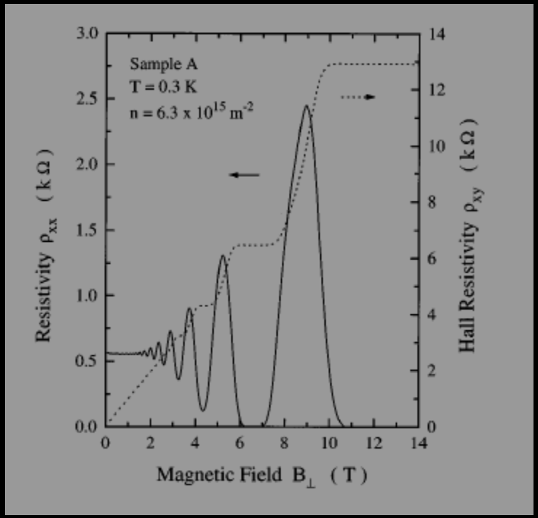
(Simon and Halperin, '93)

Lower AB

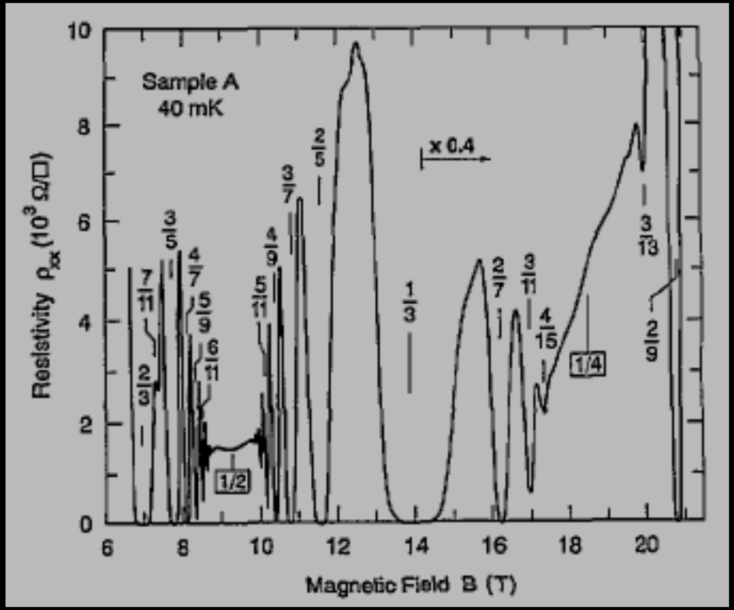
Composite fermions in the Shubnikov deHaas regime

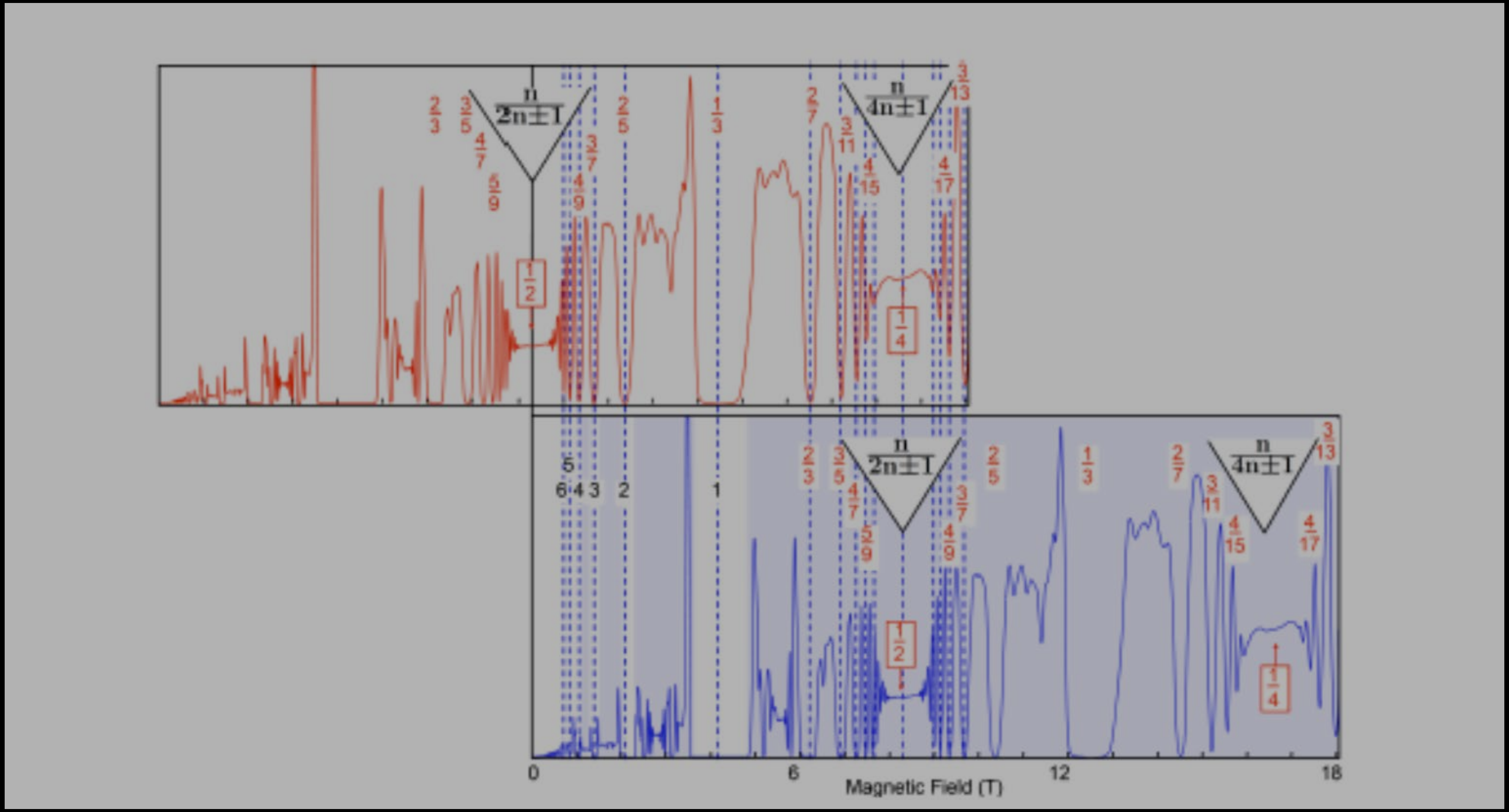
- ρ_{xy} is not quite quantized
- ρ_{xx} oscillates with magnetic field and chemical potential, but does not get all the way to zero. Oscillations originate from oscillating density of states.

Herfort et al., 1994



Stormer et al., 1994





H. Stormer and JK Jain, private communication.

Even weaker ΔB

Do composite fermions exist semi-classically?

Geometric resonances between cyclotron radius of composite fermion $\mathrm{cp_F}/e\Delta B$ and wavelength of a surface acoustic wave

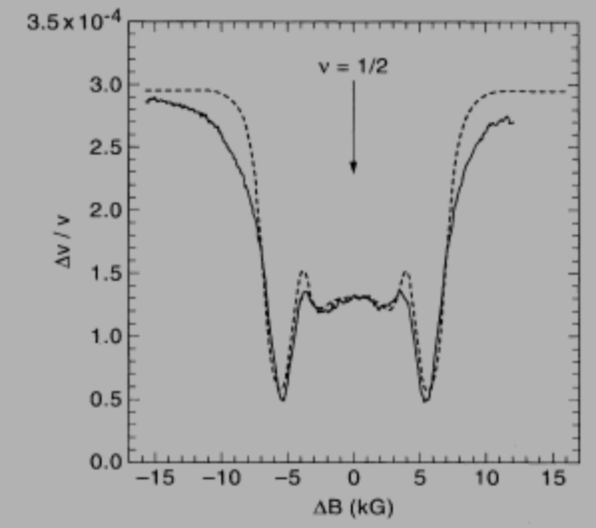


FIG. 1. Sound velocity shift versus magnetic field for 10.7 GHz surface acoustic waves near filling factor $\frac{1}{2}$. Both principle and secondary resonances are present. Temperature is ~ 130 mK. The dashed line shows the theoretical fit to the data using parameters defined in the text.

(Willett 1993)

The cyclotron radius of composite fermions $\frac{cp_F}{e\Delta B}$ is a measurable quantity

even when their quantum Hall effect is not.

Note, $\frac{cp_F}{e\Delta B} = \frac{cp_F}{e^* B}$ -

a fractional charge in the original, physical, field.

The smallest possible ΔB - $\nu = \frac{1}{2}$

Do composite fermions exist even when $\Delta B = 0$?

If so, how do they behave?

Electric dipoles moving in straight lines in a strong magnetic field

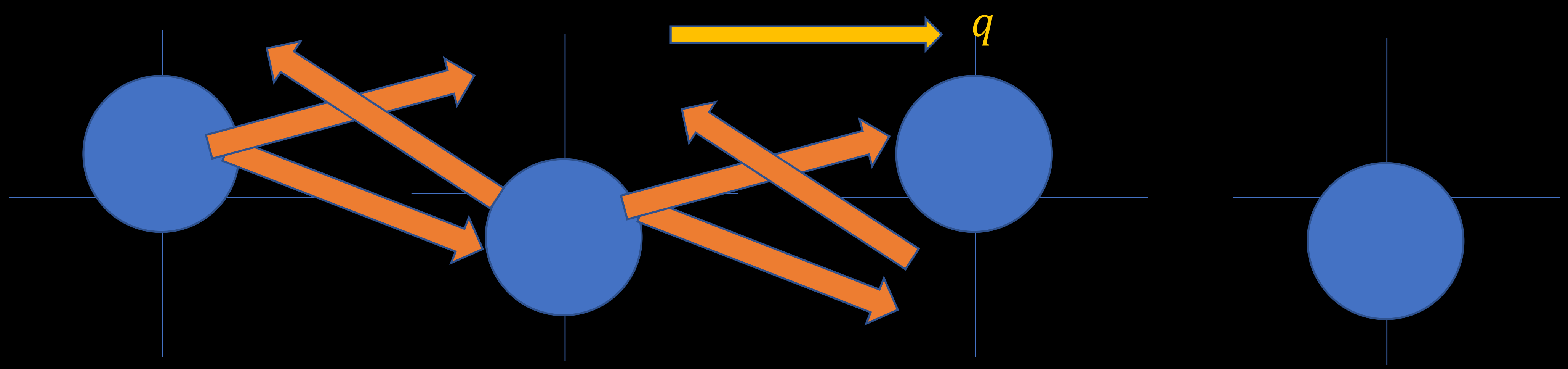
A Fermi surface of composite fermions?

Identifying a Fermi surface

- Cyclotron radius
- Anomalous skin Hall effect

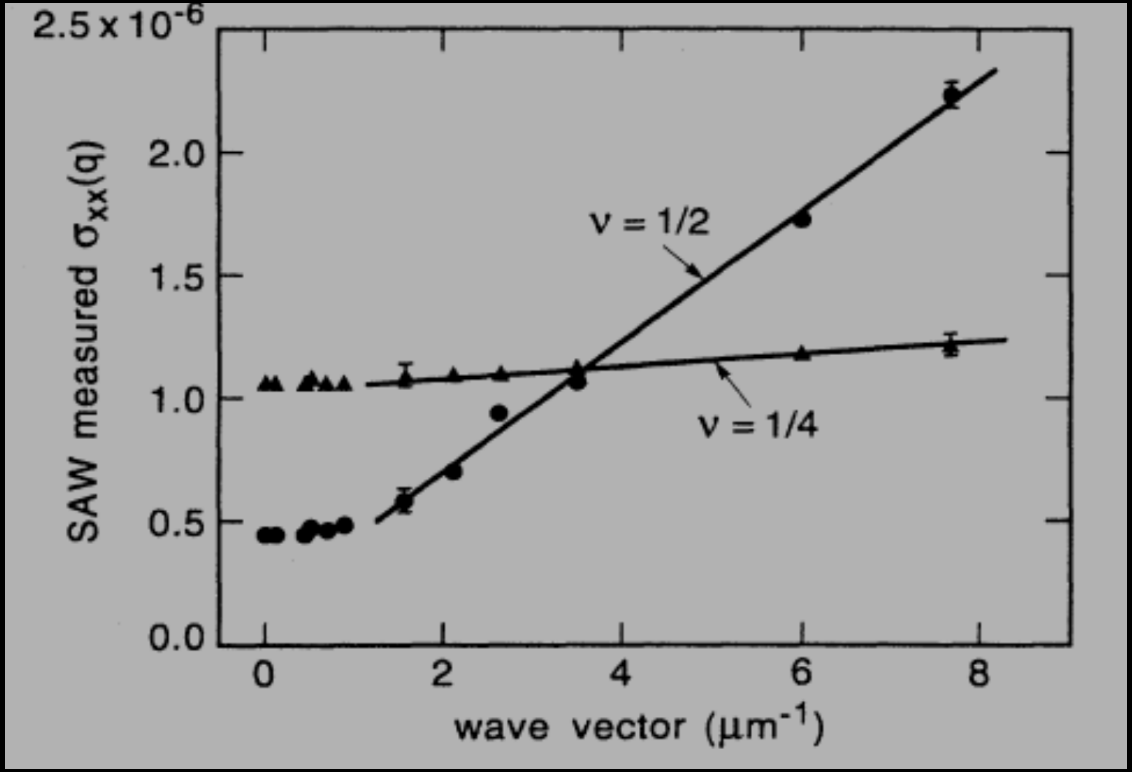
Think the resistivity at non-zero q, in the direction perpendicular to q.

Drude resistivity at $q = 0$ is $\rho_{xx} = \rho_{yy} = \frac{m}{ne^2}\frac{1}{\tau}$, where τ is the time for a current to decay in the absence of a driving force.



Current decays even without impurities. Decay time $\sim 1/qv_F$.

Resistivity $\frac{m}{ne^2}\frac{1}{\tau} \sim \frac{h}{e^2}\frac{q}{k_F}$, completely geometric!



Willett, 1993

Does a Fermi surface imply a Fermi liquid?

- Composite fermions interact with a fluctuating magnetic field, which is proportional to the fluctuating electron density.
- Most “dangerous” – interaction with slow dynamics, which is what we have here.

$$\mathcal{L} = \Psi^{+}(r)(\mathrm{i}\partial_t - a_0)\Psi + \Psi^{+}(r)\frac{1}{2m}\left(-i\nabla - \frac{eB\times r}{c} + a(r)\right)^2 \Psi(r) + \frac{1}{\alpha\Phi_0} a_0 \nabla \times a + (\alpha\phi_0)^2 \int dr' \nabla \times a(r) V(r - r') \nabla \times a(r')$$

The source of the slow dynamics – slow charge relaxation in a strong magnetic field.

The relaxation dispersion depends on the range of electron-electron interaction.

$$i\omega \sim q^2 V(q) \sigma_{xx}(q) \sim q^3 V(q)$$

- Self energy ⇒ effective mass ⇒ cyclotron gap
- Need to change an energy scale!
- Small parameter is not the interaction scale, it is the inverse of the number of filled composite fermions Landau levels

$$E_g = \frac{\pi\sqrt{2}}{\alpha^{3/2}} \frac{e^2}{l_B} \frac{1}{(\alpha p + 1)\log(\alpha p + 1)}$$

What other states can composite fermions at $\Delta B = 0$ form?
Future will tell…

A concluding comment –

Charge fractionalization is manifest beyond the limits in which it is “justified”.

Lecture II:

- Reminder of the first lecture
- Composite fermion theory and Jain’s wave functions
- Bi-layer systems in the quantum Hall regime
- Stripe states in a partially filled Landau level

In the first lecture –

1. Laughlin's wave function
2. Composite fermion theory
    1. FQHE of electrons $\Longrightarrow$IQHE of composite fermions
    2. Shubnikov deHaas regime of composite fermions
    3. Semiclassical physics of composite fermions
    4. $\nu = 1/2$ state

Wave functions

$\nu = \frac{1}{3}$:

$$\prod_{i<j} z_{ij} e^{-\frac{\sum_i |z_i|^2 \Delta B}{4\Phi_0}}$$

$$\prod_{i<j} \frac{z_{ij}^2}{|z_{ij}^2|}$$

Wave functions

$$e^{-\frac{\sum_i |z_i|^2 \Delta B}{4\Phi_0}} \quad e^{2i\sum_{i<j} \arg(z_j)}$$

traded for

$$e^{-\frac{\sum_i |z_i|^2 B}{4\Phi_0}} \qquad z_{ij}^2$$

Multiply the mean field wave function by

$$\prod_{i<j} \left|z_{ij}\right|^2 \, e^{-\frac{\sum_i |z_i|^2 B_{1/2}}{4\Phi_0}}$$

Clarifies how electrons are kept away from one another

Generalization – Jain's wave functions:

Trial wave function $= \; e^{-\frac{\sum_i |z_i|^2 B_{1/2}}{4\Phi_0}} \quad z_{ij}^2 \quad |p\ filled\ LLs\ at\ \Delta B\rangle$

projected to the electronic lowest Landau level.

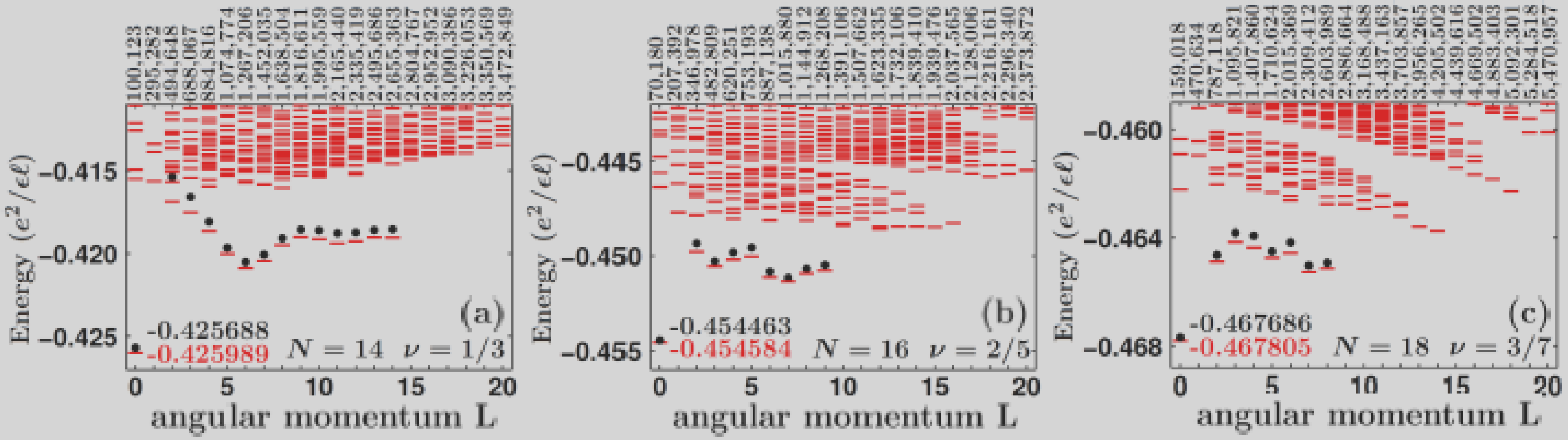


Fig. 5. These figures show a comparison between the energies (per particle) predicted by the CF theory (dots) and the exact Coulomb energies (dashes), both obtained without any adjustable parameters. Panels (a)-(c) show spectra for $(N, 2Q) = (14, 39)$, $(16, 36)$, and $(18, 37)$, which are finite size representations of the 1/3, 2/5 and 3/7 states. The wave function for ground state at $\nu = 1/3$ is the same as the Laughlin wave function. Source: A. C. Balram, A. Wójs, and J. K. Jain, Phys. Rev. B. **88**, 205312 (2013),[70] and J. K. Jain, Annu. Rev. Condens. Matter Phys. **6**, 39-62, (2015).[71]

Bi-layer quantum Hall systems

Bi-layers in the quantum Hall regime

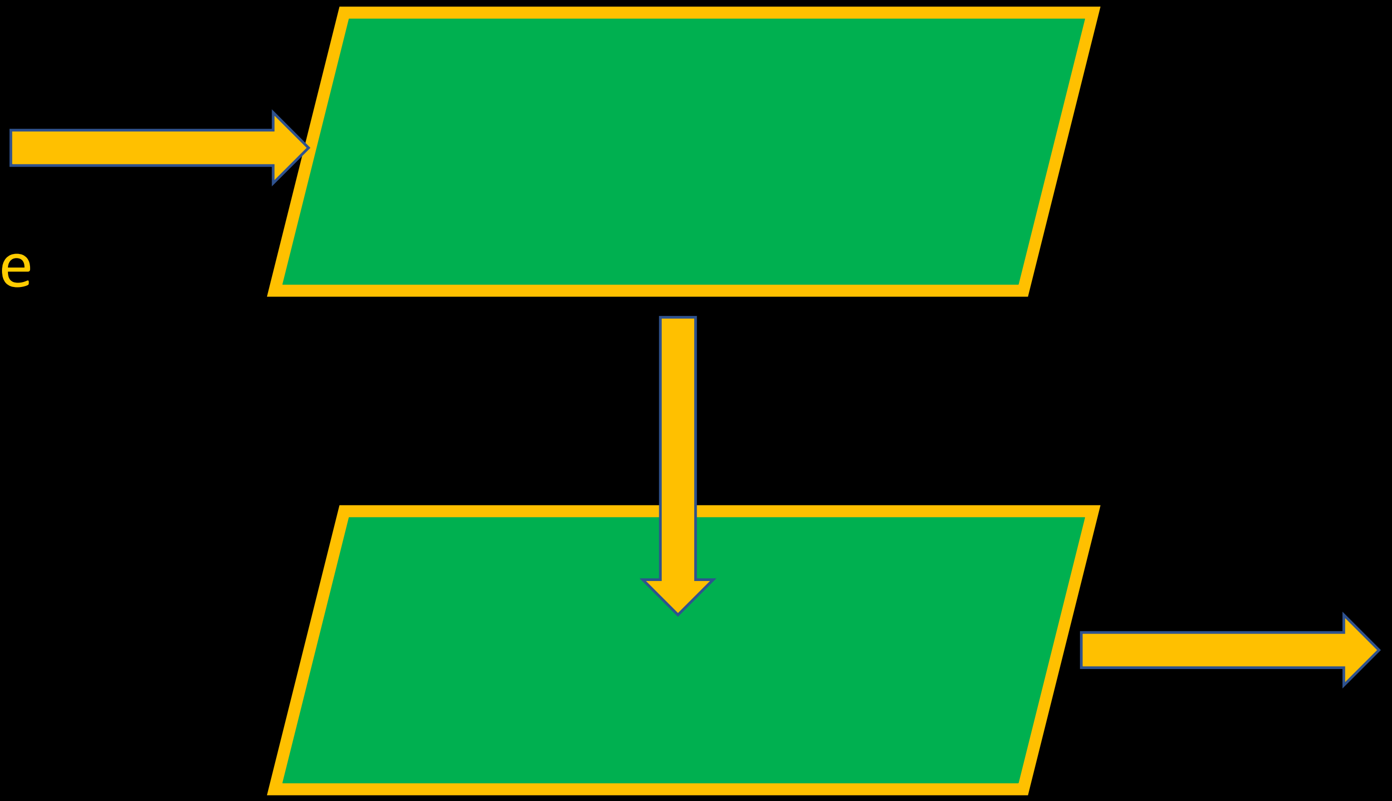
Important parameters:

- Inter-layer tunneling
- Inter-layer Coulomb coupling (depends on distance d)
- Densities in both layers $k_{F,1}, k_{F,2}$
- The usual suspects – magnetic field, temperature

Bi-layers in the quantum Hall regime

Measurements:

- Tunneling resistance
- “conventional” resistance
- Coulomb drag
- Counterflow



Naively, just a big resistance.

Voltage dependence gives much information on the two layers

Bi-layers in the quantum Hall regime

Measurements:

- Tunneling resistance
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Measurements:

- Tunneling resistance
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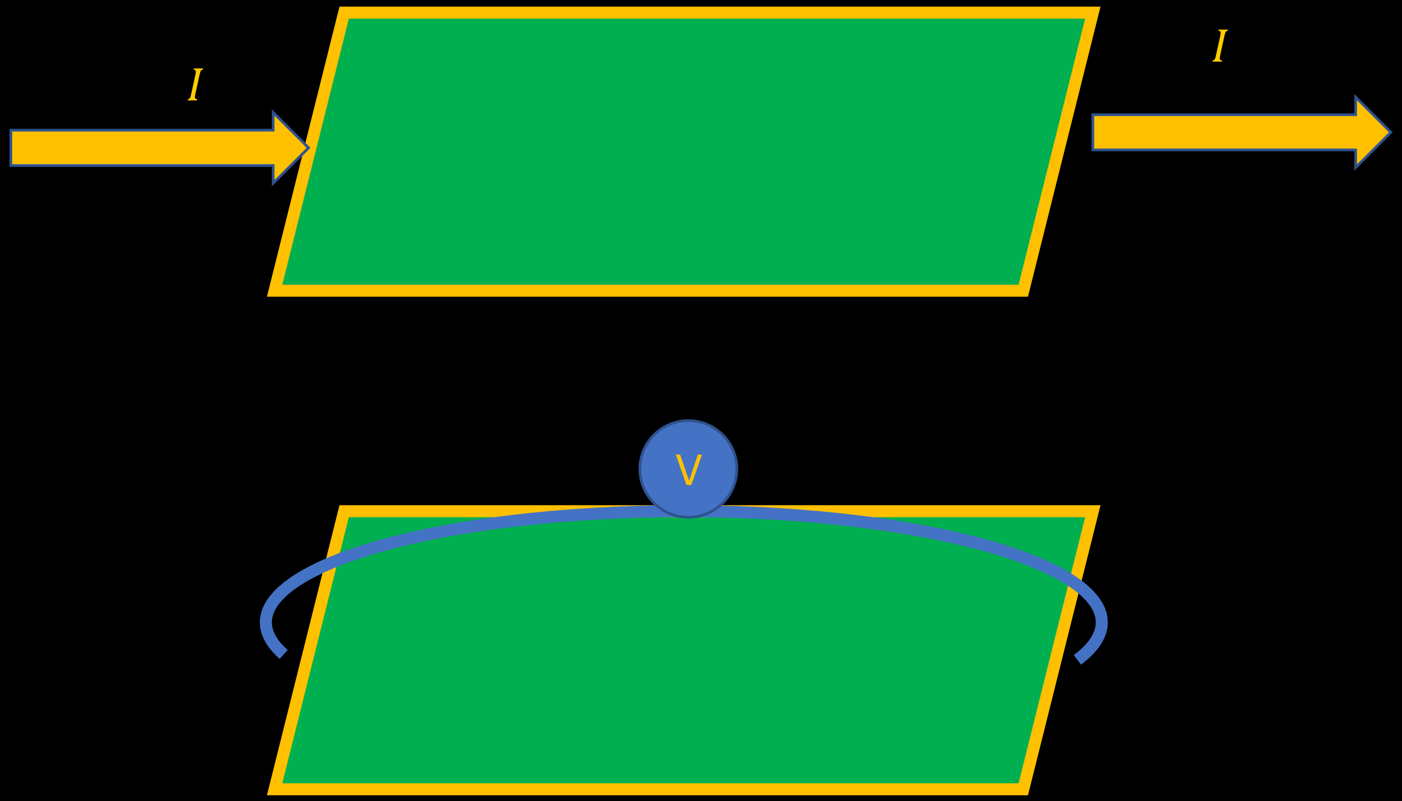
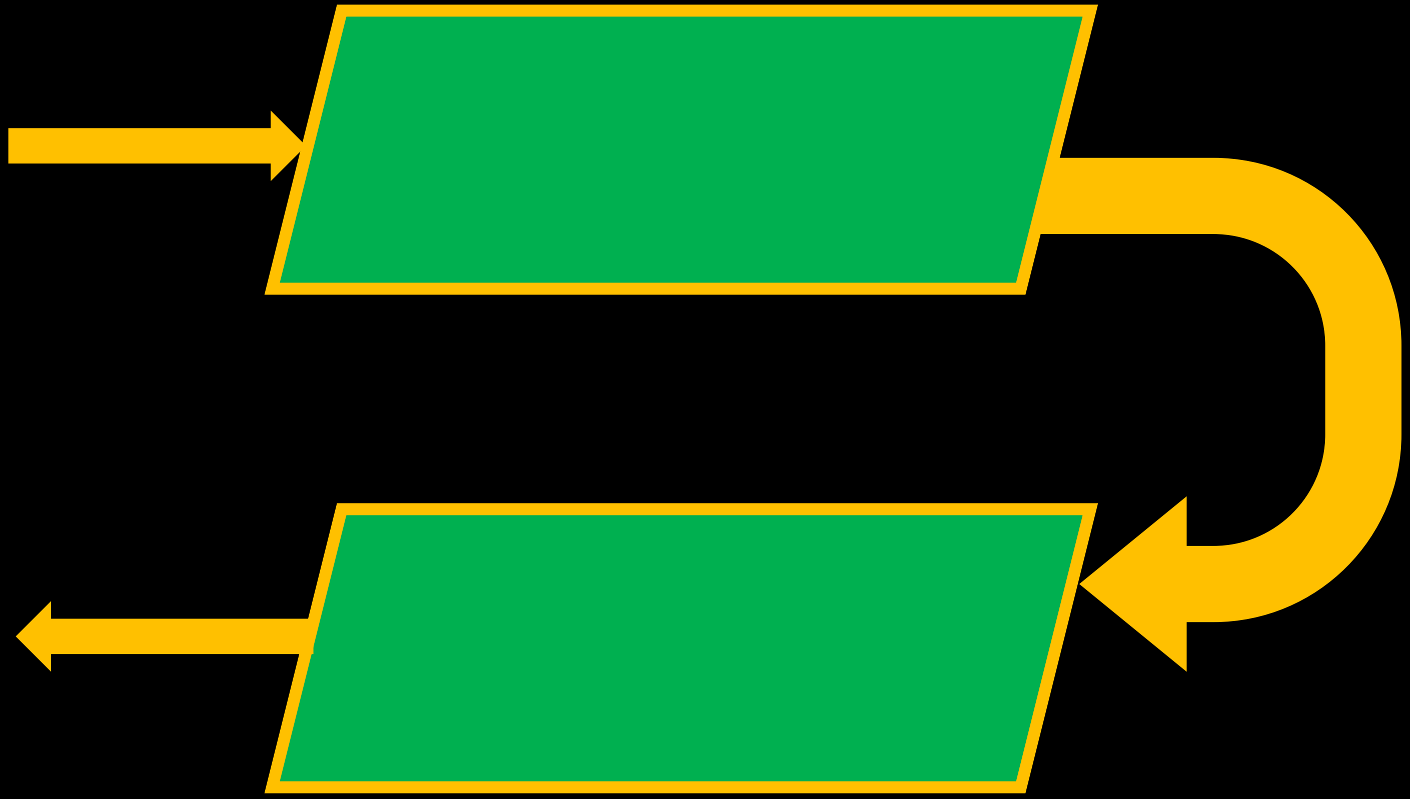


Naively, resistance due to friction

Bi-layers in the quantum Hall regime

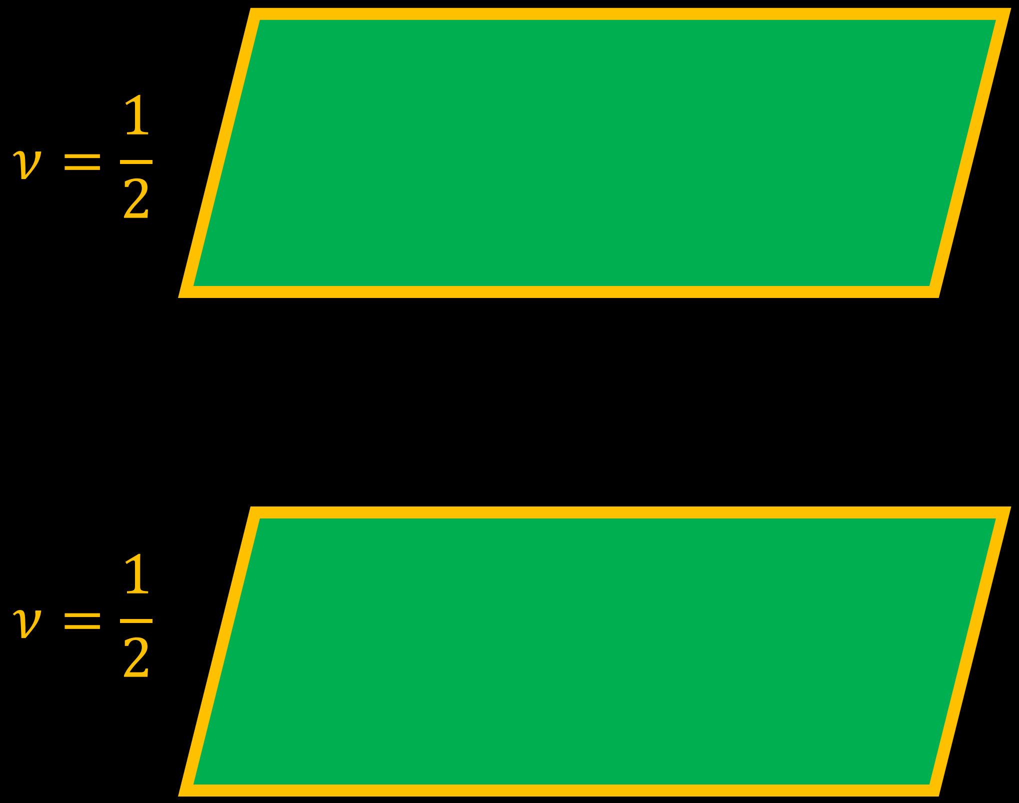
Measurements:

- Tunneling resistance
- “conventional” resistance
- Coulomb drag
- Counterflow

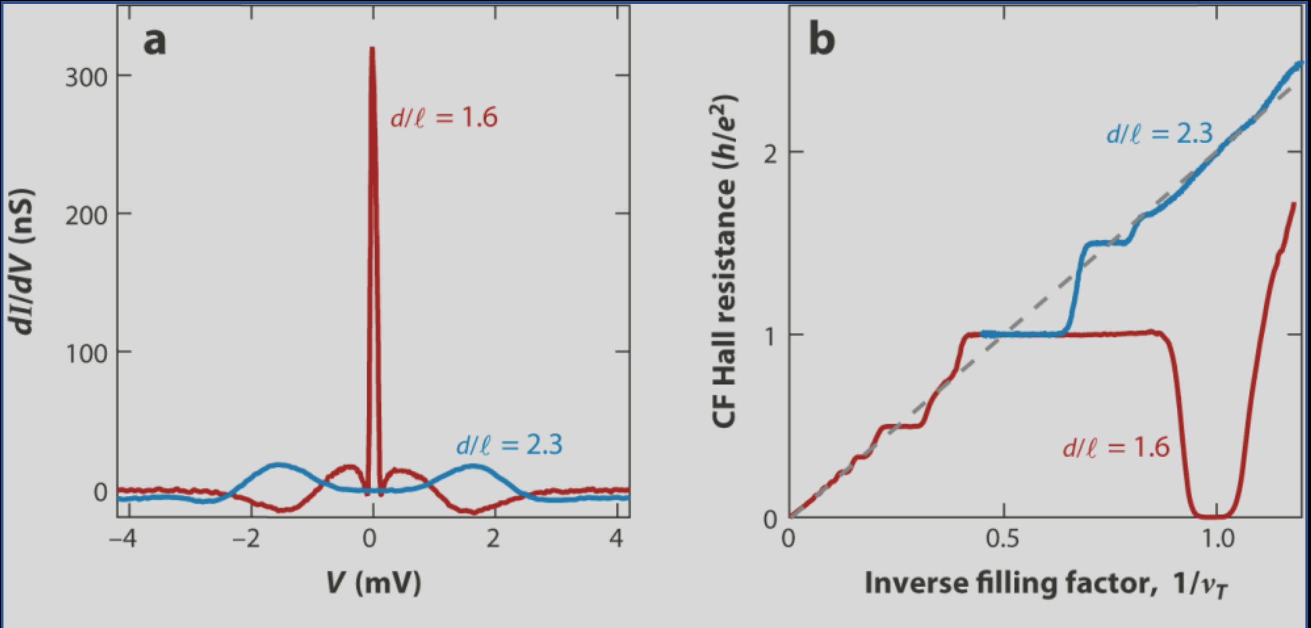


Naively, resistors in series

Bi-layer at $\boldsymbol{\nu_T = 1}$



- Assume very weak tunneling, strong magnetic fields.
- The only dimensionless parameter - d/ℓ



Eisenstein JP. 2014.
Annu. Rev. Condens. Matter Phys. 5:159–181

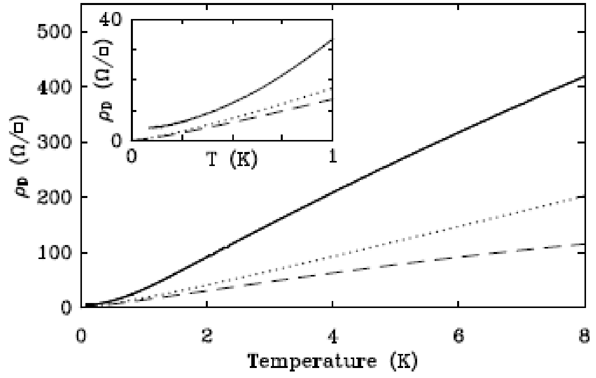


FIG. 2. Temperature dependence of ρ_D at $\nu = 1/2$ (solid line). The broken lines are calculations [16,17] of ρ_D assuming two different CF effective masses (dotted, $m^* = 12m_b$, dashed, $m^* = 4m_b$, where m_b is the GaAs band mass).

Inter-layer tunneling

counter-flow resistance

Coulomb drag

(Lilly et al.)

Large d/ℓ - weakly coupled layers

Two disconnected $\nu = 1/2$ states are mapped onto two weakly coupled Fermi liquids of composite fermions.

$$\rho_D = \frac{1}{2(2\pi)^2} \frac{h}{e^2} \frac{1}{T n_1 n_2} \int \frac{d\mathbf{q}}{(2\pi)^2} \int_0^\infty \frac{\hbar\, d\omega}{\sinh^2 \frac{\hbar\omega}{2T}} q^2 \left|U_{\mathrm{sc}}(\mathbf{q},\omega)\right|^2 \mathrm{Im}\Pi^{(1)}(\mathbf{q},\omega)\mathrm{Im}\Pi^{(2)}(\mathbf{q},\omega)\,,$$

Typically, $\frac{1}{\tau} \propto T^2$ but here the slow mode $\omega \propto iq^3$ leads to a stronger scattering

$$\frac{1}{\tau} \propto T^{\frac{4}{3}}$$

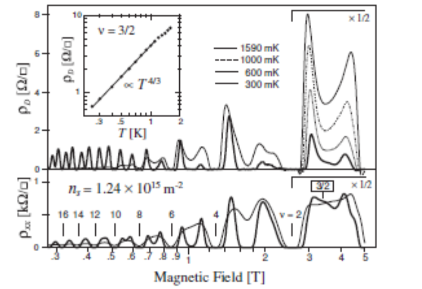
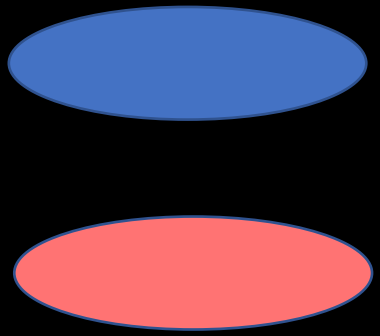


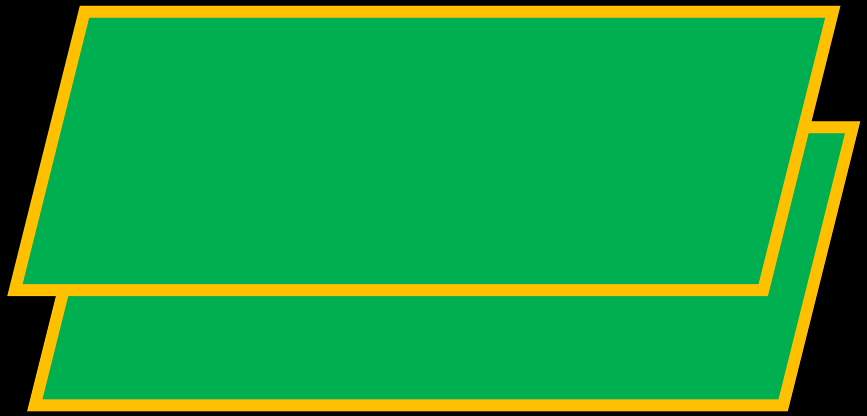
FIG. 1: Drag resistivity ρ_D (top) and single-layer resistivity ρ_{xx} of the front layer (bottom) for matched densities of $n_s = 1.24 \times 10^{15}\,\mathrm{m}^{-2}$. For visibility, ρ_D and ρ_{xx} for $\nu < 2$ are multiplied by 0.5. Inset: T dependence of ρ_D at $\nu = 3/2$, demonstrating the $T^{4/3}$ power law.

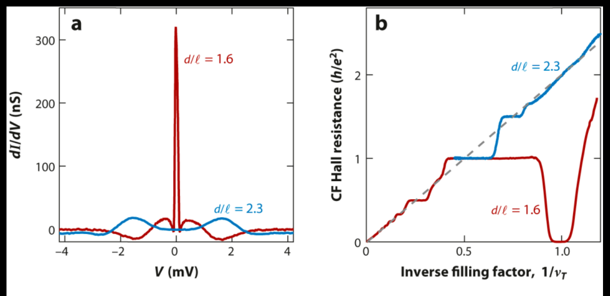
(Muraki et al.)

Tunneling: typically, tunneling between two identical Fermi liquids is "easy"

Either $\frac{dI}{dV}$ independent of V or peak at zero voltage.







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Here strong suppression of the tunneling.

The reasons:

- the tunneling object is an electron, not a composite fermion.
- The tunneling charge needs to disperse away to the edges of the system, and charge relaxation is slow.

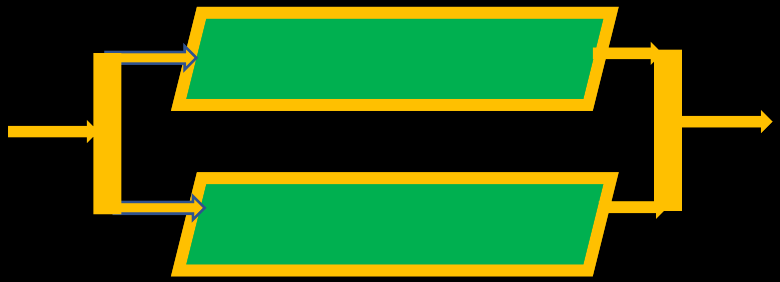
Small $k_F d$ – strongly Coulomb-coupled layers

- Cancelling the magnetic field by a different flux attachment – attaching one flux quantum to each electron, with the electrons interacting with flux quanta of both layers.
- Said differently - $\alpha = 1$
- Mapping the $\nu = \frac{1}{2} \oplus \frac{1}{2}$ state to two coupled superfluids
- Wave function

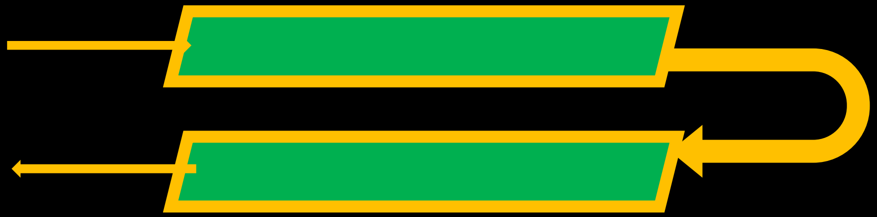
$$(z_i - z_j)(w_i - w_j)(z_i - w_j)\, e^{-\frac{|z|^2}{4l_H^2}}$$

Consequences:

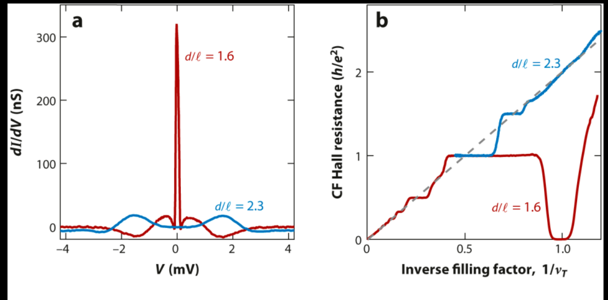
- For symmetric current – quantized Hall state



- For anti-symmetric current - superfluidity



- Large zero bias tunneling peak (Josephson-like)
- Goldstone mode which is a Plasma mode of anti-symmetric density
- BKT transition when vortices proliferate



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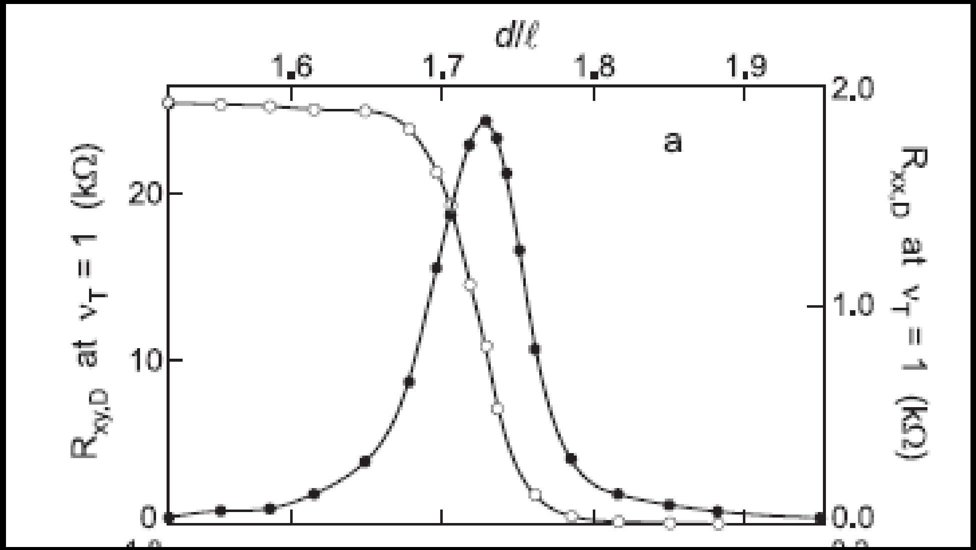
The transition between the two limits



The longitudinal drag resistance develops a very large peak at the transition

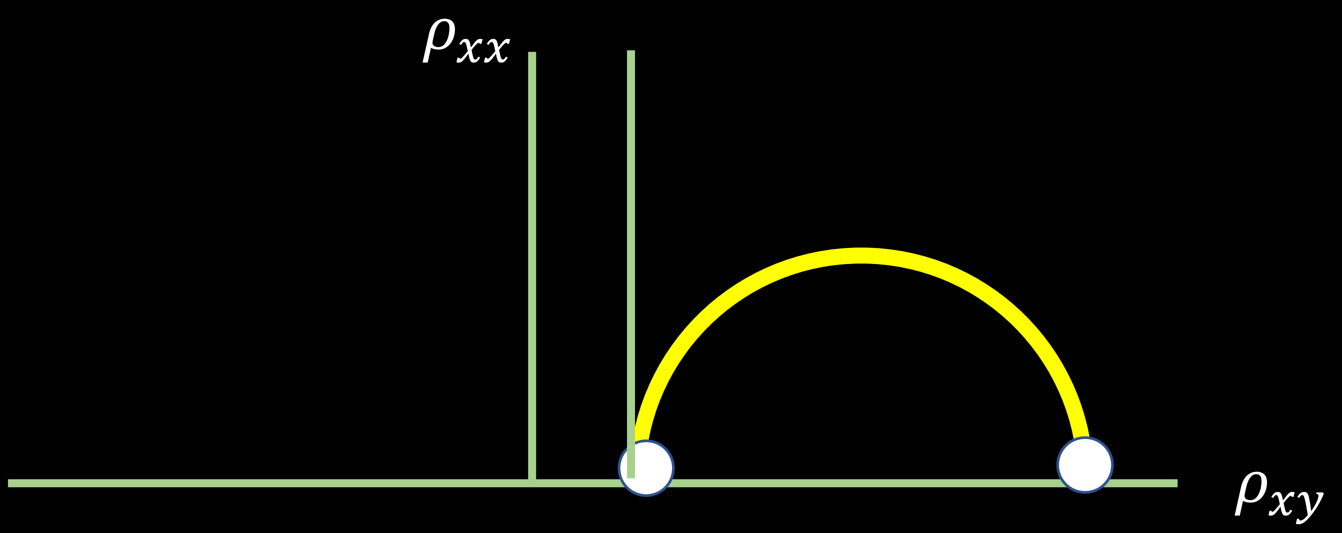
The peak at the transition and transport in a mixed system

Consider a system made of two phases, each one with its own resistivity matrix ρ_1, ρ_2.
If transport is governed by a local Ohm relation

$$\nabla \times E = 0$$
$$\nabla \cdot J = 0$$
$$E(r) = \rho(r) J(r)$$

Then the macroscopic resistivity satisfies

$$\rho_{xx}^2 + \left(\rho_{xy} - \rho_{xy,0}\right)^2 = \rho_{xx,0}$$



$$\rho_{xx}^2 + \left(\rho_{xy} - \rho_{xy,0}\right)^2 = \rho_{xx,0}$$

When the two phases are on the two sides of the semi-circle, a transition between them involves a peak in dissipation

And now for something completely different…
Breaking of translation symmetry

Half filled Landau level:

at $\nu = \frac{1}{2}, \frac{3}{2}$ a compressible state

at $\nu = \frac{5}{2}, \frac{7}{2}$ an incompressible quantum Hall state

at $\nu = \frac{9}{2}, \frac{11}{2}$ … breaking of translation symmetry

Experimental indication –

- large anisotropy in the longitudinal resistivity
- Non-monotonic ρ_{xy}.

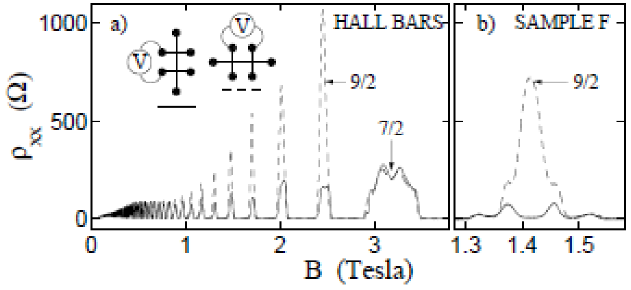
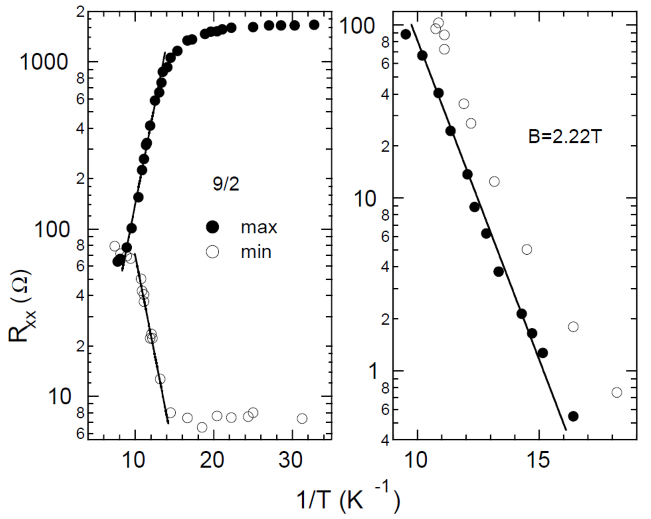
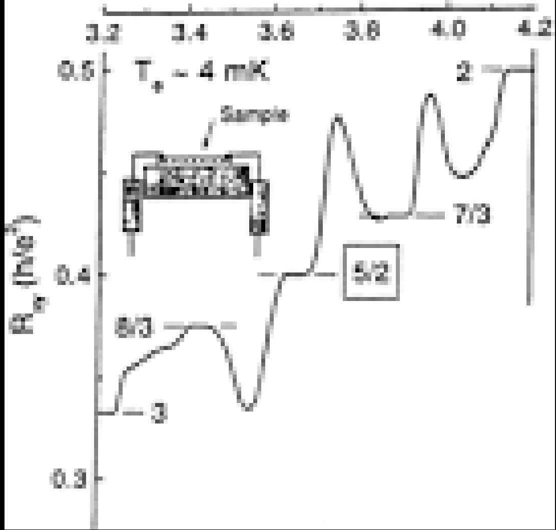


FIG. 4. a.) Anisotropy of ρ_{xx} at T=25mK measured using Hall bar samples B (dashed) and C (solid). b.) Anisotropy of ρ_{xx} at T=15mK measured in the low density sample F. Current flow configurations as in Fig. 3.





(Eisenstein group, Stormer group)

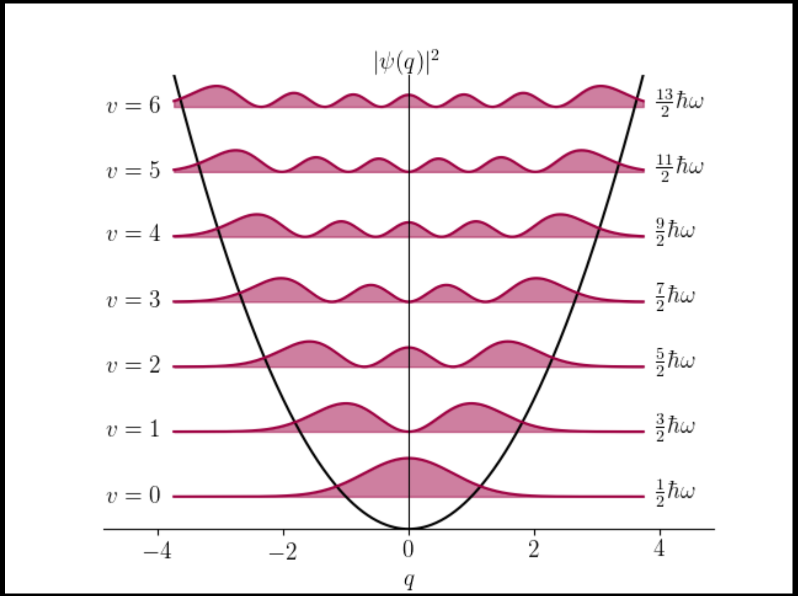
The picture – a different way interactions break the ground state degeneracy of a partially filled Landau level

The tool – Hartree-Fock analysis of the projected Coulomb interaction (Fogler et al. Moessner et al.)

The Hamiltonian $H = \sum_q V(q)\rho(q)\rho(-q)$

$$\langle k|e^{i\boldsymbol{q}\cdot\boldsymbol{r}}|k'\rangle = \delta_{k'+q_y,k}\int dx\,\phi(x+kl_B^2)e^{iq_x x}\phi(x+k'l_B^2)$$

$$\langle k|\boldsymbol{e^{iq\cdot r}}|k'\rangle = \delta_{k'+q_y,k}\int dx\phi^*(x+kl_B^2)e^{iq_x x}\phi(x+k'l_B^2)$$



Oscillations with qR_c, decay for $ql_B \gg 1$.

The projected Coulomb interaction

$\tilde{V}(q) = V_0(q)|F_N(q)|^2$ with $F_N(q) \propto J_0(qR_c)$.

Transforms the Hamiltonian from real space density $\rho(q)$ to “guiding center” space density $\tilde{\rho}(q) \propto \sum_x e^{-iq_x x}\, c^+_{x+\frac{q_y}{2}}\, c_{x-\frac{q_y}{2}}$.

The Hartree Fock energy

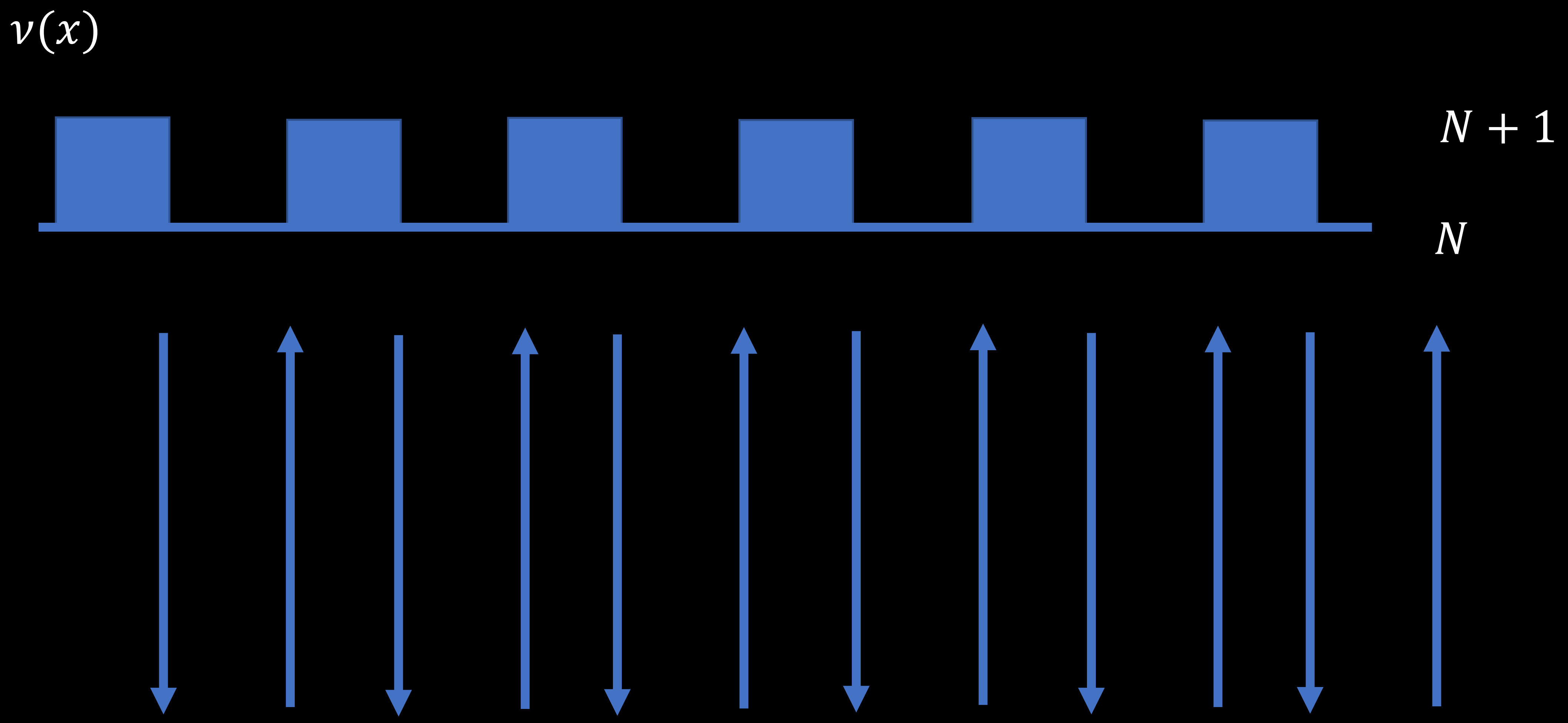
- Hartree: always positive, charging energy due to non-uniform charge density

  The Hartree potential $V_0(q)|F_N(q)|^2\langle\rho_L(-q)\rangle$
  May be minimized to zero

- Fock: always negative

- The key – make the charging energy vanish. Possible due to the oscillations of $F_N(q)$

At half filled levels – a striped phase $(q_y = 0)$



Edge states pattern leads to anisotropy

Beyond Hartree-Fock, finite temperature

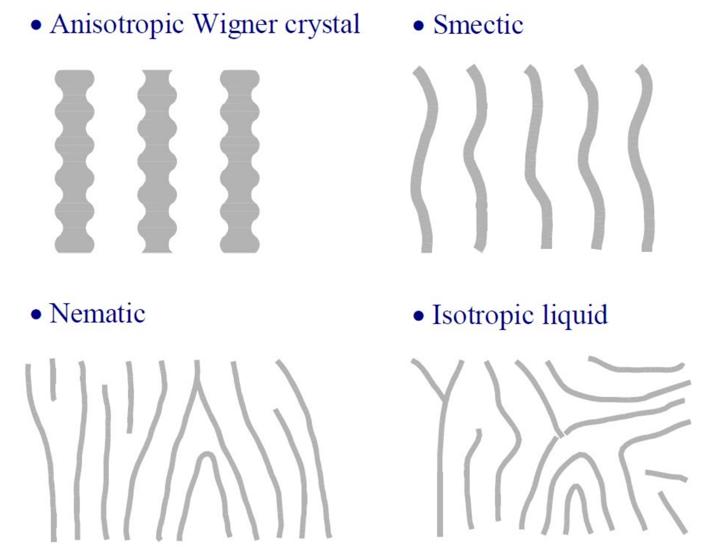


Fig. 9. Sketches of possible stripe phases

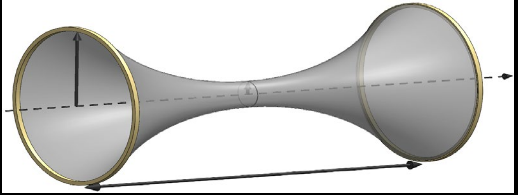
(Fogler)

Away from the half-filled level – insulating bubble phases.

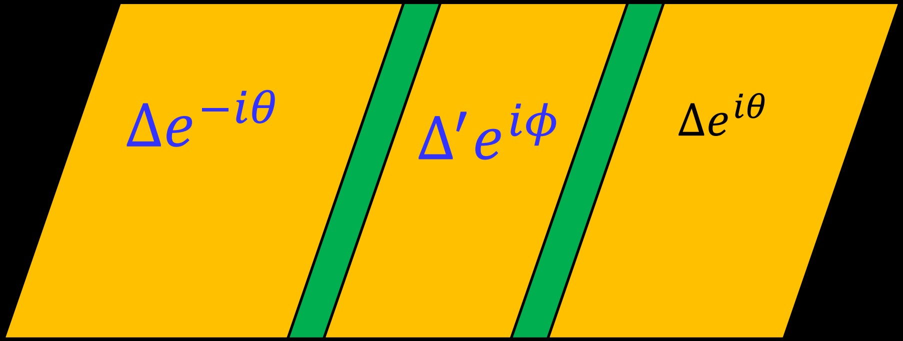
Before going off stage, I will go off topic…

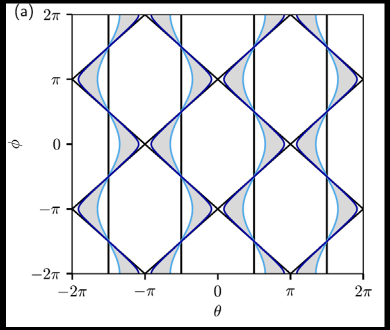
Electron hydrodynamics (together with Scaffidi, Reuven, and the Ilani group)

- Drude theory – momentum loss to impurities $R \propto L$
- Landauer-Sharvin theory – resistivity with no impurities
- Scattering of electrons off one another – spread and erase $R \propto 1/L$



A magnetic-field-free setup to create 1D topological superconductivity (with Lesser and Oreg)





Using symmetries to flatten Dirac cones (with Sheffer and Queiroz)

- Making the Dirac velocity vanish for a Dirac cone on the surface of a 3D topological insulator
- Constructing new models for chiral-limit-based perfectly flat bands